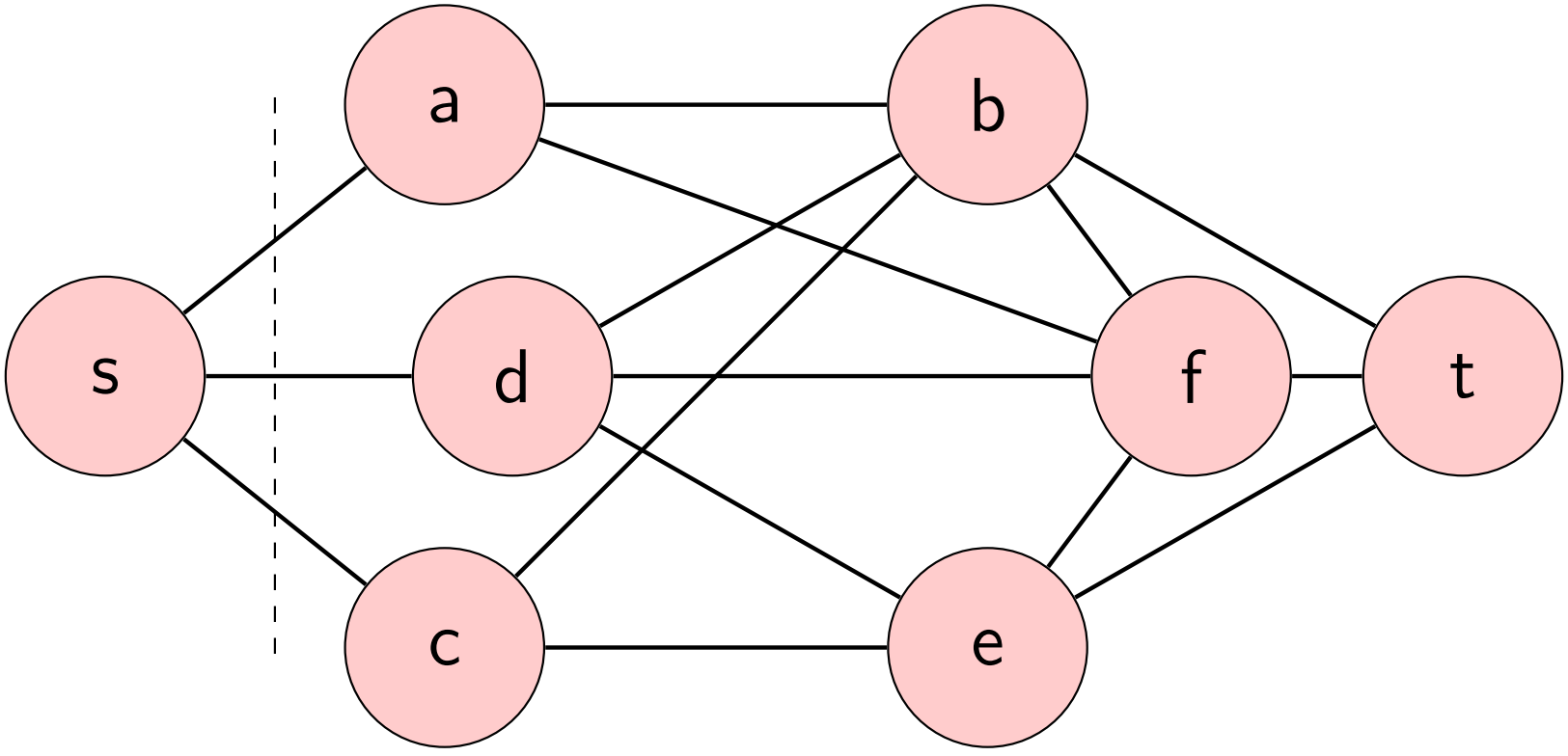
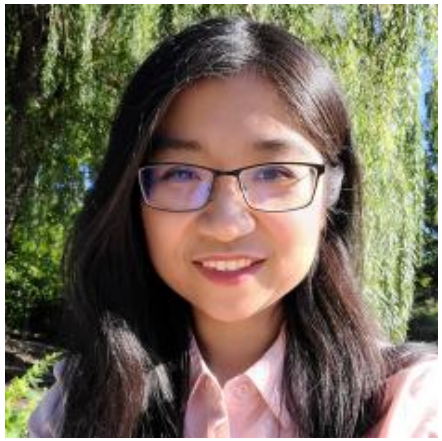


Lipschitz Graph Algorithms via Proximal Gradient Analysis

Q.C. Liu, G. Velegkas, Y. Yoshida, F. Zhou

Yale University, National Institute of Informatics

Collaborators



Quanquan C. Liu
(Yale)

Grigoris Velegkas
(Yale)

Yuichi Yoshida
(NII)

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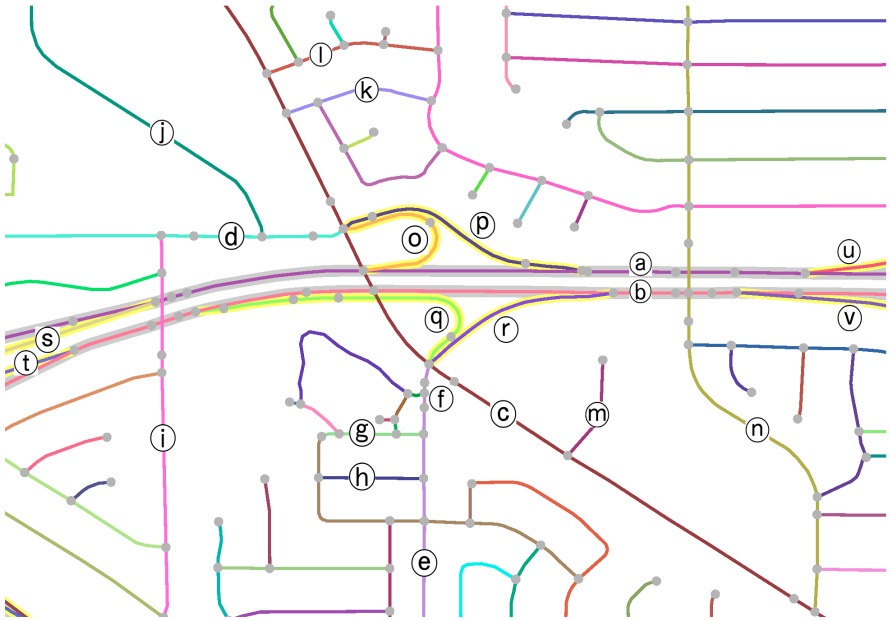
Example

Bottlenecks in a traffic network



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Bottlenecks in a traffic network

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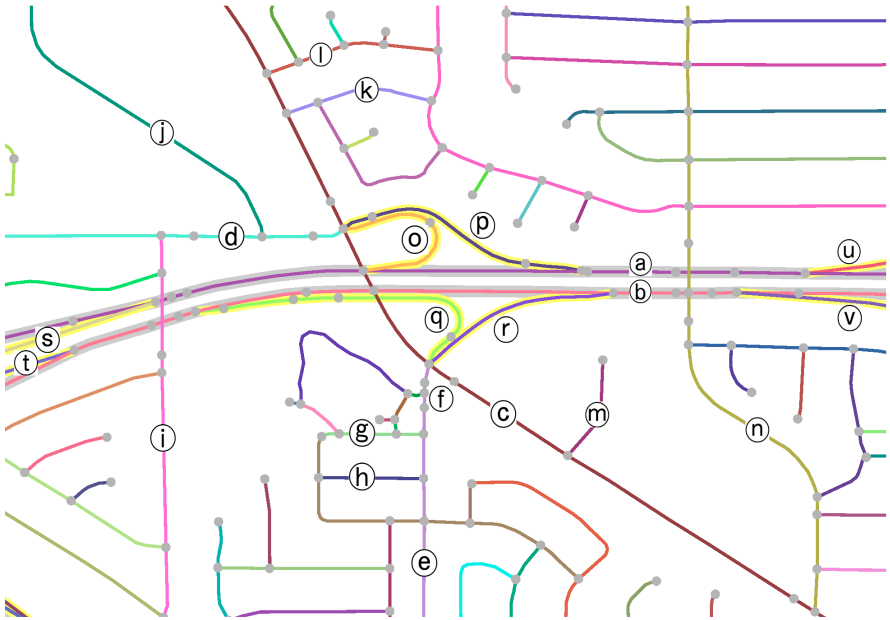

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Bottlenecks in a traffic network

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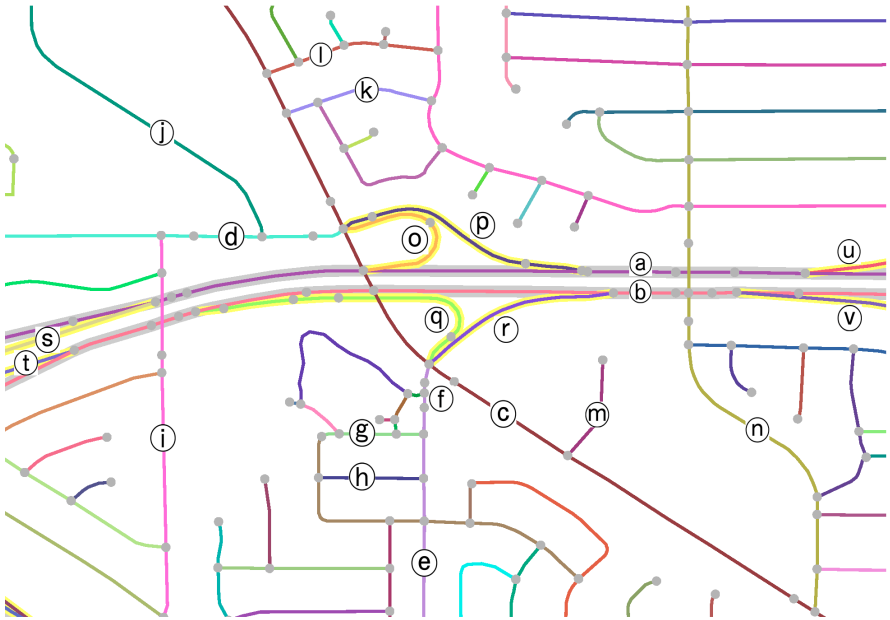

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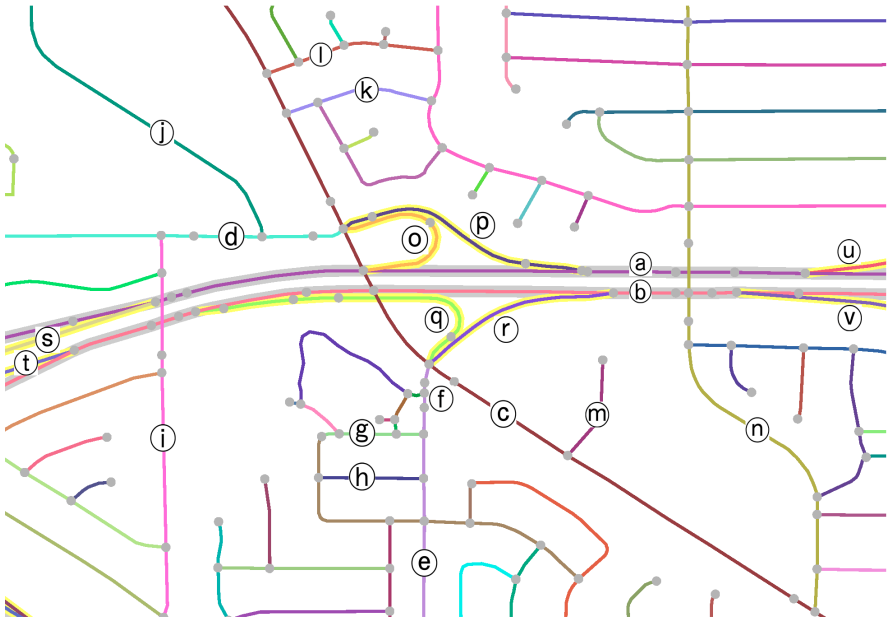

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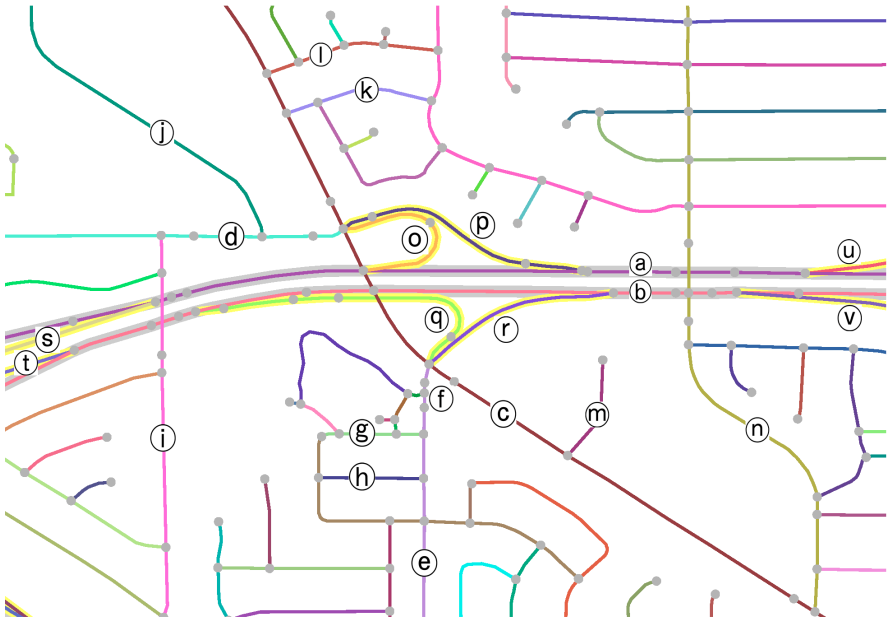
- Bottlenecks as min s-t cuts



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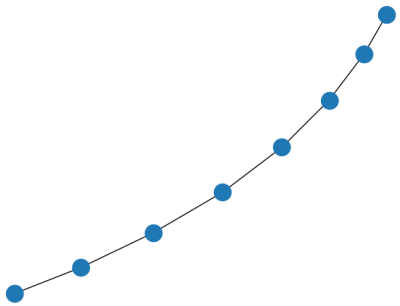

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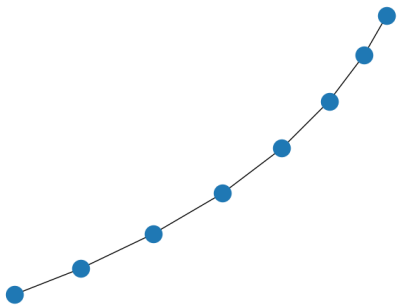

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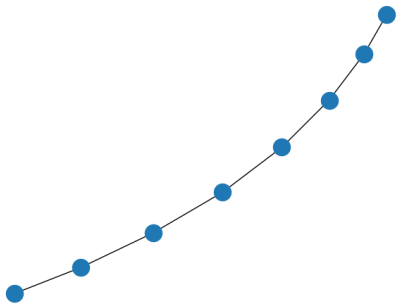

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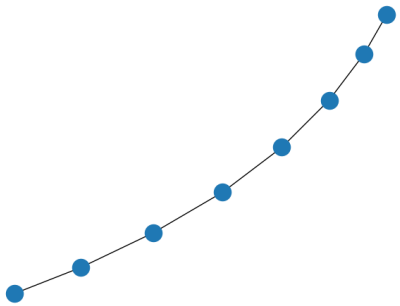

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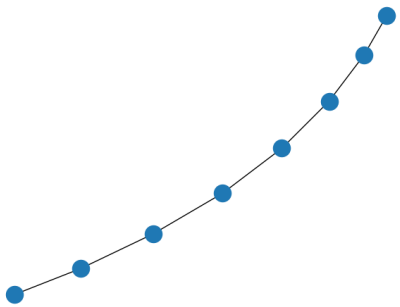

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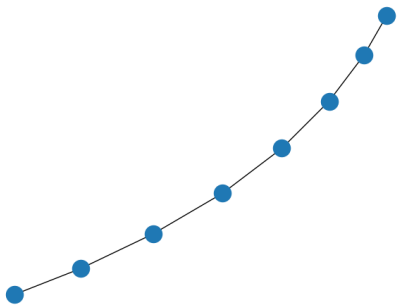

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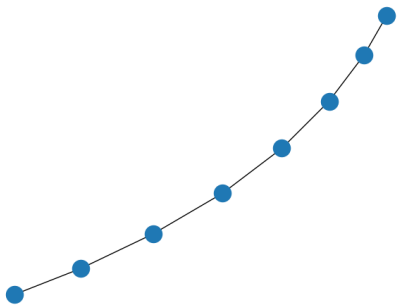

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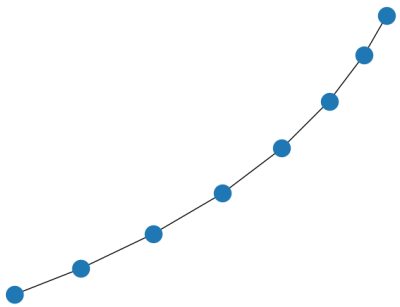

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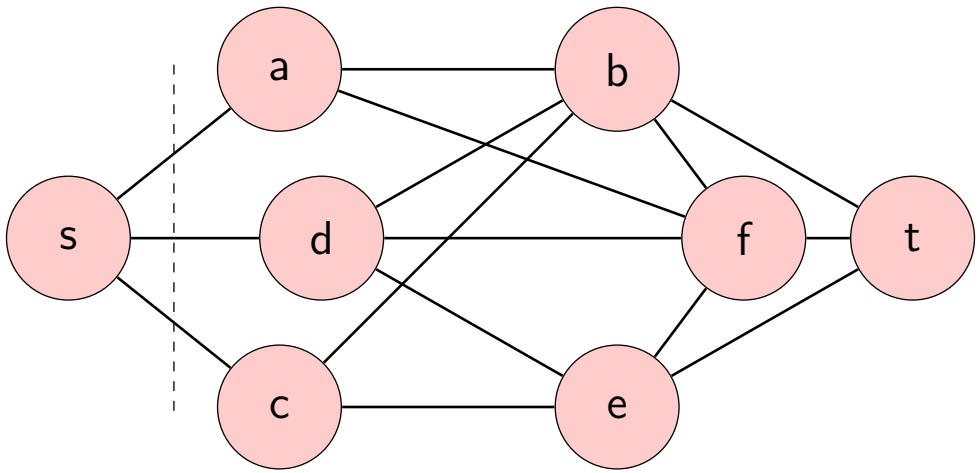
(Undirected) Minimum s-t Cut

Given:

- Graph $G = (V, E)$

Want:



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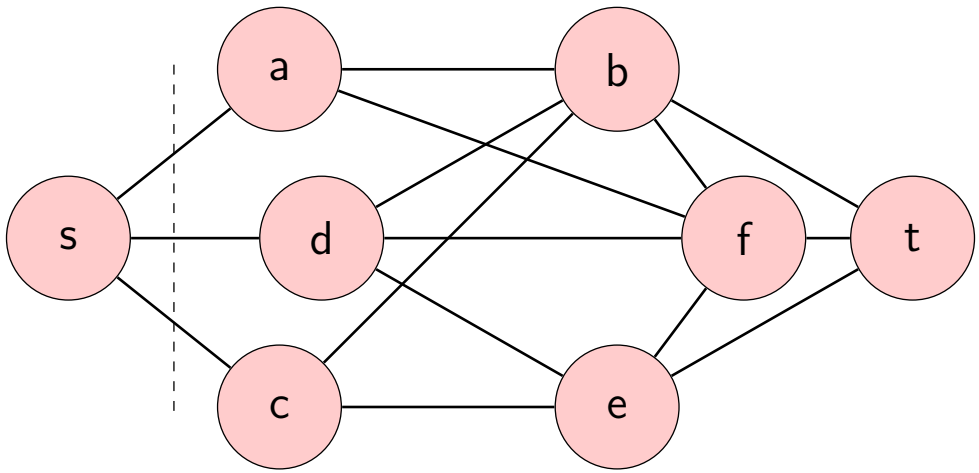

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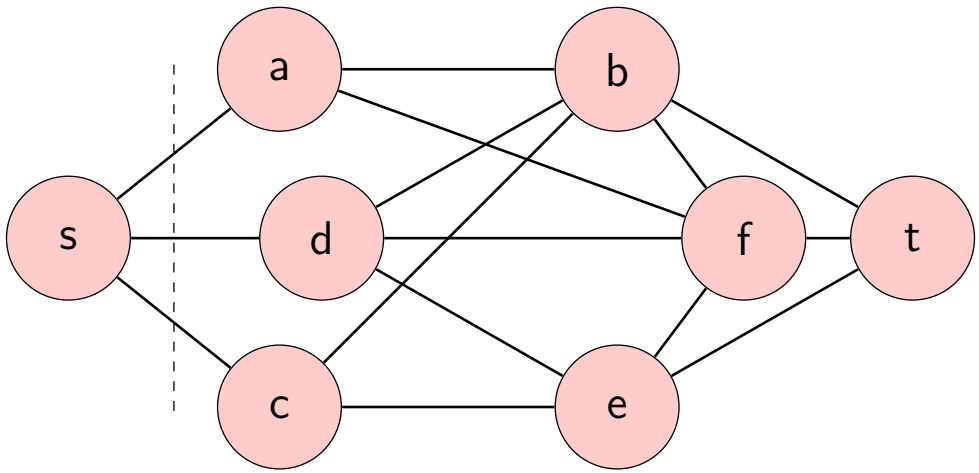

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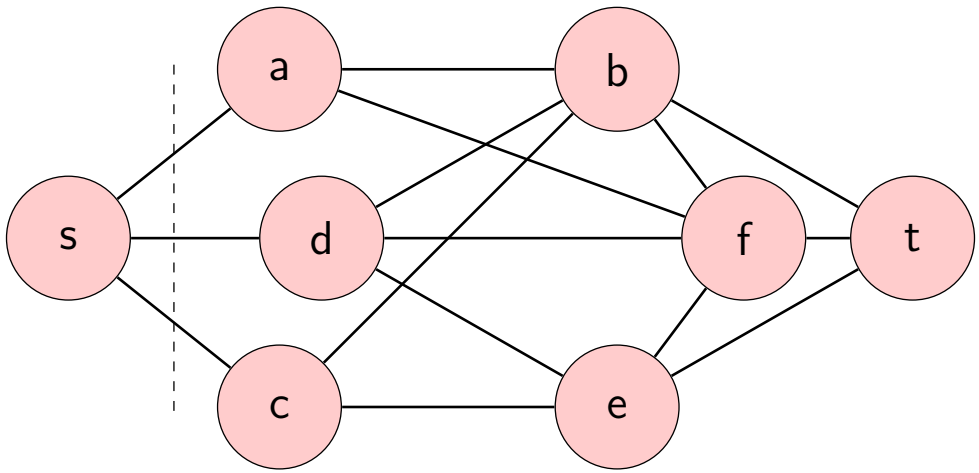

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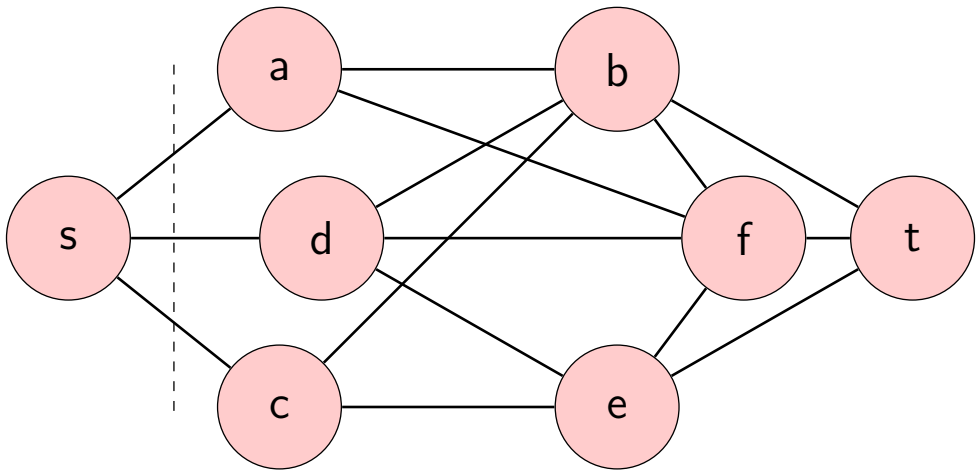

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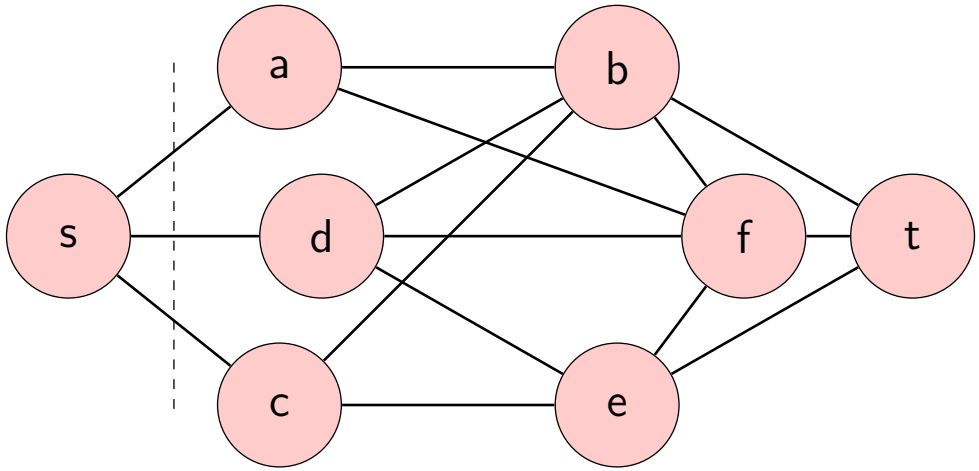

Pointwise Lipschitz Constant of Output

And: small pointwise Lipschitz constant

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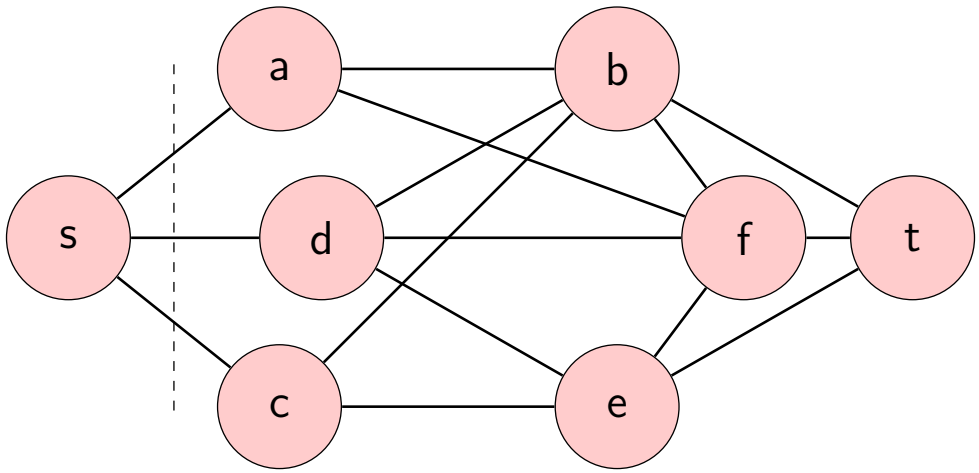

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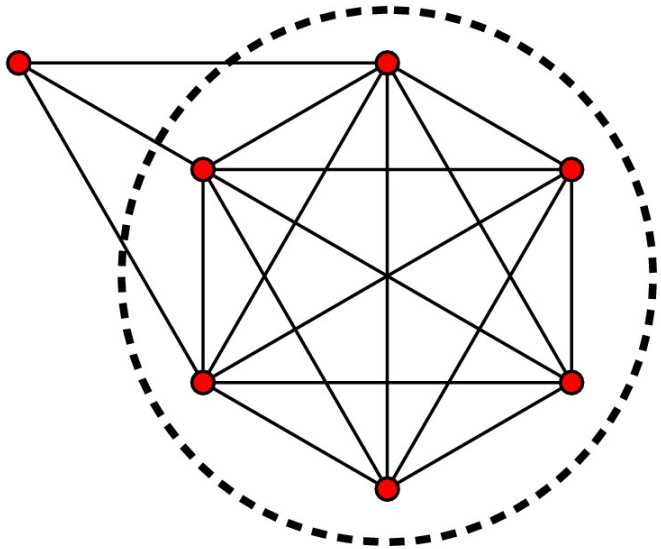

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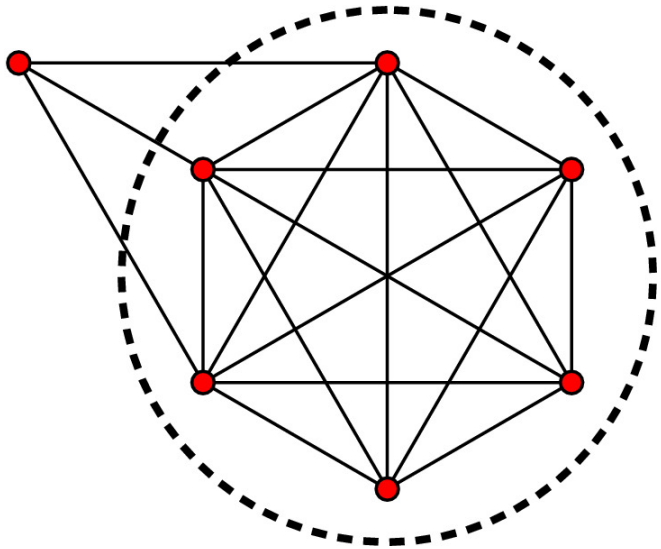

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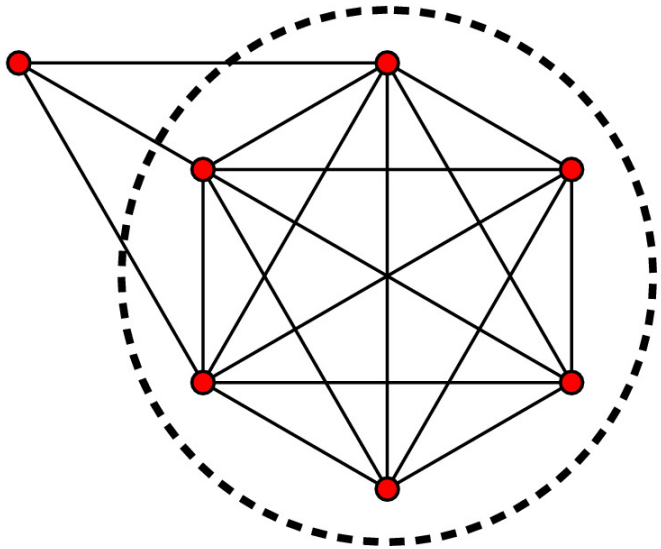

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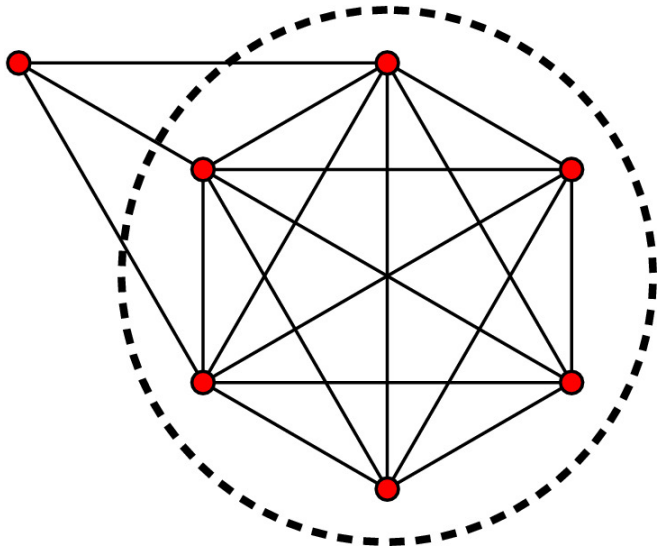

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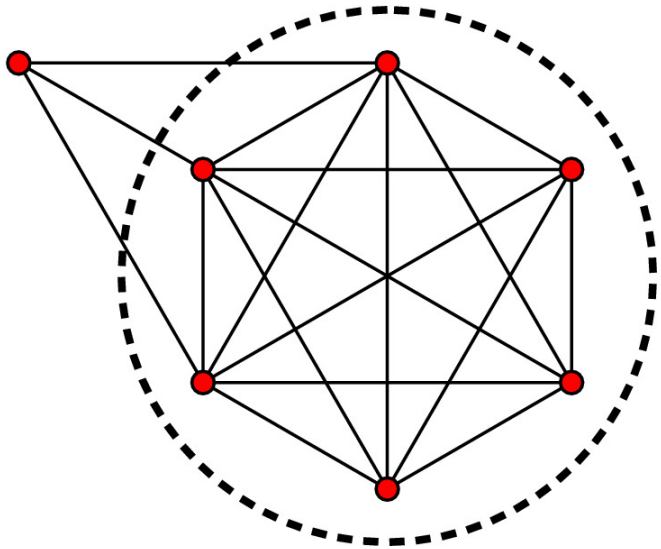

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Definition introduced by Kumabe & Yoshida [KY23a, FOCS].
Considered weighted metric on output space

- Minimum spanning tree
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Briefly studied unweighted metric (our focus)

- Minimum spanning tree
  - $(1+\varepsilon)$-approximation
  - $O(n/\varepsilon \operatorname{OPT})$ pointwise Lipschitz constant
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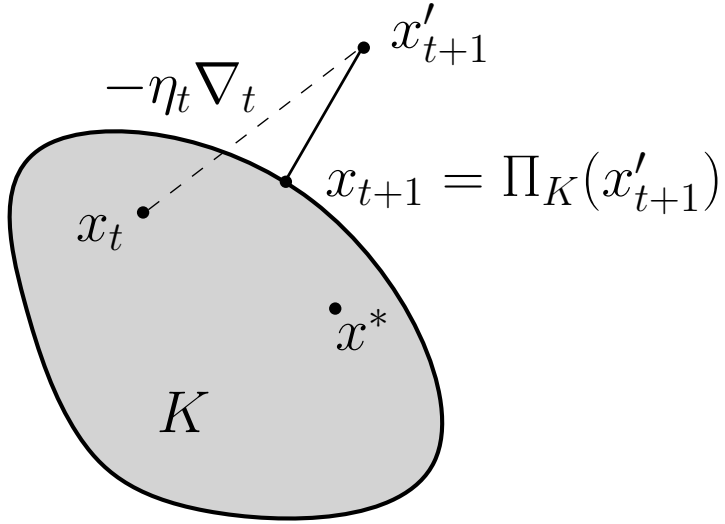
Proximal Gradient Analysis

Recall proximal gradient method:

- Want to solve composite $\min_{x \in K} f(x) + g(x)$



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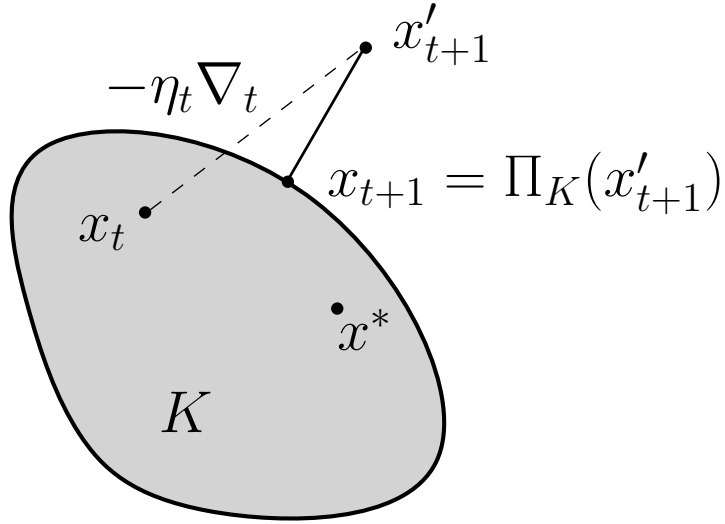

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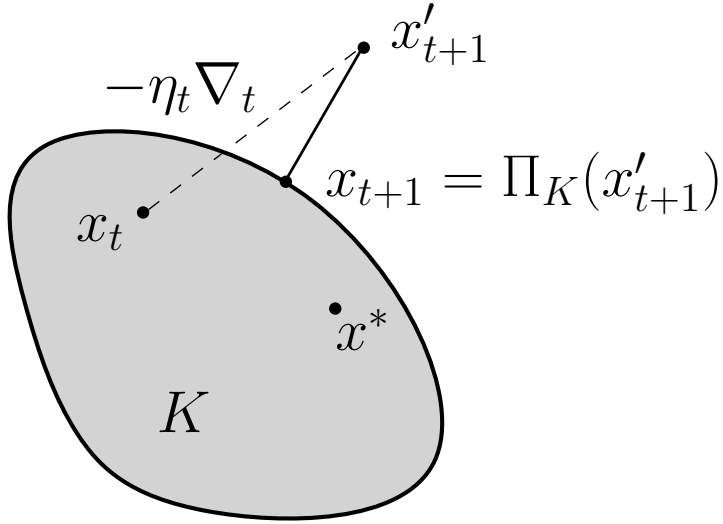

Proximal Gradient Analysis

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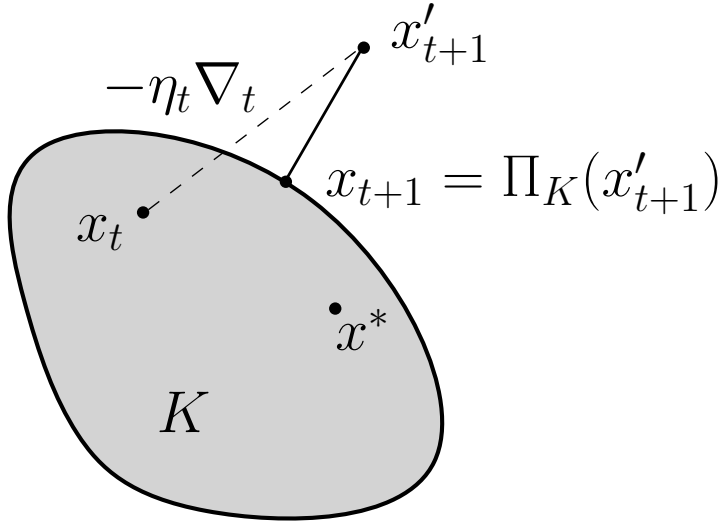
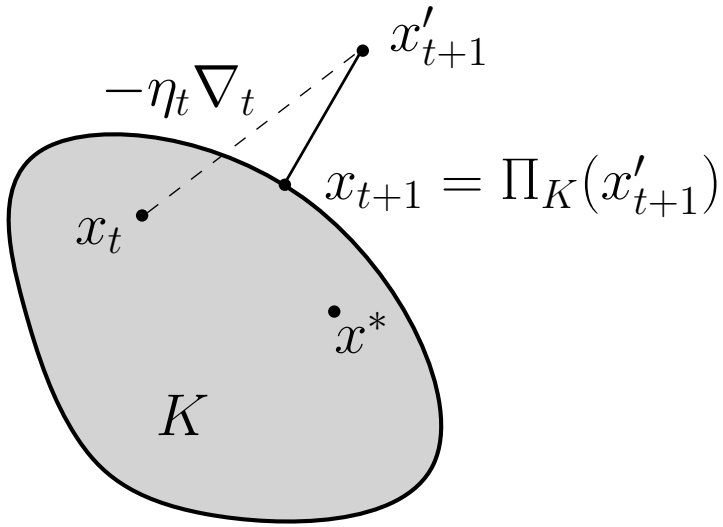


Proximal Gradient Analysis

Iterate

$$x^{t+1} \leftarrow \text{prox}_{L,f}\left(x^t - \frac{1}{L}\nabla g(x^t)\right)$$

$$\text{prox}_{L,f}(y) := \text{argmin}_{z \in K}\, f(z) + \frac{L}{2}\|z - x\|_2^2$$



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THEOREM (INFORMAL; LVYZ '24)

If $\|\nabla g - \nabla \tilde{g}\|_2$ is "small" and $\left\|\operatorname{prox}_{L,f} - \operatorname{prox}_{L,\tilde{f}}\right\|_2$ is "small", then $\|x^\star - \tilde{x}^\star\|_2$ is "small".

Application to Minimum s-t Cut

- $\varepsilon > 0$

minimize
$\sum_{uv \in E} w_{uv} |y_u - y_v|$
$+ \frac{\varepsilon}{2} \sum_{uv \in E} w_{uv} (y_u - y_v)^2$

subject to
$y_s - y_t = 1$
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We have $\|y^\star - \tilde{y}^\star\|_2 = O(\delta/\varepsilon\lambda_2)$

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- $\|y^\star - \tilde{y}^\star\|_1 = O(\delta\sqrt{n}/\varepsilon\lambda_2)$
- $\sum_{uv\in E} w_{uv}|y^\star_u - y^\star_v| \le (1+\varepsilon)\,\mathrm{OPT}$

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Threshold cut

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- But S_τ is s-t cut w.p. only $\approx 1/2$ ...

Rounding a Fractional Solution

Exponential mechanism + threshold cut

1. Solve LP(i) for each $i \in [k]$

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$$\sum_{uv \in E} w_{uv} |y_u - y_v| + \frac{\varepsilon}{2} y^\top \mathcal{L}(G, w) y$$

subject to

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Rounding a Fractional Solution

Exponential mechanism + threshold cut

1. Solve LP(i) for each $i \in [k]$
2. Sample $i^\star = M_\eta(\mathrm{OPTLP} \in \mathbb{R}^k)$ w.p. $\propto \exp(-\eta\,\mathrm{OPTLP}(i))$ (exponential mechanism M_η)

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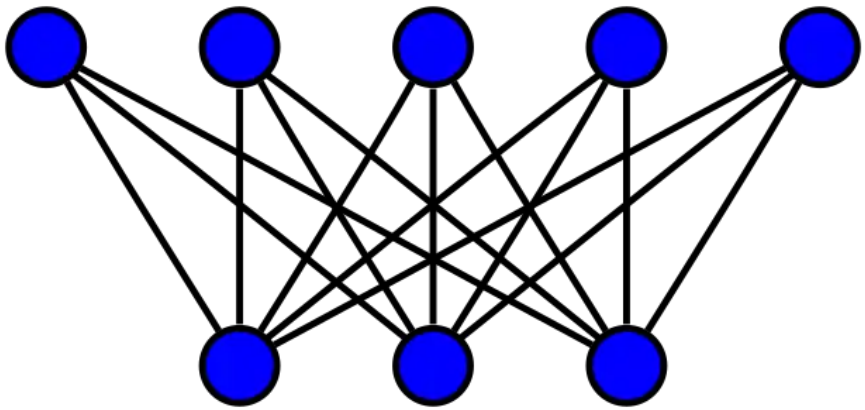
-

$$
\begin{aligned}
&\mathbb{E}_{M,\tilde{M},\tau}[|S_\tau \Delta \tilde{S}_\tau|] \\
&\le (1 - \mathrm{TV}(M, \tilde{M})) \cdot O(\|y^\star - \tilde{y}^\star\|_1) + \mathrm{TV}(M, \tilde{M}) \cdot n \\
&\le O\left(\frac{\delta\sqrt{n}}{\varepsilon\lambda_2}\right) + O(\eta k \delta) \cdot n \\
&= O\left(\frac{n}{\lambda_2}\right) \qquad\qquad (\eta = 1/k\lambda_2, \varepsilon = 1/\sqrt{n})
\end{aligned}
$$

Remarks

- $\lambda_2 \neq \tilde{\lambda}_2$, need to sample from $U[\lambda_2/2, \lambda_2]$



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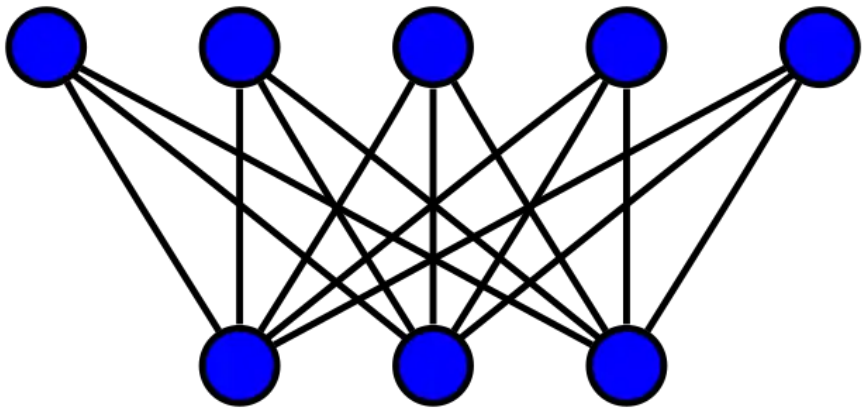

Remarks

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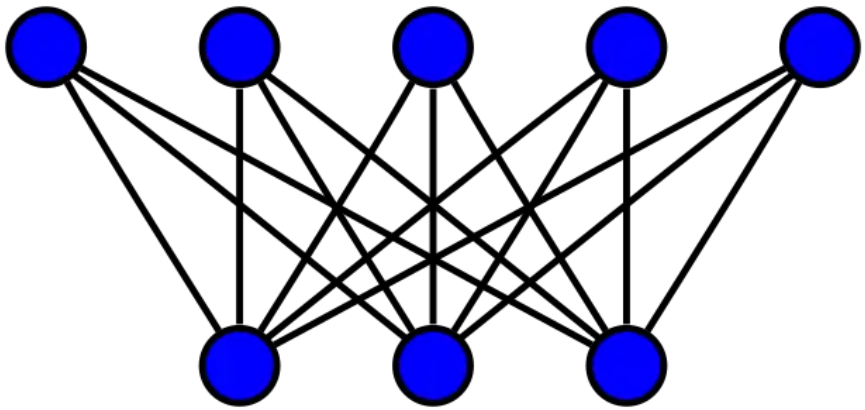
- Lower bound attained by a complete bipartite graph



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Thank You!

arxiv.org/abs/2405.08938