On the Complexity of Nucleolus Computation for Bipartite b-Matching Games

Jochen Könemann, Justin Toth, Felix Zhou

Cooperative Game Theory

- Setting: Finite collection N of players, any subset $S \subseteq N$ can collaborate to generate revenue.

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  - $\nu(S)$ is revenue of coalition S.

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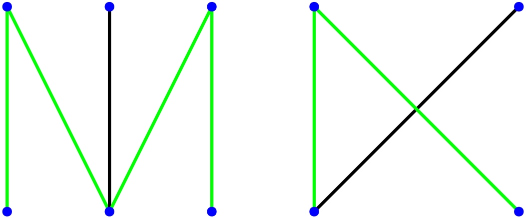
- *Allocation:* $x \in \mathbb{R}^N : x(N) = \nu(N)$.
- *Imputation:* Subset of allocations such that $x(i) \geq \nu(\{i\})$ for all $i \in N$.

(Non-simple) b-Matching Problem

- Graph $G = (N, E)$.



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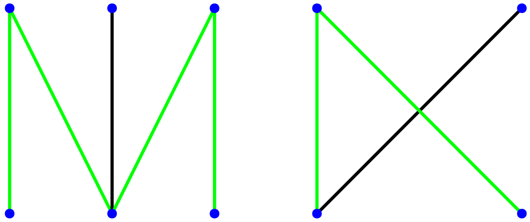

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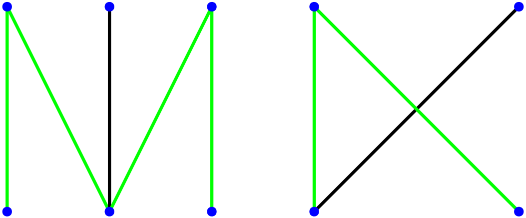

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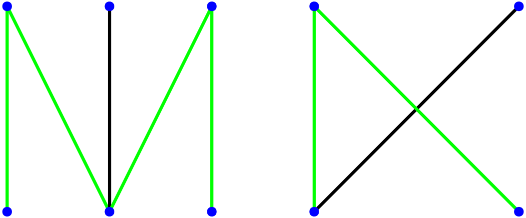

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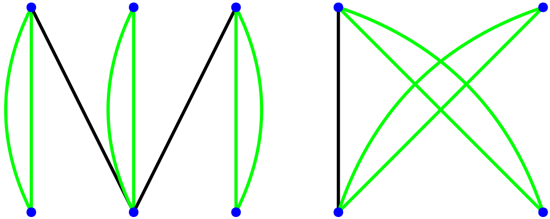

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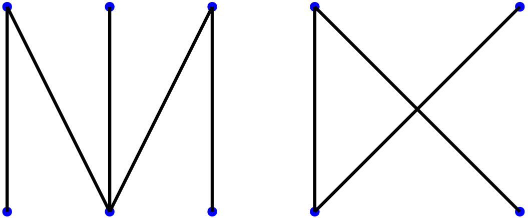

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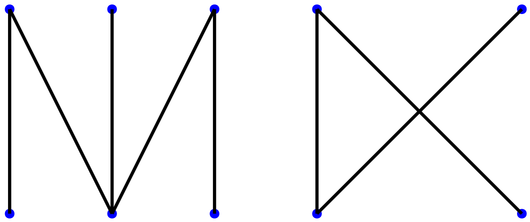

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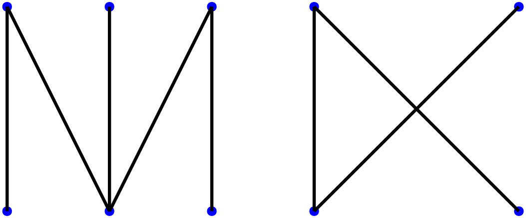
- Characteristic Function: $\nu(S)$ is the weight of a maximum weight b-matching in $G[S]$.



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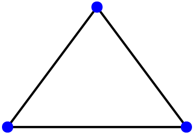
- Imputations: Non-negative singleton excess.
- *Core:* Subset of imputations such that $e(S, x) \geq 0$ for all $S \subseteq N$.

Core

- 2012; Biro, Kern, Paulusma: Stable matchings with payments (variant of stable marriage problem) correspond to core allocations.



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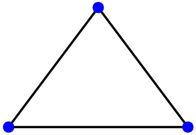

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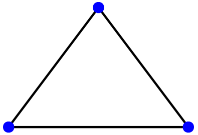

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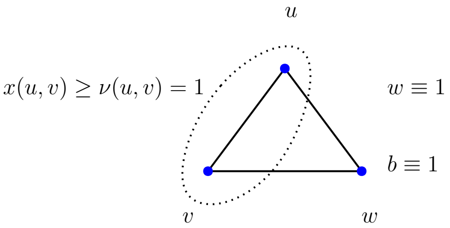
- 1999; Deng, Ibaraki, Nagamochi: The core is non-empty in bipartite b-matching games.
  - The core of a combinatorial optimization game is non-empty if and only if the fractional LP of the underlying optimization problem has integral optimal solutions.
- The core can be empty, even for 1-matching games.



Nucleolus

- Alternative definition of "fairness"?

$$\begin{aligned} & \max \epsilon && LP_1 \\ x(N) &= \nu(N) && \\ x(S) &\geq \nu(S) + \epsilon && \forall \varnothing \neq S \subsetneq N \end{aligned}$$

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- Alternative definition of “fairness”?
- Idea: Maximize the satisfaction among the worst-case coalitions.

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- Why stop at the worst-case coalitions?
- $\Theta(x) \in \mathbb{R}^{2^n-2}$: Entries are $e(S,x), \varnothing \neq S \subset N$, sorted in non-decreasing order.
- *Nucleolus:* (Unique) imputation maximizing $\Theta(x)$ lexicographically.

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- The nucleolus is unique.
- If core is non-empty, nucleolus is a member of the core.

Kopelowitz Scheme

- How can we compute the nucleolus?

$$LP_1 : \max \epsilon$$

$$x(N) = \nu(N)$$

$$x(S) \geq \nu(S) + \epsilon \quad \forall \varnothing \neq S \subsetneq N$$

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- How can we compute the nucleolus?
- Idea: Solve a sequence of recursively defined linear programs $LP_k, k \geq 1$.

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- Define LP_{k+1} by fixing new tight constraints.

$$LP_{k+1} : \max \epsilon$$

$$x(S) = \nu(S) + \epsilon_k \quad \forall S \in \mathcal{J}_r, 1 \leq r \leq k$$

$$x(S) \geq \nu(S) + \epsilon \quad \forall S \in 2^N \setminus \bigcup_{r=1}^{k} \mathcal{J}_r$$

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- Maschler's scheme: Variant of Kopelowitz scheme which guarantees termination after $O(|N|)$ iterations.

Main Results

Theorem [Könemann, Toth, Zhou '21]

Deciding whether an allocation is the nucleolus of an unweighted bipartite 3-matching game is NP-hard, even in graphs of maximum degree 7.

Theorem [Könemann, Toth, Zhou '21]

Computing the nucleolus of a bipartite b-matching game is NP-hard, even when $b \leq 3$ and the underlying graph is sparse.

Positive Results

Theorem [**Könemann, Toth, Zhou '21**]

Let $G = (N, E), w$, and $b \leq 2$ be an instance of b-matching. Suppose G has bipartition $N = A \cup B$. Let $k \geq 0$ be a universal constant.

- Suppose $b_v = 2$ for all $v \in A$ but $b_v = 2$ for at most k vertices of B, then the nucleolus of the b-matching game in G is polynomial-time computable.

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- If $b \equiv 2$, then the nucleolus of the non-simple b-matching game on G is polynomial-time computable.

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- 2018; Könemann, Pashkovich, Toth: The nucleolus is computable in polynomial time.

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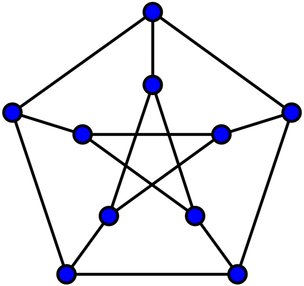
  - Surprisingly, our work answers this in the negative.

Hardness Proof Overview

- Cubic Subgraph Problem: Given a graph $G = (N, E)$, does it contain a subgraph where each vertex has degree 3?



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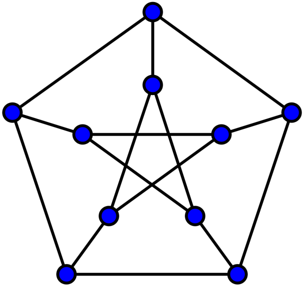

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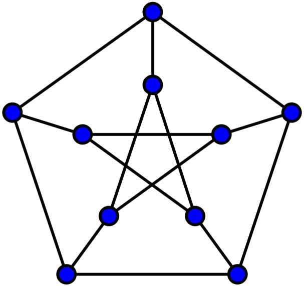
- Two From Cubic Subgraph Problem: Given a graph $G = (N, E)$, does it contain a subgraph where every vertex has degree 3 *except for two vertices of degree 2*?



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  - "Either $V(Y) \subseteq V(X)$ or $V(Y) \cap V(X) = \varnothing$".

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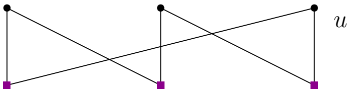

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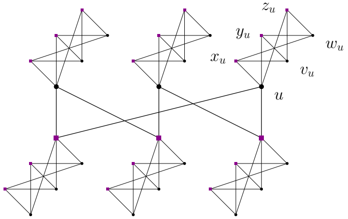

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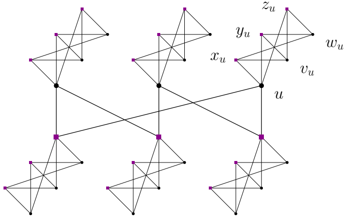
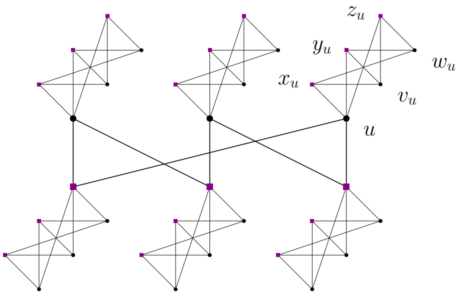
- Create gadget graph $G^* = (N^*, E^*)$ by "adding a $K_{3,3}$" to every vertex.
- The nucleolus of the unweighted 3-matching game on G^* is "some specific allocation" if and only if G does not contain a two from cubic subgraph.



Gadget Graph

- The maximum cardinality 3-matching on G^* has size $\frac{3}{2}|N^*|$.



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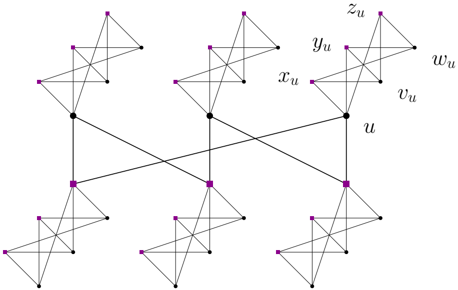

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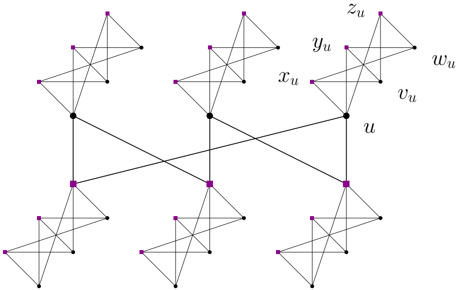

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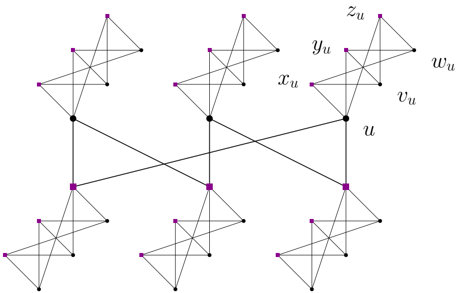
  - "some allocation" resides in the core of game on G^* if and only if G has no cubic subgraph.



The Reduction

Theorem [**Könemann, Toth, Zhou '21**]

$x \equiv \frac{3}{2}$ is the nucleolus of the 3-matching game on G^* if and only if G does not contain a two from cubic subgraph.

The Reduction

- Let (x^*, ϵ_k) be an optimal solution to each LP_k of Kopelowitz scheme.

$$
\begin{aligned}
&\max \epsilon && LP_k \\
&x(S) = \nu(S) + \epsilon_r && \forall S \in \mathcal{J}_r, 0 \le r \le k-1 \\
&x(S) \ge \nu(S) + \epsilon && \forall S \in \mathcal{J} \setminus \bigcup_{r=0}^{k-1} \mathcal{J}_r
\end{aligned}
$$

The Reduction

- Let (x^*, ϵ_k) be an optimal solution to each LP_k of Kopelowitz scheme.
- If there is no two from cubic subgraph, $\epsilon_1 = 0, \epsilon_2 = \frac{3}{2}$, and $x^* \equiv \frac{3}{2}$ is the unique optimal solution to LP_2.

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- If there is a two from cubic subgraph, $\epsilon_1 = 0$ and $x \equiv \frac{3}{2}$ is not optimal in LP_2.

$$
\begin{array}{ll}
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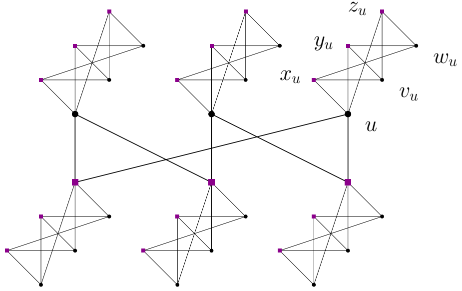
$$

Case I: No Two From Cubic Subgraph

$\epsilon_1 = 0$

- Core is non-empty for bipartite graphs, so $\epsilon_1 \geq 0$.



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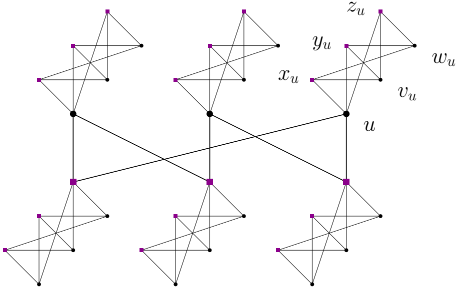

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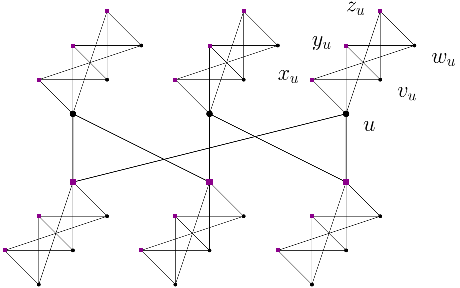

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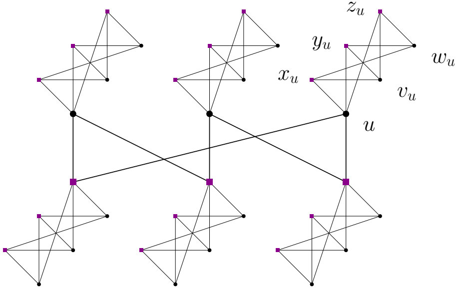
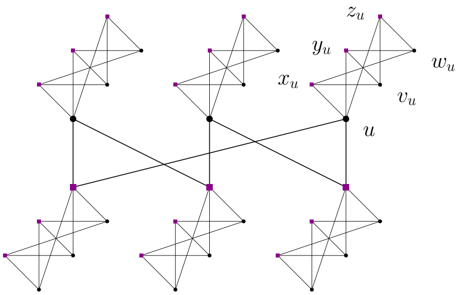
- The only coalitions fixed in LP_1 are the union $K_{3,3}$ gadgets.



Case I: No Two From Cubic Subgraph

$\epsilon_2 = \frac{3}{2}$

- Since $x(N^*) = \frac{3}{2}|N^*|$, $\epsilon_2 \leq \min_{v \in N^*} x^*(v) \leq \frac{3}{2}$.



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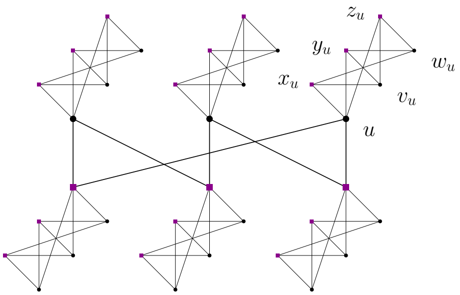

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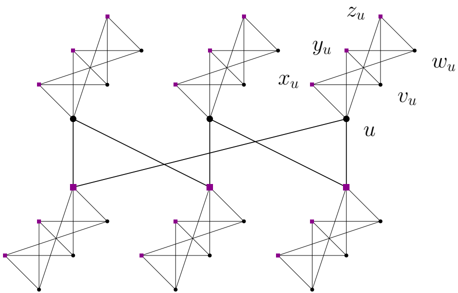
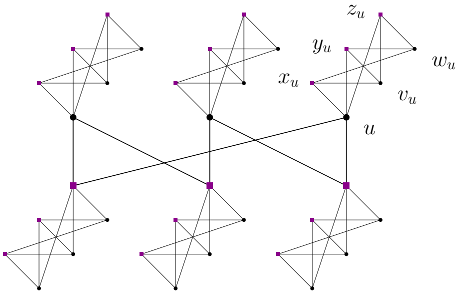
  - Uses fact that G does not contain a two from cubic subgraph.



Case II: Contains Two From Cubic Subgraph

Converse when G does contain a two from cubic subgraph is similar.

- $\epsilon_1 = 0$



Case II: Contains Two From Cubic Subgraph

Converse when G does contain a two from cubic subgraph is similar.

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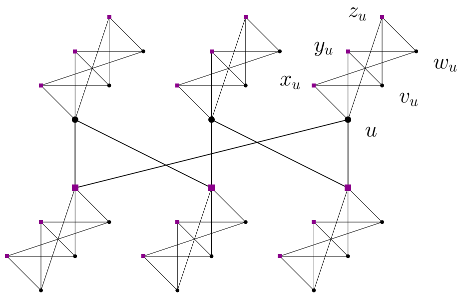
- Construct allocation which is feasible in LP_2 with strictly greater objective than $x \equiv \frac{3}{2}$.



Positive Results

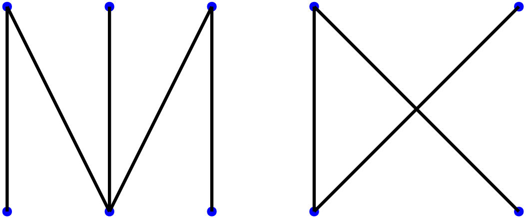
Theorem [**Könemann, Toth, Zhou '21**]

Let $G = (N, E), w$, and $b \leq 2$ be an instance of b-matching. Suppose G has bipartition $N = A \cup B$. Let $k \geq 0$ be a universal constant.

- Suppose $b_v = 2$ for all $v \in A$ but $b_v = 2$ for at most k vertices of B, then the nucleolus of the b-matching game in G is polynomial-time computable.
- If $b \equiv 2$, then the nucleolus of the non-simple b-matching game on G is polynomial-time computable.

Positive Results

- Prune constraints from Kopelowitz scheme which are “not necessary”.



$\nu = 4$ $\nu = 2$

$w \equiv 1$

$b \equiv 2$

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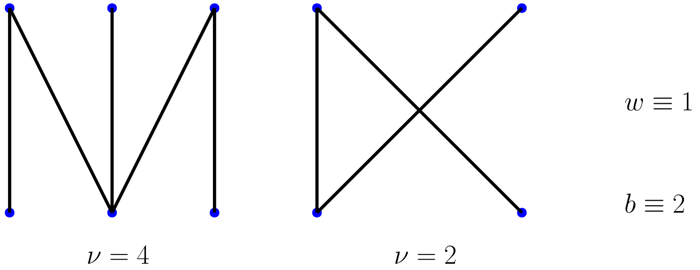
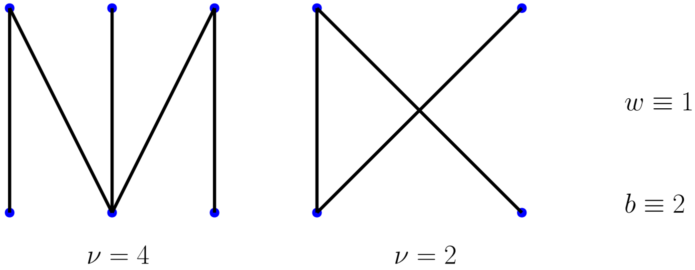


Positive Results

- Prune constraints from Kopelowitz scheme which are "not necessary".
- If the remaining constraints are polynomial-sized, the nucleolus can be computed in polynomial time.
- If core is non-empty and there is some maximum b-matching of $G[S]$ that is disconnected, S can be omitted.



Simple b-Matching Games

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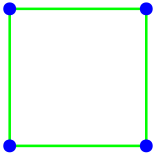
- Run Kopelowitz scheme with $O\left(|N|^{2k+3}\right)$ constraints.

Non-simple b-Matching Games

- Suppose $b \equiv 2$.



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Non-simple b-Matching Games

- Suppose $b \equiv 2$.
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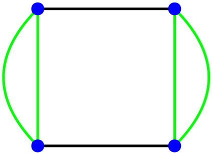

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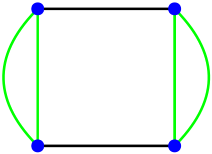

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- Can we compute the nucleolus for b-matching games in general graphs when $b \leq 2$ in polynomial time?
- Is there a combinatorial algorithm to compute the nucleolus for b-matching games?

Thanks!