PMATH 453: Functional Analysis

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Contents

Chapter 1

Topology

This is a shallow introduction to weak topologies and nets which is vital to the study of functional analysis.

1.1 Toplogy

Definition 1.1.1 (Topology)
A topology on a set X is a collection of sets $\tau \subseteq 2^X$ such that

(i) $X, \varnothing \in \tau$

(ii) Closed under arbitrary unions

(iii) Closed under finite intersections

$G \in \tau$ are referred as *open* sets while *closed* sets F are ones such that $X \setminus F \in \tau$.

Note that the arbitrary intersection of topologies is once again a topology.

Example 1.1.1
trivial topology $\tau = \{\varnothing, X\}$

discrete topology $\tau = 2^X$

metric topology

co-finite topology $\tau = \{\varnothing\} \cup \{Y \subseteq X : |X \setminus Y| < \infty\}$

Definition 1.1.2 (Neighbourhood)
Let (X, τ) be a topological space and $x \in X$. $U \subseteq X$ is a neighbourhood of x if there is some $G \in \tau$ such that $x \in G \subseteq U$.

The *neighbourhood system* at x is

$$\mathcal{U}_x := \{U \subseteq X : U \text{ is a neighbourhood of } x\}.$$

Theorem 1.1.2
Let (X, τ) be a topological space and $x \in X$. Then

(a) $U \in \mathcal{U}_x$ implies $x \in U$

(b) $U, V \in \mathcal{U}_x$ implies $U \cap V \in \mathcal{U}_x$

(c) For all $U \in \mathcal{U}_x$, there is $V \in \mathcal{U}_x$ such that $U \in \mathcal{U}_y$ for all $y \in V$

(d) $U \in \mathcal{U}_x$ and $U \subseteq V$ implies $V \in \mathcal{U}_x$

(e) $G \subseteq X$ is open if and only if G contains a neighbourhood of each of its points.

Conversely, suppose we have a set X, a non-empty collections $\mathcal{U}_x$ for each $x \in X$ satisfying (a)-(d), and we define open as in (e), the result is a topology on X in which the neighbourhood system at x is precisely $\mathcal{U}_x$.

Definition 1.1.3 (Neighbourhood Base)
Let (X, τ) be a topological base. $\mathcal{B}_x \subseteq \mathcal{U}_x$ is a neighbourhood base at x if $U \in \mathcal{U}_x$ implies that there is $B \in \mathcal{B}_x$ such that $B \subseteq U$.

We refer to elements of $\mathcal{B}_x$ as basic neighbourhoods of x.

Example 1.1.3
Let (X, d) be a metric space and $V_r(x)$ the open ball of radius r about $x \in X$. For any $\{r_n\}_{n \geq 1}$ such that $r_n \to 0$,

$$\mathcal{B}_x := \{V_{r_n}(x) : n \geq 1\}$$

is a neighbourhood base at x.

Definition 1.1.4 (Base)
Let (X, τ) be a topological space. A base for τ is a collection $\mathcal{B} \subseteq \tau$ such that every open set is a union of elements of $\mathcal{B}$.

A *subbase* for the topology is a collection $\mathcal{S} \subseteq \tau$ such that the set of finite intersections of elements of $\mathcal{S}$ forms a base for τ.

Any collection $\mathcal{C} \subseteq X$ serves as subbase for some topology on X, called the *topology generated by* $\mathcal{C}$.

Example 1.1.4

neighbourhood bases Let $\mathcal{B}_x \subseteq \tau$ be a neighbourhood base at each $x \in X$. Then $\bigcup_x \mathcal{B}_x$ is a base for τ

open intervals The set of all finite open intervals form a base for $\mathbb{R}$ under the usual topology, a subbase is intervals of the form $(-\infty, a), (b, \infty)$ for $a, b \in \mathbb{R}$

1.2 Separation

Let (X, τ) be a topological space.

Definition 1.2.1 (T_0)
(X, τ) is T_0 if for every $x \neq y \in X$, either there is some neighbourhood $U_x \in \mathcal{U}_x$ with $y \notin U_x$ or there is a neighbourhood $U_y \in \mathcal{U}_y$ such that $x \notin U_y$.

Definition 1.2.2 (T_1)
(X, τ) is T_0 if for every $x \neq y \in X$, either there are neighbourhoods $U_x \in \mathcal{U}_x$ with $y \notin U_x$ and $U_y \in \mathcal{U}_y$ such that $x \notin U_y$.

Definition 1.2.3 (T_2 / Hausdorff)
(X, τ) is T_0 if for every $x \neq y \in X$, either there are disjoint neighbourhoods $U_x \in \mathcal{U}_x$ and $U_y \in \mathcal{U}_y$.

We say that $A, B \subseteq X$ can be *separated* by τ if there are $U, V \in \tau$ such that $A \subseteq U, B \subseteq V$ and $U \cap V$.

Definition 1.2.4 (Regular)
(X, τ) is regular if whenever $F \subseteq X$ is closed and $x \notin F$, $F, \{x\}$ can be separated.

Definition 1.2.5 (Normal)
(X, τ) is normal if whenever $F_1, F_2 \subseteq X$ are closed and disjoint, F_1, F_2 can be separated.

(X, τ) is T_3 if it is T_1 and regular.

(X, τ) is T_4 if it is T_1 and normal.

Theorem 1.2.1
Every metric space equipped with the metric topology is T_4.

Proof
Define $g : X \to \mathbb{R}$ by $g(x) := d(x, F_1) - d(x, F_2)$. Then $F_1 \subseteq g^{-1}(-\infty, 0)$ and $F_2 \subseteq g^{-1}(0, \infty)$.

1.3 Compactness

Let (X, τ) be a topological space Recall that an *open cover* of X is some $\mathcal{G} \subseteq \tau$ such that $X = \bigcup_{G \in \mathcal{G}} G$. A *finite subcover* is a finite subset $G' \subseteq \mathcal{G}$ that remains a cover. We say (X, τ) is compact if every open cover of X admits a finite subcover.

Theorem 1.3.1
Every compact Hausdorff topological space is T_3 and T_4.

Proof
Let $x \notin F \subseteq X$. Then for every $y \in F$, we can find some $U_y \in \mathcal{U}_x, V_y \in \mathcal{U}_y$ that are disjoint.

Then $F \subseteq \bigcup_y U_y$ is a open cover of the compact set F and we extract a finite subcover $F_i, i \in [n]$. Then $x \in \bigcap_i U_y, F \subseteq \bigcup_i V_y$.

Repeat this idea for T_4.

Recall that (X, τ) is *separable* if it admits a countably dense subset. That is, every non-empty open set contains a member of this subset.

Proposition 1.3.2
Every compact metric space is separable.

1.4 Nets & Continuity

Definition 1.4.1 (Directed Set)
A set Λ is directed if it is equiped with a relation $\leq$ such that

(i) $\lambda \leq \lambda$ for all $\lambda \in \Lambda$

(ii) $\lambda_1 \leq \lambda_2$ and $\lambda_2 \leq \lambda_3$ implies $\lambda_1 \leq \lambda_3$

(iii) Every $\lambda_1, \lambda_2 \in \Lambda$ has some $\lambda_3 \in \Lambda$ such that $\lambda_1, \lambda_2 \leq \lambda_3$

$\leq$ is referred to as a *direction* on Λ.

Let (X, τ) be a topological space.

Definition 1.4.2 (Net)
A net in X is a function $P : \Lambda \to X$ with Λ being a directed set.

We typically write $(x_\lambda)_{\lambda \in \Lambda}$ to denote a net.

Let $\varphi : M \to \Lambda$ be a function between directed sets. We say φ is *increasing* if $\mu_1 \leq \mu_2 \implies \varphi(\mu_1) \leq \varphi(\mu_2)$. Also, we say φ is *cofinal* if for each $\lambda \in \Lambda$, there is some $\mu \in M$ so that $\lambda \leq \varphi(\mu)$.

Definition 1.4.3 (Subnet)
A subnet of a net $P : \Lambda \to X$ is the composition $P \circ \varphi$, where $\varphi : M \to \Lambda$ is an increasing cofinal function.

We typically write $(x_{\lambda_\mu})_\mu$.

Definition 1.4.4 (Net Convergence)
The net $(x_\lambda)_\lambda$ converges to $x \in X$ if for every $U \in \mathcal{U}_x$, there is some $\lambda_0 \in \Lambda$ so that $\lambda \geq \lambda_0$ implies $x_\lambda \in U$.

We write $\lim_\lambda x_\lambda = x$.

Example 1.4.1
sequences $\mathbb{N}$ is a directed set under the usual $\leq$ thus every sequence is a net

inclusion Let $\Lambda = 2^X$ and $\leq = \subseteq$. This is a directed set

For $U_1, U_2 \in \mathcal{U}_x$, define $U_1 \leq U_2 \iff U_2 \subseteq U_1$, then $(\mathcal{U}_x, \leq)$ forms a directed set. In this case, $\lim_U x_U = x$

Note that a subsequence is a net but a subnet need NOT be a sequence since its domain need not be any countable set.

The property that (X, τ) is Hausdorff is equivalent to the condition that limits of nets in X are unique.

Definition 1.4.5 (Continuous Function)
Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be such that $f^{-1}(G)$ is open for every $G \in \tau_Y$. Then f is said to be continuous.

Theorem 1.4.2
$f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous if and only if whenever $(x_\lambda)_{\lambda \in \Lambda}$ is a net converging to $x \in X$, the net $(f(x_\lambda))_{\lambda \in \Lambda}$ is a net in Y converging to $f(x)$.

1.5 Weak Topology

Definition 1.5.1
Let $\varnothing \neq X$ be a set and $\{(Y_\gamma, \tau_\gamma)\}_{\gamma \in \Gamma}$ be a family of topological spaces. Let $\mathcal{F} = \{f_\gamma : X \to Y_\gamma \mid \gamma \in \Gamma\}$ be a collection of functions.
Then

$$\mathcal{S} := \{f_\gamma^{-1}(G) : G \in \tau_\gamma, \gamma \in \Gamma\} \subseteq 2^X$$

is a subbase for the weak topology induced by $\mathcal{F}$, often written as $\sigma(X, \mathcal{F})$.

Proposition 1.5.1

(a) If τ is a topology and $f_\gamma : (X, \tau) \to (Y_\gamma, \tau_\gamma)$ is continuous for all $\gamma \in \Gamma$, then $\sigma(X, \mathcal{F}) \subseteq \tau$

(b) Let (Z, τ_Z) be a topological space. Then $G : (Z, \tau_Z) \to (X, \sigma(X, \mathcal{F}))$ is continuous if and only if $f_\gamma \circ g : Z \to Y_\gamma$ is continuous for all $\gamma \in \Gamma$

Definition 1.5.2 (Cartesian Product)
Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a collection of topological spaces. The cartesian product is

$$\prod_{\alpha \in A} X_\alpha := \{x : A \to \bigcup_\alpha X_\alpha : x(\alpha) \in X_\alpha\}.$$

The map $\pi_\beta \prod X_\alpha \to X_\beta$ given by $\pi_\beta(x) := x_\beta$ is the *β-th projection map.*

Definition 1.5.3 (Product Topology)
The product topology on $\prod X_\alpha$ is the weak topology on $\prod_\alpha X_\alpha$ induced by the family $\{\pi_\beta\}_{\beta \in A}$.

The product topology has as a base the collection $\mathcal{B} := \{\prod_\alpha U_\alpha\}$, where $U_\alpha \in \tau_\alpha$ for each α and for all but finitely many α, $U_\alpha = X_\alpha$.

It suffices to take $U_\alpha \in \mathcal{B}_\alpha$ where B_α is a fixed base for τ_α. Observe that if $U_\alpha \in \tau_\alpha$ and $U_\alpha = X_\alpha$ for all α except $\alpha_i, i \in [n]$. Then

$$\prod_\alpha U_\alpha = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}).$$

From this, we know that $\{\pi_\alpha^{-1}(U_\alpha) : U_\alpha \in \mathcal{B}_\alpha, \alpha \in A\}$ is a subbase for the product topology, where $\mathcal{B}_x$ is a fixed base or even subbase for τ_α.

Example 1.5.2
The product topology on $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$ is just the usual topology on $\mathbb{R}^n$.

Chapter 2

Normed Linear Spaces

2.1 Normed Linear Spaces

Definition 2.1.1 (Norm)
Let X be a vector space over $\mathbb{K}$. A seminorm is a map $\nu : X \to \mathbb{R}$ such that

(i) $\nu(x) \geq 0$ for all $x \in X$

(ii) $\nu(\lambda x) = |\lambda|\nu(x)$ for all $x \in X, \lambda \in \mathbb{K}$

(iii) $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in X$

(iv) $\nu(x) = 0$ if and only if $x = 0$

We usually write $\nu(x) = \|x\|$. If ν only satisfies (a)-(c), it is a *seminorm*.

The *norm topology* on $(X, \|\cdot\|)$ is the topology induced by the metric induced by the norm. Namely, the set of open balls form a base.

We say $(X, \|\cdot\|)$ is *complete* if the corresponding metric space is complete.

Example 2.1.1
Define $c_{00}^{\mathbb{K}}(\mathbb{N})$ to be the set of finitely supported sequences in $\mathbb{K}$. $(c_{00}^{\mathbb{K}}, \|\cdot\|_\infty)$ is a normed linear space under the sup norm but is not complete.

The space $c_0^{\mathbb{K}}(\mathbb{N})$ of sequences in $\mathbb{K}$ converging to 0 under the same norm is however a complete normed linear space.

We refer to vector subspaces as *linear manifolds* as they may not be closed. *Subspaces* are closed linear manifolds. Note that $c_{00}^{\mathbb{K}}(\mathbb{N})$ is a dense linear manifold in $c_0^{\mathbb{K}}(\mathbb{N})$ but not a subspace.

> **Example 2.1.2 (Polynomials)**
> Let $\mathcal{P}_{\mathbb{K}}([0,1])$ denote the set of polynomials $[0,1] \to \mathbb{K}$. Then the sup norm is a norm on $\mathcal{P}_{\mathbb{K}}([0,1])$.

Recall that the Stone-Weirstrauss theorem states that $\mathcal{P}_{\mathbb{K}}([0,1])$ is a dense linear manifold in the normed linear space $\mathcal{C}([0,1],\mathbb{K})$ under the sup norm. If we choose $x_0 \in [0,1]$, then $\nu(f) := |f(x_0)|$ defines a seminorm.

> **Example 2.1.3 (p-Norm)**
> Let $1 \leq p < \infty$. Then
> $$\|x\|_p := \left(\sum_{i=1}^n |x_k|^p\right)^{\frac{1}{p}}$$
> is a norm on $\mathbb{K}^n$.
>
> We can also define
> $$\|x\|_\infty := \max_{i\in[n]} |x_i|$$
> which is also a norm.

We typically write $\ell_p^n = (\mathbb{K}^n, \|\cdot\|_p)$.

For $1 \leq p < \infty$, we can similarly define

$$\ell_{\mathbb{K}}^p(\mathbb{N}) := \{(x_n)_{n\geq 1} : x_n \in \mathbb{K}, \sum_{n\geq 1} |x_n|^p < \infty\}$$

and imbue it with the ℓ_p norm as well as

$$\ell_{\mathbb{K}}^\infty(\mathbb{N}) := \{(x_n)_{n\geq 1} : x_n \in \mathbb{K}, \sup_n |x_n| < \infty\}$$

similary with the sup norm. These spaces are complete, unlike $c_{00}^{\mathbb{K}}(\mathbb{N})$ or $\mathcal{P}_{\mathbb{K}}([0,1])$.

2.2 Banach Spaces

Recall that a *Banach* space is a complete normed linear space.

Example 2.2.1
The set of continuous functions $[0,1] \to \mathbb{K}$, denoted $\mathcal{C}([0,1],\mathbb{K})$ is a Banach space under the supremum norm.

Example 2.2.2
Let $D \subseteq \mathbb{C}$ be the unit disc and $\mathbb{T}$ its boundary. Then the disc algebra

$$\mathcal{A}(D) := \left\{ f \in \mathcal{C}(\bar{D}) : f\Big|_D \text{ is holomorphic} \right\}$$

is a Banach space under the norm $\|p\|_\infty := \sup_{z\in\bar{D}}|p(z)|$.

Note that $\mathcal{A}(D)$ is closed under multiplication, hence it is actually a *Banach algebra*.

By the Maximum Modulus Principle, the map $\Gamma : \mathcal{A}(D) \to \mathcal{C}(\mathbb{T})$ given by $\Gamma(f) := f\big|_{\mathbb{T}}$ is isometric and hence we identify $\mathcal{A}(D)$ with the algebra

$$\{f \in \mathcal{C}(\mathbb{T}) : f \text{ extends to a holomorphic function on } D\}$$

equipped with the infinity norm on $\mathbb{T}$.

Recall that an *inner product space* is equippted with an inner product, a function $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ satisfying linearity in the first argument, conjugate symmetry, and positive semidefiniteness. If $\mathcal{H}$ is complete with respect to the metric induced by the inner product, we say it is a *Hilbert space*.

Example 2.2.3
$\ell^2_{\mathbb{K}}$ is a Hilbert space with inner product $\langle x, y\rangle = \sum_{n\geq 1} x_n\bar{y}_n$

$L^2(X,\mu)$ is a Hilbert space with inner product $\langle f, g\rangle = \int_X f\bar{g}d\mu$

Proposition 2.2.4
The operations $(x,y) \mapsto x+y$ and $(\lambda, x) \mapsto \lambda x$ is continuous wiht respect to the product topologies on $X \times X, \mathbb{K} \times X$ and the norm topology on X.

This implies that for all $0 \neq \lambda \in \mathbb{K}$ and $y \in X$, $\sigma_y(x) := x + y, \mu_\lambda(x) := \lambda x$ are homeomorphisms and so $G \subseteq X$ is open if and only if $G + y, \lambda G$ are open.

Let $(X_n, \|\cdot\|_n)_{n\geq 1}$ be a countable family of Banach spaces and $X := \prod_n X_n$.

Definition 2.2.1 (ℓ^p-Direct Sum)
For each $1 \leq p < \infty$,

$$\sum_{n\geq 1} \oplus_p X_n := \left\{ (x_n) \in X : \|(x_n)\|_p = \left(\sum_{n\geq 1} \|x_n\|_n^p \right)^{\frac{1}{p}} < \infty \right\}$$

is a Banach space.

Definition 2.2.2 (ℓ^∞-Direct Product)

$$\prod_{n\geq 1} X_n := \left\{ (x_n) \in X : \|(x_n)\|_\infty = \sup_{n\geq 1} \|x_n\|_n < \infty \right\}$$

is a Banach space.

Definition 2.2.3 (ℓ^∞-Direct Sum)

$$\sum_{n\geq 1} \oplus_\infty X_n := \left\{ (x_n) \in X : \lim_n x_n = 0 \right\}$$

is a Banach space.

Recall that if X is a vector space and $\|\cdot\|_1, \|\cdot\|_2$ are norms, we say the norms are *equivalent* if there are constants $\kappa_1, \kappa_2 > 0$ os that

$$\kappa_1 \|x\|_1 \leq \|x\|_2 \leq \kappa_2 \|x\|_1$$

for all $x \in X$.

Recall that all norms on finite dimensional vector spaces are equivalent.

Example 2.2.5
Let $X = \mathcal{C}([0,1], \mathbb{K})$. Then $\|x^n\|_\infty = 1$ while $\|x^n\|_1 = \frac{1}{n+1}$. Thus the L_1, L_∞ norms are inequivalent.

Proposition 2.2.6
Two norms are equivalent on a vector space if and only if they generate the same metric topology.

Corollary 2.2.6.1
Equivalence of norms is an equivalence relation for norms on a vector space.

Definition 2.2.4 (Absolutely Summable)
Let $(X, \|\cdot\|)$ be a normed linear space. A series in X is absolutely summable if $\sum_{n\geq 1}\|x_n\| < \infty$.

Proposition 2.2.7
Let $(X, \|\cdot\|)$ be a normed linear space. Then X is a Banach space if and only if every absolutely summable sequence in X is summable.

The idea is to relate a Cauchy sequence to an appropriate series.

2.3 Quotient Spaces

Theorem 2.3.1
Let $(X, \|\cdot\|)$ be a normed linear space and $M \subseteq X$ a linear manifold. Then

$$p(x + M) := \inf\{\|x + m\| : m \in M\}$$

defines a seminorm on the quotient space X/M. Moreover, p is a norm if and only if M is closed.

Let M be a linear manifold of a normed linear space X. We denote the canonical quotient map as $q : X \to X/M$. Clearly, $\|q(x)\|_{X/M} \leq \|x\|_X$, so q is continuous. In fact, it is an *open map*, one which maps open maps to open maps.

Theorem 2.3.2
Let X be a normed linear space and M a closed subspace of X.

(a) If X is complete, then so are $M, X/M$

(b) If M and X/M are complete, then so is X.

Proof
(a) Any closed subspace of a complete metric space is complete. To see that X/M is complete, we relate an absolutely summable series $\sum_n q(x_n)$ to an absolutely summable

sequence

$$\sum_n \|x_n + m_n\| \leq \sum_n \|q(x_n)\| + 2^{-n} < \infty$$

with $m_n \in M$. Then $\sum_n x_n + m_n$ is summable by assumption and we can apply the the continuity of q.

(b) Let (x_n) be Cauchy in X. We see that $(q(x_n))$ is Cauchy in X/M and thus converges to some $q(y)$. We can choose $m_n \in M$ so that

$$\|y - (x_n + m_n)\| \leq \|q(y) - q(x_n)\| + 2^{-n}.$$

But then $(x_n + m_n)$ converges to y in X and is Cauchy.

But since $(x_n), (x_n + m_n)$ are both Cauchy, so is (m_n). Since M is complete, $m_n \to m \in M$ and

$$y - m = \lim_n (x_n + m_n) - m = \lim_n x$$

exists.

Proposition 2.3.3
Let X be a normed linear space and M a closed subspace of X with $q : X \to X/M$ the canonical quotient map.

(a) A subset $W \subseteq X/M$ is open if and only if $q^{-1}(W)$ is open in X

(b) The map q is an open map

Proof
(a) If $W \subseteq X/M$ is open, so is $q^{-1}(M)$ since q is continuous. Conversely, suppose $q^{-1}(W)$ is open. Let $q(x) \in W$ so that $x \in q^{-1}(W)$. There is some $\delta > 0$ for which $V_\delta(x) \subseteq q^{-1}(W)$. Now, if $\|q(y) - q(x)\| < \delta$, there is some $m \in M$ for which $\|y - x + m\| < \delta$ and $q(y) = q(y + m) \in q(V_\delta(x)) \subseteq W$. In other words, $V_\delta(q(x)) \subseteq W$ and W is open.

(b) Suppose G is open. Then $q^{-1}(q(G)) = G + M$ is a union of open sets and is thus open. q is open by (a).

Proposition 2.3.4
Let M be a finite dimensional linear manifold in a normed linear space X. Then M is closed in X.

Proposition 2.3.5
Let X be a normed linear space. If M, N are closed subspaces of X and $\dim N < \infty$, then $M + N$ is closed in X.

Proof
Let $q : X \to X/M$ denote the canonical quotient map. $N, q(N)$ are both closed by the previous proposition, Since q is continuous, $M + N = q^{-1}(q(N))$ is closed in X.

Chapter 3

Operators

3.1 Operators

Definition 3.1.1 (Bounded Operator)
Let X, Y be normed linear spaces and $T : X \to Y$ be linear.
T is a bounded operator if there is some $k \geq 0$ such that $\|Tx\| \leq k\|x\|$ for all $x \in X$.

Recall the *operator norm*. If T is bounded, we define

$$\|T\| := \inf\{k \geq 0 : \forall x \in X, \|Tx\| \leq k\|x\|\}.$$

Theorem 3.1.1
Let X, Y be normed linear spaces and $T : X \to Y$ linear.

(a) T is continuous on X

(b) T is continuous at 0

(c) T is bounded

(d) $\kappa_1 := \sup\{\|Tx\| : x \in X, \|x\| \leq 1\} < \infty$

(e) $\kappa_2 := \sup\{\|Tx\| : x \in X, \|x\| = 1\} < \infty$

(f) $\kappa_3 := \sup\left\{\frac{\|Tx\|}{\|x\|} : 0 \neq x \in X\right\} < \infty$

Moreover, if any of the above holds, $\kappa_1 = \kappa_2 = \kappa_3$.

Computing the operator norm is not always easy. Here are some examples of easy ones.

Example 3.1.2 (Multiplication Operator)
Let $X = (\mathcal{C}([0,1],\mathbb{C}), \|\cdot\|_\infty)$ and suppose that $f \in X$. Define $M_f(g) := fg$. Then $\|M_f\| = \|f\|_\infty$.

Example 3.1.3
Let $\mathcal{H} = L^2(X, d\mu)$ where $d\mu$ is a positive, regular Borel measure. For $f \in L^\infty(X, d\mu)$, define $M_f(g) := fg$. Again, we can show that $\|M_f\| = \|f\|_\infty$.

Note that computing the operator norm depends in general on the underlying norms.

Example 3.1.4 (Diagonal Operator)
Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $f \in \ell^\infty(\mathbb{N})$. Then $\|M_f\| = \sup_{|f(n)|:n\geq 1}$.

Example 3.1.5 (Weighted Shifts)
Let $\mathcal{H} = \ell^2(\mathbb{N})$ and $(w_n)_n \in \ell^\infty(\mathbb{N})$. Consider the maps $W, V : \mathcal{H} \to \mathcal{H}$

$$W(x_n)_n := (0, w_1x_1, w_2x_2, \dots) \qquad \text{unilateral forward weighted shift}$$
$$V(x_n)_n := (w_1x_2, w_2x_3, \dots) \qquad \text{unilateral backward weighte shift}$$

Also takke $\mathcal{H} = \ell^2(\mathbb{Z})$ and $(u_n)_n \in \ell^\infty(\mathbb{Z})$. The map $U : \mathcal{H} \to \mathcal{H}$ given by

$$U(x_n)_n := (u_{n-1}x_{n-1})_n \qquad \text{bilateral weighted shift}$$

Both V, W, U are bounded with $\|V\|, \|W\| = \sup\{|w_n| : n \geq 1\}$.

Example 3.1.6 (Differentiation Operator)
Consider $P(\mathbb{D})$, the space of polynomials over a compact set $\mathbb{D}$, equipped with the infinity norm. The map $D : \mathcal{P}(\mathbb{D}) \to \mathcal{P}(\mathbb{D})$ given by

$$p \mapsto p'$$

is the differentiation operator.

In particular, $D(z^n) = nz^{n-1}$ so D is unbounded.

The set of bounded linear operators from $X \to Y$ is denoted $\mathcal{B}(X, Y)$. If $X = Y$, we write $\mathcal{B}(X)$.

Proposition 3.1.7
Let X, Y be normed linear spaces. Then $\mathcal{B}(X, Y)$ is a vector space and the operator norm is a norm on $\mathcal{B}(X, Y)$.

Theorem 3.1.8
Let X, Y be normed linear spaces and Y be complete. Then $\mathcal{B}(X, Y)$ is a Banach space.

Proof
Absolutely summable sequences are summable.

3.2 Dual Spaces

Definition 3.2.1 (Dual)
Let X be a normed linear space. The dual of X is $X^* := \mathcal{B}(X, \mathbb{K})$, the space of continuous linear functionals on X.

Note that when $\mathbb{K}$ is complete, so is X^*. Hence we can define the *n-th iterated dual spaces* $X^{(n)} : +(X^{(n-1)})^*$, each of which are Banach spaces.

Definition 3.2.2 (Schauder Basis)
A collection $\{e_n\}_{n\geq 1}$ in a Banach space X is a Schauder basis if every $x \in X$ can be uniquely written as a norm convergent series

$$x = \sum_{n\geq 1} x_n e_n$$

for some $x_n \in \mathbb{K}$.

Example 3.2.1
$\{e_n\}$ is a Schauder basis for c_0, ℓ^p for $1 \leq p < \infty$. It is the *standard Schauder basis* for these spaces.

There is a Schauder basis for $(\mathcal{C}[0,1], \mathbb{R})$ discovered by Schauder, but the description of such a basis is extremely non-trivial.

Example 3.2.2
Consider the Banach space c_0 equipped with the sup norm. c_0^* is isometrically isomorphic to $\ell^1(\mathbb{N})$ through the map

$$\Theta(z) := \varphi_z$$

where

$$\varphi_z := \sum_{n\geq 1} x_n z_n.$$

Example 3.2.3
Let $1 \leq p < \infty$. Recall from real analysis that there is an isometric linear bijection $\Theta : (\ell^q, \|\cdot\|_q) \to (\ell^p, \|\cdot\|_p)^*$ given by $z \mapsto \varphi_z$ where

$$\varphi_z(x) := \sum_{n\geq 1} x_n z_n.$$

Here q is the Lebesgue conjugate of p.

Example 3.2.4
We can extend the example above to an isometric linear bijection $\Theta : L^q(X,\mu) \to L^p(X,\mu)^*$, given by $g \mapsto \varphi_g$,

$$\varphi_g(f) := \int_X fg d\mu.$$

Here μ is a σ-finite, positive, regular Borel measure on $L^p(X,\mu)$.

It we drop the hypothesis that μ is σ-finite, the result still holds for $1 < p < \infty$.

In the special case of $p = 2$, we consider the related map $\Omega : L^2(X,\mu) \to L^2(X,\mu)^*$ given by $g \mapsto \varphi_g$ such that

$$\varphi_g(f) = \int_X f\bar{g} d\mu$$

is a conjugate-linear, isometric bijection between the two spaces.

Example 3.2.5 (Functions of Bounded Variation)
$f : [0,1] \to \mathbb{K}$ is of *bounded variation* if there is some $\kappa > 0$ such that for every partition $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0,1]$,

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \leq \kappa.$$

The infimum of all such κ's is the *variation* of f.

If f is a function of bounded variation, then the left and right limits always exist (but may not be unique). Moreover, it admits at most a countable number of discontinuities in $[0, 1]$. In addition, for $g \in \mathcal{C}([0, 1], \mathbb{K})$, the *Riemann-Stieltjes integral* exists.

Let

$$BV[0, 1] := \{f : [0, 1] \to \mathbb{K} \mid \|f\|_v < \infty f \text{ is left-continuous on } (0, 1), f(0) = 0\}.$$

Then $(BV[0, 1], \|\cdot\|_v)$ is a Banach space with norm given by the variation.

The dual of $(\mathcal{C}[0, 1], \|\cdot\|_\infty)$ is isometrically isomorphic to $BV[0, 1]$. For $f \in BV[0, 1]$ and $g \in \mathcal{C}[0, 1]$, we define a functional $\varphi_f \in \mathcal{C}([0, 1], \mathbb{K})$ by

$$\varphi_f(g) := \int_0^1 g df.$$

Proposition 3.2.6
Let X be a normed linear space. There exists a contractive linear map $J : X \to X^{**}$.

Proof
Let $z \in X$ and define $\hat{z} : X^* \to \mathbb{K}$ via $\hat{z}(x^*) := x^*(z)$.

J is the *canonical embedding* of X into X^{**}. Once we have proven the Hahn-Banach theorem, we can actually show that J is isometric.

If J is an isometric bijection onto X^{**}, then X is said to be *reflexive*, a topic we shall revisit.

Chapter 4

Hilbert Spaces

4.1 Hilbert Spaces

Recall that a *Hilbert space* $\mathcal{H}$ is an inner product space such that it is complete under the induced norm. In any inner product space, the Cauchy-Schwartz inequality holds:

$$|\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2$$

for all $x, y \in \mathcal{H}$. We say that x, y are *orthogonal* if $\langle x, y\rangle = 0$ and write $x \perp y$.

Example 4.1.1
If (X, μ) is any measure space, then $L^2(X, \mu)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)}d\mu(x).$$

A special case of the above is the following example.

Example 4.1.2
$\ell^2 := \left\{(x_n)_n \in \mathbb{K}^{\mathbb{N}} : \sum_{n\geq 1}|x_n|^2 < \infty\right\}$ is a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_{n\geq 1} x_n\bar{y}_n.$$

Example 4.1.3 (Weighted ℓ^2 Space)
Fix a sequence of positive integers $(r_n)_n$. Then $\ell^2_{(r_n)_n} := \left\{(x_n)_n \in \mathbb{K}^{\mathbb{N}} : \sum_n r_n x_n \bar{y}_n < \infty\right\}$

is a Hilbert space with the inner product

$$\langle x, y\rangle = \sum_n r_n x_n \bar{y}_n.$$

Theorem 4.1.4
Let $\mathcal{H}$ be a hilbert space and $x_1, \dots, x_n \in \mathcal{H}$.

Pythagorean Theorem If the vectors are pairwise orthogonal, then

$$\left\| \sum_{i=1}^n x_i \right\|^2 = \sum_{i=1}^n \|x_i\|^2.$$

Paralleogram Law

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\|x_1\|^2 + 2\|x_2\|^2.$$

Theorem 4.1.5
Let $\mathcal{H}$ be a Hilbert space and $K \subseteq \mathcal{H}$ a closed, non-empty convex subset of $\mathcal{H}$. Given $x \in \mathcal{H}$, there is a unique point $y \in K$ satisfying

$$\|x - y\| = \operatorname{dist}(x, K) = \min_{z \in K} \|x - z\|.$$

Proof
By translation, it suffices to consider $x = 0$.

Define $d := \operatorname{dist}(0, K)$ and choose $k_n \in K$ so that $\|0 - k_n\| < d + \frac{1}{n}$. By the Parallelogram Law,

$$\begin{aligned} \left\| \frac{k_n - k_m}{2} \right\|^2 &= \frac{1}{2}\|k_n\|^2 + \frac{1}{2}\|k_m\|^2 - \left\| \frac{k_n + k_m}{2} \right\|^2 \\ &\leq \frac{1}{2}\left(d + \frac{1}{n}\right)^2 + \frac{1}{2}\left(d + \frac{1}{m}\right)^2 - d^2. && \frac{k_n + k_m}{2} \in K \end{aligned}$$

It follows that $(k_n)_n$ is Cauchy and this converges to some $k \in K$. Since K is closed and $\mathcal{H}$ is complete,

$$d = \lim_n \|k_n\| = \|k\|.$$

For uniqueness, suppose that $z \in K$ such that $\|z\| = d$. Then

$$\begin{aligned} 0 &\leq \left\| \frac{k-z}{2} \right\|^2 \\ &= \frac{1}{2}\|k\|^2 + \frac{1}{2}\|z\|^2 - \left\| \frac{k+z}{2} \right\|^2 \\ &\leq \frac{1}{2}d^2 + \frac{1}{2}d^2 - d^2 \\ &= 0 \end{aligned}$$

so $k = z$.

Theorem 4.1.6
Let $\mathcal{H}$ be a Hilbert space and $M \subseteq \mathcal{H}$ a closed subspace. For $x \in \mathcal{H}, m \in M$, the following are equivalent.

(a) $\|x - m\| = \operatorname{dist}(x, M)$

(b) $x - m$ is orthogonal to M so that $\langle x - m, y \rangle = 0$ for all $y \in M$.

Proof
Suppose that $\|x - m\| = \operatorname{dist}(x, M)$ but that there is some $y \in M, \|y\| = 1$ such that $k := \langle x - m, y \rangle \neq 0$. Then

$$\begin{aligned} \|x - (m + ky)\|^2 &= \|x - m\|^2 - \langle x - m, ky \rangle - \langle ky, x - m \rangle + |k|^2 \|y\|^2 \\ &= \|x - m\|^2 - |k|^2 \\ &< \operatorname{dist}(x, M) \end{aligned}$$

which is a contradiction.

Now suppose that $x - m \in M^\perp$. If $z \in M$ is arbitrary, then $y := z - m \in M$ so that

$$\begin{aligned} \|x - z\|^2 &= \|(x - m) - y\|^2 \\ &= \|x - m\|^2 + \|y\|^2 \\ &\geq \|x - m\|^2. \end{aligned}$$

Proposition 4.1.7
Let $\varnothing \neq S \subseteq \mathcal{H}$ and define

$$S^\perp := \{y \in \mathcal{H} : \forall x \in S, \langle x, y\rangle = 0\}.$$

Then $S^\perp$ is a norm-closed subspace of $\mathcal{H}$ and

$$\left(S^\perp\right)^\perp \supseteq \overline{\operatorname{span} S}.$$

Recall from linear algebra that if V is a vector space and W is a linear manifold $W \subseteq V$, then there is a linear manifold $X \subseteq V$ such that

(i) $W \cap X = \{0\}$

(ii) $V = W + X := \{w + x : w \in W, x \in X\}$

Definition 4.1.1 (Algebraically Complemented)
We say that W is algebraically complemented by X.

The existence of such a X for each W says that every linear manifold is algebraically complemented.

Definition 4.1.2 (Topologically Complemented)
If X is a Banach space and Y is a closed subspace of X, we say that Y is topologically complemented if there is a closed subspace $Z \subseteq X$ such that Z is an algebraic complement for Y in X.
We write $X = Y \oplus Z$.

Here the crucial issue is that both Y, Z must be closed subspaces.

Theorem 4.1.8 (Phillips)
The closed subspace c_0 of ℓ^∞ is NOT topologically complemented in ℓ^∞.

Proposition 4.1.9
Let $\mathcal{H}$ be a hilbert space and $M \subseteq \mathcal{H}$ a closed subspace. Then

$$\mathcal{H} = M \oplus M^\perp.$$

Moreover, this topological complement is unique.

We refer to M as the *orthogonal complement* since every element of $M^\perp$ is perpendicular to

M.

Proof
The proof of $\mathcal{H} = M \oplus M^\perp$ is easy. For uniqueness, we know that $x - m \in M^\perp$.

Since $\mathcal{H} = M \oplus M^\perp$, we can write $x = m_1 + x_2$ with $m_1 \in M, m_2 \in M^\perp$ uniquely for all $x \in \mathcal{H}$. Consider the map $P : \mathcal{H} \to M$ given by

$$x \mapsto m_1.$$

P is linear and idempotent with $\|P\| = 1$. We refer to P as the *orthogonal projection* of $\mathcal{H}$ onto M. The map $Q := I - P$ is the orthogonal projection onto $M^\perp$ and given that $M \subsetneq \mathcal{H}$, then $\|Q\| = 1$ as well.

Let $\varnothing \neq S \subseteq \mathcal{H}$. We know that $S^{\perp\perp} \supsetneq \overline{\operatorname{span} S}$. In fact, if we let $M := \overline{\operatorname{span} S}$, then M is a closed subspace of $\mathcal{H}$ and thus

$$\mathcal{H} = M \oplus M^\perp.$$

It is routine to check that $S^\perp = M^\perp$.

Proposition 4.1.10
$S^{\perp\perp} = \overline{\operatorname{span} S}$.

Proof
If there is some $0 \neq x \in S^{\perp\perp}$ with $x \notin M$, then we can write $x = m_1 + m_2$ for some $m_1 \in M$ and $0 \neq m_2 \in M^\perp = S^\perp$. But then

$$\langle m_2, x\rangle = \langle m_2, m_1\rangle + \langle m_2, m_2\rangle = \|m_2\|^2 \neq 0.$$

This contradicts the assumption that $x \in S^{\perp\perp}$. It follows that $S^{\perp\perp} = \overline{\operatorname{span} S}$.

Suppose that M admits an orthonormal basis $\{e_k\}_{k=1}^n$. Let $x \in \mathcal{H}$ and P the orthogonal projection onto M. From our work above, Px is the unique element of M so that $x - Px$ lies in $M^\perp$. Consider the vector $w := \sum_{k=1}^n \langle x, e_k\rangle e_k$. Then

$$\begin{aligned}
\langle x - w, e_j\rangle &= \langle x, e_j\rangle - \sum_{k=1}^n \langle \langle x, e_k\rangle e_k, e_j\rangle \\
&= \langle x, e_j\rangle - \sum_{k=1}^n \langle x, e_k\rangle \langle e_k, e_j\rangle \\
&= \langle x, e_j\rangle - \langle x, e_j\rangle \|e_j\|^2 \\
&= 0.
\end{aligned}$$

It follows that $x - w \in M^\perp$, and thus $w = Px$. That is, $Px = \sum_{k=1}^n \langle x, e_k\rangle e_k$.

Theorem 4.1.11 (Riesz Representation)
Let $\{0\} \neq \mathcal{H}$ be a Hilbert space over $\mathbb{K}$ and $\varphi \in \mathcal{H}^*$. Then there is a unique vector $y \in \mathcal{H}$ so that

$$\forall x \in \mathcal{H}, \varphi(x) = \langle x, y \rangle.$$

Moreover, $\|\varphi\| = \|y\|$.

Proof
The map $\Theta(y) := \langle \cdot, y \rangle$ is an isometric, conjugate-linear bijection.

The only difficulty is in showing it is surjective. Let $\varphi \in \mathcal{H}^*$. If $\varphi = 0$, then $\varphi = \Theta(0)$. Let $M := \ker \varphi$ and recall that $\mathcal{H}/M \cong \operatorname{Im}(\varphi) = \mathbb{K}$. Also, since $\mathcal{H} = M \oplus M^\perp$, $\mathcal{H}/M = \{M + m_2 : m_2 \in M^\perp\}$. Thus $\dim M^\perp = \dim \mathcal{H}/M = 1$.

Let $\{e\}$ be an orthonormal basis of $M^\perp$ and P the orthogonal projection onto M, eith $I - P = \langle \cdot, e \rangle$ the orthogonal projection onto $M^\perp$. Thus

$$x = Px + (I-P)x = Px + \langle x, e \rangle e$$

and for every $x \in \mathcal{H}$,

$$\varphi(x) = \varphi(Px) + \langle x, e \rangle \varphi(e) = \langle x, \overline{\varphi(e)} e \rangle.$$

Thus $\varphi = \Theta(\overline{\varphi(e)} e)$ as desired.

4.2 Orthonormal Bases

Definition 4.2.1 (Orthonormal)
$\{e_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{H}$ is orthonormal if each has unit norm and they are pair-wise orthogonal.

An *orthonormal basis* is a maximal orthonormal set with respect to inclusion. Given any orthonormal set E, Zorn's lemma guarantees the existence of an orthonormal basis in $\mathcal{H}$ containing E.

Example 4.2.1
In $\mathcal{H} = \ell^2$, the standard Schauder basis $\{e_n\}_{n \geq 1}$ for ℓ^2 is an orthonormal basis for $\mathcal{H}$.

If $\mathcal{H} = L^2(\mathbb{T}, d\mu)$ where $\mathbb{T}$ is the unit circle and $d\mu$ is the normalised Lebesgue measure, then $\{f_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T}, d\mu)$, where $f_n(z) = z^n$ for all $z \in \mathbb{T}$ and

$n \in \mathbb{Z}$.

Theorem 4.2.2 (Gram-Schmidt Orthogonalisation)
If $\mathcal{H}$ is a Hilbert space over $\mathbb{K}$ and $\{x_n\}_{n\geq 1}$ is a linearly independent set in $\mathcal{H}$, then we can find an orthonormal set $\{y_n\}_{n\geq 1}$ in $\mathcal{H}$ so that $\text{span}\{x_1, \ldots, x_k\} = \{y_1, \ldots, y_k\}$ for all $k \geq 1$.

Proof
Set $y_1 := \frac{1}{\|x_1\|} x_1$ and

$$y_k := \frac{x_k - \sum_{j=1}^{k-1} \langle x_k, y_j \rangle y_j}{\|x_k - \sum_{j=1}^{k-1} \langle x_k, y_j \rangle y_j\|}$$

for $k \geq 2$.

Theorem 4.2.3 (Bessel)
If $\{e_n\}_{n\geq 1}$ is an orthonormal set in $\mathcal{H}$, then for all $x \in \mathcal{H}$,

$$\sum_{n\geq 1} |\langle x, e_n \rangle|^2 \leq \|x\|^2.$$

Proof
The inequality holds for partial sums since orthogonal projections are contractive. Since the series is monotonic and bounded, the series converges and the inequality holds for the infinite sum as well.

Definition 4.2.2 (Unconditionally Summable)
Let $\mathfrak{X}$ be a Banach space and $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathfrak{X}$. Let $\mathcal{F}$ denote the collection of all finite subsets of Λ, partially ordered by inclusion. For each $F \in \mathcal{F}$, define $y_F = \sum_{\lambda\in F} x_\lambda$, so that $(y_F)_{F\in\mathcal{F}}$ is a net in $\mathfrak{X}$.
If $y := \lim_{F\in\mathcal{F}} y_F$ exists, then we write $y = \sum_{\lambda\in\Lambda} x_\lambda$ and say that $\{x_\lambda\}_\lambda$ is unconditionally summable.

Corollary 4.2.3.1
Let $\mathcal{H}$ be a Hilbert space and $\mathcal{E} \subseteq \mathcal{H}$ an orthonormal set.

(a) Given $x \in \mathcal{H}$, the set $\{e \in \mathcal{E} : \langle x, e \rangle \neq 0\}$ is countable

(b) For all $x \in \mathcal{H}$, $\sum_{e\in\mathcal{E}} |\langle x, e \rangle|^2 \leq \|x\|^2$

Proof
(a) Each of $\mathcal{F}_k := \{e \in \mathcal{E} : |\langle x, e\rangle| \geq \frac{1}{k}\}$ is necessarily finite. Thus the desired set is a countable union of finite sets and is therefore countable.

Lemma 4.2.4
Let $\mathcal{H}$ be a Hilbert space, $\mathcal{E} \subseteq \mathcal{H}$ an orthonormal set, and $x \in \mathcal{H}$. Then $\sum_{e\in\mathcal{E}} \langle x, e\rangle e$ converges in $\mathcal{H}$.

Proof
We may find a countable subset $\{e_n\}_{n\geq 1} \subseteq \mathcal{E}$ such that any basis elements outside are orthogonal to x. Let $\epsilon > 0$. By Bessel's inequality, $\sum_{n\geq N+1} |\langle x, e_k\rangle|^2 < \epsilon$ for sufficiently large N. Then for $F, G \geq [N]$,

$$\|y_F - y_G\|^2 \leq \epsilon.$$

Hence $(y_F)_{F\in\mathcal{F}}$ is a Cauchy net which necessarily converges in complete spaces.

Theorem 4.2.5
Let $\mathcal{E} \subseteq \mathcal{H}$ be orthonormal. The following are equivalent.

(a) $\mathcal{E}$ is an orthonormal basis for $\mathcal{H}$

(b) $\operatorname{span} \mathcal{E}$ is norm-dense in $\mathcal{H}$

(c) For all $x \in \mathcal{H}$, $x = \sum_{e\in\mathcal{E}} \langle x, e\rangle e$

(d) For all $x \in \mathcal{H}$, $\|x\|^2 = \sum_{e\in\mathcal{E}} |\langle x, e\rangle|^2$ (Parceval's Identity)

Proposition 4.2.6
If $\mathcal{H}$ is a Hilbert space, then any two orthonormal bases for $\mathcal{H}$ have the same cardinality.

Proof
Let $\mathcal{E}, \mathcal{F}$ be two orthonormal bases for $\mathcal{H}$. Given $e \in \mathcal{E}$, let $\mathcal{F}_e := \{f \in \mathcal{F} : \langle e, f\rangle \neq 0\}$ and recall that this is countable. Moreover, for every $f \in \mathcal{F}$, there is at least one $e \in \mathcal{E}$ for which $\langle e, f\rangle \neq 0$, otherwise $f \perp \overline{\operatorname{span} \mathcal{E}}$ which is a contradiction.

Thus $\mathcal{F} = \bigcup_{e\in\mathcal{E}} \mathcal{F}_e$ and $|\mathcal{F}| \leq \aleph_0 |\mathcal{E}| = \mathcal{E}$.

We can thus define the *dimension* of a Hilbert space as the cardinality of any orthonormal basis for $\mathcal{H}$.

We say two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are said to be *isomorphic* if there is a linear bijection

$U : \mathcal{H}_1 \to \mathcal{H}_2$ so that

$$\langle Ux, Uy \rangle = \langle x, y \rangle$$

for all $x, y \in \mathcal{H}_1$. We write $\mathcal{H}_1 \cong \mathcal{H}_2$ to denote this isomorphism.

linear maps implementing the above isomorphism are said to be *unitary operators*. It is clear that unitary operators are isometries. Moreover, the inverse map is also linear and inner product preserving. Furthermore, if $\mathcal{L} \subseteq \mathcal{H}_1$ is a closed subspace, then $\mathcal{L}$ is complete and so $U\mathcal{L}$ is complete and thus closed in $\mathcal{H}_2$.

Theorem 4.2.7
Two Hilbert spaces over $\mathbb{K}$ are isomorphic if and only if they have the same dimension.

Proof
$\underline{(\implies)}$ Suppose $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary. Then U maps orthonormal bases to orthonormal bases.

$\underline{(\impliedby)}$ Let $\{e_\lambda : \lambda \in \Lambda\}$ and $\{f_\lambda : \lambda \in \Lambda\}$ be orthonormal bases of $\mathcal{H}_1, \mathcal{H}_2$, respectively.

The map $U : \mathcal{H}_1 \to \ell^2(\Lambda)$ given by

$$Ux := (\langle x, e_\lambda \rangle)_{\lambda \in \Lambda}$$

is unitary. Moreover, $\operatorname{ran} U$ is dense since we certainly reach all finitely supported elements of $\ell^2(\Lambda)$. But $U\mathcal{H}_1$ is closed and thus necessarily equals all of $\ell^2(\Lambda)$.

Similarly, $\mathcal{H}_2 \cong \ell^2(\Lambda)$.

Corollary 4.2.7.1
The spaces $\ell^2(\mathbb{N}), \ell^2(\mathbb{Q}), \ell^2(\mathbb{Z})$ and $L^2([0,1], d\mu)$, where $d\mu$ is the Lebesgue measure are all isomorphic.

Chapter 5

Topological Vector Spaces

5.1 Topological Vector Spaces

Definition 5.1.1 (Compatible)
Let W be a vector space over $\mathbb{K}$ and $\mathcal{T}$ a topology on W. We say that $\mathcal{T}$ is compatible with the vector space structure on W if the maps $\sigma : W \times W \to W$ given by

$$(x, y) \mapsto x + y$$

and $\mu : \mathbb{K} \times W \to W$ given by

$$(k, x) \mapsto kx$$

are continuous, where $\mathbb{K} \times W, W \times W$ carry the respective product topologies.

Definition 5.1.2 (Topological Vector Space)
A pair $(W, \mathcal{T})$ where W is a vector space with a compatible Hausdorff topology $\mathcal{T}$.

Example 5.1.1
Any normed linear space is a topological vector space with the topology induced by the norm.

Let $(V, \mathcal{T})$ be a topological vector space and $U \in \mathcal{U}_0$ a neighbourhood of 0 in V. $\sigma^{-1}(U)$ is a neighbourhood of $(0, 0) \in V \times V$. Thus $\sigma^{-1}(U)$ contains a basic neighbourhood $N_1 \times N_2$ of $(0, 0)$ where N_1, N_2 are open in V. Observe that we can take $N := N_1 \cap N_2$ so that

$$\sigma(N \times N) = N + N \subseteq U.$$

Similarly, we can find a neighbourhood $V_\epsilon(0)$ of $0 \in \mathbb{K}$ and $N \in \mathcal{U}_0^V$ open so that $V_\epsilon(0) \times N \subseteq \mu^{-1}(U)$. Equivalently,

$$\{kn : n \in N, 0 \leq |k| < \epsilon\} \subseteq U.$$

Proposition 5.1.2
Consider a topological vector space V. If $U \in \mathcal{U}_0$, then

$$V = \bigcup_{n \geq 1} nU.$$

Proof
Let $x \in V$. Consider the continuous function $f : \mathbb{R} \to V$ defined by $f(t) = tx$. The continuity of f at 0 implies there exists $\delta > 0$ such that $|t| < \delta$ forces $f(t) = tx \in U$. Choosing $n > \frac{1}{\delta}$ yields $\frac{1}{n}x \in U$. In other words, $x \in nU$. As such, if $(k_n)_n$ is a sequence in $\mathbb{N}$ with $\lim_n k_n = \infty$, then

$$V = \bigcup_{n \geq 1} k_n U.$$

This phenonmenon is often referred to as saying that any neighbourhood of 0 in a topological vector space is *absorbing*.

Definition 5.1.3 (Balanced)
A neighbourhood N of 0 in a topological vector space V is balanced if $kN \subseteq N$ for all $k \in \mathbb{K}$ satisfying $|k| \leq 1$.

Example 5.1.3
The open ball of radius δ in a normed linear space is a balanced neighbourhood of 0.

Proposition 5.1.4
Every neighbourhood of 0 in a topological vector space containes a balanced open neighbourhood of 0.

Proof
For any $U \in \mathcal{U}_0$, we can find $\epsilon > 0$ and $N \in \mathcal{U}_0$ open such that $k \in \mathbb{K}, 0 < |k| < \epsilon$ implies $kN \subseteq U$. Take

$$M := \bigcup_{0 < |k| < \epsilon} kN.$$

Let $(W, \mathcal{T})$ be a topological vector space, $w_0 \in W$ and $k_0 \in \mathbb{K}$. Define $t_{w_0} : W \to W$ via $t_{w_0}(x) := w_0 + x$. By the continuity of addition, we get that t_{w_0} is continuous and clearly t_{w_0}

is a bijection. The inverse is also a translation and is so t_{w_0} is actually a homeomorphism. In other words, $N \in \mathcal{U}_0^W$ if and only if $w_0 + N \in \mathcal{U}_{w_0}^W$. That is, **the neighbourhood system at any point in W is determined by the neighbourhood system at $\mathbf{0}$**.

Proposition 5.1.5
Let V be a vector space with a topology for which

(i) Addition is continuous

(ii) Scalar multiplication is continuous

(iii) Points in V are closed in the $\mathcal{T}$-topology

Then $\mathcal{T}$ is a Hausdorff topology and $(V, \mathcal{T})$ is a topology vector space.

Proposition 5.1.6
Let $(W, \mathcal{T})$ be a topological vector space and Y a linear manifold in W. Then

(a) Y is a topological vector space with the relative topology induced by $\mathcal{T}$

(b) $\bar{Y}$ is a closed subspace of W

We make the remark that if $C \subseteq V$ in a topological vector space V, then C being convex implies that $\bar{C}$ is also convex. If $E \subseteq V$ is balanced then so is $\bar{E}$.

Definition 5.1.4 (Cauchy Net)
Let $(V, \mathcal{T})$ be a topological vector space and $(x_\lambda)_\lambda$ be a net in V. We say that $(x_\lambda)_\lambda$ is a Cauchy net if for all $U \in \mathcal{U}_0$, there is some $\lambda_0 \in \Lambda$ such that $\lambda_1, \lambda_2 \geq \lambda_0$ implies that $x_{\lambda_1} - x_{\lambda_2} \in U$.

Note that every convergent net is necessarily Cauchy.

Definition 5.1.5 (Cauchy Complete)
We say $K \subseteq V$ of a topological vector space is Cauchy complete if every Cauchy net in K converges to some element of K.

Example 5.1.7
If $(X, \|\cdot\|)$ is a normed linear space, then X is Cauchy complete if and only if X is complete.

Lemma 5.1.8
Let V be a topological vector space and $K \subseteq V$ be complete. Then K is closed in V.

5.2 Quotient Spaces

Let $(V, \mathcal{T})$ be a topological vector space and W a closed subspace of V. Then V/W exists as a quotient space of vector spaces. Let $q : V \to V/W$ denote the canonical quotient map. We can establish a topology on V/W using the $\mathcal{T}$ topology on V by defining a subset $G \subseteq V/W$ to be open if $q^{-1}(G)$ is open in V.

We refer to this topology as the *quotient topology* on V/W. This quotient map is the continuous with respect to this topology by design. In fact, it is the largest topology for this to hold.

We first define an *open map* as one which maps open sets to open sets.

Proposition 5.2.1
The canonical quotient map $q : V \to V/W$ is an open map.

Proposition 5.2.2
V/W is a topological vector space.

5.3 Finite Dimensional Topological Vector Spaces

Our goal is to prove that there is only one topology that one can impose on a finite-dimensional vector space to make it into a topological vector space.

Lemma 5.3.1
Let $(\mathcal{V}, \mathcal{T})$ be a one-dimensional topological vector space over $\mathbb{K}$. Let $\{e\}$ be a basis for $\mathcal{V}$. Then $\mathcal{V}$ is homeomorphic to $\mathbb{K}$ via the map $\tau : \mathbb{K} \to \mathcal{V}$ given by

$$k \mapsto ke.$$

Proof
τ is a bijection and the continuity of scalar multiplication in a topological vector space makes it continuous as well. It suffices to show the inverse map is also continuous. Since τ is linear, we need only show it is net continuous at 0.

Suppose $\lim_\lambda k_\lambda e = 0$. Pick $\delta > 0$. Then $\delta e \neq 0$. Since $\mathcal{V}$ is Hausdorff, we can find a balanced neighboourhood V of 0 such that $\delta e \notin V$.

By convergence, there is some λ_0 such that $\lambda \geq \lambda_0$ implies that $k_\lambda e \in V$. Suppose towards a contradiction that there is some $\beta \geq \lambda_0$ wth $|k_\beta| \geq \delta$. Then

$$\delta e = \left(\frac{\delta}{k_\beta}\right) k_\beta e \in V$$

as V is balanced.

This contradiction shows that $\lambda \geq \lambda_0$ implies that $|k_\lambda| < \delta$. But $\delta > 0$ was arbitrary, thus $\lim_\lambda k_\lambda = 0$.

Proposition 5.3.2
Let $n \in \mathbb{Z}_{++}$ and $(\mathcal{V}, \mathcal{T})$ an n-dimensional topological vector space over $\mathbb{K}$ with basis $B := \{e_1, \dots, e_n\}$. The map $\tau : \mathbb{K}^n \to \mathcal{V}$ given by

$$(k_1, \dots, k_n) \mapsto \sum_{j=1}^n k_n e_j$$

is a homeomorphism.

Proof
We have already proven the case for $n = 1$. We argue by induction and assume that the result holds for all $1 \leq n < m$.

let $\mathcal{Y} := \operatorname{span} E$ where $E \subsetneq B$. Then $\mathcal{Y} \cong \mathbb{K}^n$ for some $n < m$ and it is thus closed (complete). The canonical map $q_\mathcal{Y} : \mathcal{V} \to \mathcal{V}/\mathcal{Y}$ is continuous and $q(B \setminus E)$ is a basis for $\mathcal{V}/\mathcal{Y}$. By the induction hypothesis, there is a homeomorphism $p_\mathcal{Y} : \mathcal{V}/\mathcal{Y} \to \mathbb{K}^{m-n}$. Then $\gamma := p_\mathcal{Y} \circ q : \mathcal{V} \to \mathbb{K}^{m-n}$ is also continuous as it is a composition of continuous functions.

To complete the proof, apply the argument above with $F = \{e_m\}, \{e_1, \dots, e_{m-1}\}$ to get continuous maps $\gamma_1 : \mathcal{V} \to \mathbb{K}$ and $\gamma_2 : \mathcal{V} \to \mathbb{K}^{m-1}$. Then $\tau^{-1} = (\gamma_1, \gamma_2)$, thus it too is continuous and we are done.

Corollary 5.3.2.1
Let $n \in \mathbb{Z}_{++}$ and $\mathcal{V}$ be an n-dimensional vector space. There is a unique topology $\mathcal{T}$ which makes $\mathcal{V}$ a topological vector space.
Thus any norms on a finite dimensional vector space are equivalent.

Corollary 5.3.2.2
Let $\mathcal{V}$ be a topological vector space and $\mathcal{W}$ a finite-dimensional linear manifold of $\mathcal{V}$. Then $\mathcal{W}$ is closed in $\mathcal{V}$.

5.4 Local Compactness

Definition 5.4.1
A topologial space $(X, \mathcal{T})$ is locally compact if each point in X has a neighbourhood base consisting of compact sets.

Suppose X is locally compact and Hausdorff with $x_0 \in X$. Then for all $U \in \mathcal{U}_{x_0}$, there exists $K \in \mathcal{U}_{x_0}$ so that K is compact and $K \subseteq U$. Choose $G \in \mathcal{T}$ so that $x_0 \in G \subseteq K$. Then $\bar{G} \subseteq \bar{K} = K \subseteq U$ and so $\bar{G}$ is compact. That is, if X is Hausdorff and locally compact, then for any $U \in \mathcal{U}_{x_0}$, there exists $G \in \mathcal{T}$ so that $\bar{G}$ is compact and $x_0 \in G \subseteq \bar{G} \subseteq U$.

Example 5.4.1
Consider $(\mathbb{K}^n, \|\cdot\|_2)$. The collection of closed balls about $x \in \mathbb{K}^n$ is a neighbourhood base at x consisting of compact sets thus $(\mathbb{K}^n, \|\cdot\|_2)$ is locally compact.

Theorem 5.4.2
A topological vector space $(\mathcal{V}, \mathcal{T})$ is locally compact if and only if $\mathcal{V}$ is finite-dimensional.

Proof
If $\dim \mathcal{V} < \infty$, then $\mathcal{V} \cong \mathbb{K}^n$ and it is clearly locally compact.

Conversely, suppose that $\mathcal{V}$ is locally compact. Choose $K \in \mathcal{U}_0$ compact. We can find a neighbourhood $N \in \mathcal{U}_0^{\mathcal{V}}$ such that $N + N \subseteq K$. We may assume that N is open by taking its interior if necessary. Now,

$$K \subseteq \bigcup_{x \in K} x + N.$$

so we may apply the compactness of K to find finite subcover $x_1, \dots, x_r \in K$ so that

$$K \subseteq \{x_1, \dots, x_r\} + N.$$

Let $\mathfrak{M} := \operatorname{span}\{x_1, \dots, x_r\}$. Consider the quotient map $q : \mathcal{V} \to \mathcal{V}/\mathfrak{M}$. Recall q is both continuous and open. Moreover,

$$q(K) \subseteq \bigcup_{i=1}^{r} q(x_i + N) = \bigcup_{i=1}^{r} q(N) = q(N) \subseteq q(K).$$

since $q(x_i) = 0$ for all $i \in [r]$. Since $N + N \subseteq K$, we see that

$$2q(K) \subseteq q(K) + q(K) = q(N) + q(N) \subseteq q(K).$$

By induction, $2^m q(K) \subseteq q(K)$ for all $m \geq 1$.

But $K \in \mathcal{U}_0$ implies that it is absorbing. Thus

$$q(\mathcal{V}) \subseteq \bigcup_{m \geq 1} 2^m q(K) = q(K).$$

Since K is compact and q is continuous, we infer that $q(\mathcal{V})$ is also compact.

If $q(\mathcal{V}) \neq \{0\}$, then $q(\mathcal{V})$ contains a one-dimensional subspace $\mathbb{K}(y+\mathfrak{M})$ for some $y \in \mathcal{V} \backslash \mathfrak{M}$. We know that $\mathbb{K}(y+\mathfrak{M})$ is closed in $q(\mathcal{V})$, thus it is also compact. But $\mathbb{K}(y+\mathfrak{M})$ is homeomorphic to $\mathbb{K}$, which forces $\mathbb{K}$ to be compact as well, which is absurd.

By contradiction, $q(\mathcal{V}) = \{0\}$ so that $\mathcal{V} = \mathfrak{M}$, which is finite-dimensional.

Corollary 5.4.2.1
Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. The closed unit ball $\mathfrak{X}_1$ of $\mathfrak{X}$ is compact if and only if $\mathfrak{X}$ is finite-dimensional.

Proof
If $\mathfrak{X}_1$ is compact, then $\mathfrak{X}$ is locally compact and finite-dimensional.

The converse is trivial.

5.5 Uniform Continuity

Let $(\mathcal{V}, \mathcal{T}_{\mathcal{V}})$ and $(\mathcal{W}, \mathcal{T}_{\mathcal{W}})$ be topological vector spaces.

Definition 5.5.1 (Uniformly Continuous)
Let $f : \mathcal{V} \to \mathcal{W}$ be a map. f is uniforly continuous if for every $U \in \mathcal{U}_0^{\mathcal{W}}$, there is some $N \in \mathcal{U}_0^{\mathcal{V}}$ so that $x - y \in N$ implies that $f(x) - f(y) \in U$.

Let us extend this idea to functions defined only on subsets of $\mathcal{V}$.

Definition 5.5.2 (Uniformly Continuous)
Let $C \subseteq \mathcal{V}$ and $f : C \to \mathcal{W}$ a map. f is uniforly continuous if for every $U \in \mathcal{U}_0^{\mathcal{W}}$, there is some $N \in \mathcal{U}_0^{\mathcal{V}}$ so that $x, y \in C$ and $x - y \in N$ implies that $f(x) - f(y) \in U$.

Proposition 5.5.1
If $f : \mathcal{V} \to \mathcal{W}$ is uniformly continuous, then f is continuous on $\mathcal{V}$.

Theorem 5.5.2
Suppose $T : \mathcal{V} \to \mathcal{W}$ be linear. The following are equivalent:

(a) T is continuous at some $x_0 \in \mathcal{V}$

(b) T is uniformly continuous on $\mathcal{V}$

Let $\mathcal{V}_{\mathbb{R}}, \mathcal{W}_{\mathbb{R}}$ be the same space of vectors, viewed as vector spaces over $\mathbb{R}$ (they might originally be over $\mathcal{C}$). If $T : \mathcal{V} \to \mathcal{W}$ is conjugate-lienar then $T : \mathcal{V}_{\mathbb{R}} \to \mathcal{W}_{\mathbb{R}}$ is linear over $\mathbb{R}$. Moreover, the topologies $\mathcal{T}_{\mathcal{V}}, \mathcal{T}_{\mathcal{W}}$ are real-vector space topologies for $\mathcal{V}_{\mathbb{R}}, \mathcal{W}_{\mathbb{R}}$, respectively. This $T : \mathcal{V} \to \mathcal{W}$ is continuous if and only if $T : (\mathcal{V}_{\mathbb{R}}, \mathcal{T}_{\mathcal{V}}) \to (\mathcal{W}_{\mathbb{R}}, \mathcal{T}_{\mathcal{W}})$ is continuous. This happens precisely when T is uniformly continuous on $\mathcal{V}$ (or $\mathcal{V}_{\mathbb{R}}$). In other words, the previous theorem also holds for conjugate linear maps too.

Corollary 5.5.2.1
Suppose $n := \dim \mathcal{V} < \infty$. If $T : \mathcal{V} \to \mathcal{W}$ is linear, then T is continuous.

Proof
It suffices to show that T is continuous at 0. Let $\{e_1, \ldots, e_n\}$ be a basis for $\mathcal{V}$ and $(x_\lambda)_{\lambda \in \Lambda}$ a net converging to 0 in $\mathcal{V}$. Since $\sum_i k_i e_i \equiv (k_i) \in \mathbb{K}^n$, we see that $\lim_\lambda x_\lambda = 0 \iff \lim_{\lambda_i} k_{\lambda_i} = 0$ for each $i \in [n]$. Thus

$$\lim_\lambda T x_\lambda = \lim_\lambda T \sum_i k_i e_i = \lim_\lambda \sum_i 0 T e_j = 0 = T(\lim_\lambda x_\lambda).$$

If we restrict attention to subsets of $\mathcal{V}$, we get

Proposition 5.5.3
Let $T : \mathcal{V} \to \mathcal{W}$ be linear. suppose $0 \in \mathcal{C} \subseteq \mathcal{V}$ is balanced and convex. If $T\big|_C$ is continuous at 0, then $T\big|_C$ is uniformly continuous.

5.6 Extensions

Let A, B, C be sets with $A \subseteq B$ and $f : A \to C$. The map $g : B \to C$ *extends* f if $g\big|_A = f$.

Proposition 5.6.1
Suppose $\mathcal{W}$ is Cauchy complete. If $\mathcal{X} \subseteq \mathcal{V}$ is a linear manifold and $T_0 : \mathcal{X} \to \mathcal{W}$ is continous and linear. Then T_0 extends to a continuous linear map $T : \bar{\mathcal{X}} \to \mathcal{W}$.

Proof
Let $x \in \overline{\mathcal{X}}$ and choose $\lim_\lambda x_\lambda = x$. Define

$$Tx = \lim_\lambda T_0 x_\lambda.$$

Checking for linearity is easy and checking for uniqueness involves some net work.

Corollary 5.6.1.1
Suppose $\mathfrak{X}, \mathfrak{Y}$ are Banach spaces and $\mathfrak{M} \subseteq \mathfrak{X}$ is a linear manifold. If $T_0 : \mathfrak{M} \to \mathfrak{Y}$ is boounded, then T_0 extends to a bounded linear map $T : \bar{\mathfrak{M}} \to \mathfrak{Y}$ and $\|T\| = \|T_0\|$.

Proof
The extension is clear that so is the fact hat $\|T\| \geq \|T_0\|$. The reverse inequality requires the observation that every $x \in \bar{\mathfrak{M}}$ has some $y \in \mathfrak{M}$ of the same norm and is arbitrarily close.

Chapter 6

Seminorms and Locally Convex Spaces

6.1 Seminorms and Locally Convex Spaces

Definition 6.1.1 (Seminorm)
Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A seminorm on $\mathcal{V}$ is a map $p : \mathcal{V} \to \mathbb{R}$ satisfying

(i) $p(x) \geq 0$ for all $x \in \mathcal{V}$

(ii) $p(\lambda x) = |\lambda| p(x)$ for all $x \in \mathcal{V}, \lambda \in \mathbb{K}$

(iii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{V}$

Note that the triangle inequality implies that

$$p(x + y) - p(y) \leq p(x).$$

Or if we take $z := x + y$,

$$p(z) - p(x) \leq p(z - x).$$

This shows that

$$|p(x) - p(z)| \leq p(z - x).$$

Example 6.1.1
Let $\mathcal{V} := \mathcal{C}([0, 1], \mathcal{C})$. For each $x \in [0, 1]$, the map $p_x : \mathcal{V} \to \mathbb{R}$ given by

$$p_x(f) := |f(x)|$$

is a seminorm.

Recall a subset E of a vector space $\mathcal{V}$ is *convex* if for any two points $x, y \in E$, the line segment between them is also in E, $[x, y] \subseteq E$.

Proposition 6.1.2
In a topological vector space $\mathcal{V}$, $E \subseteq \mathcal{V}$ being convex implies that $\bar{E}$ is also convex.

Proposition 6.1.3
If $C \subseteq \mathcal{V}$ is convex and $T : \mathcal{V} \to \mathcal{W}$ is linear, then $T(C)$ is also convex.

Proposition 6.1.4
If $C \subseteq \mathcal{V}$ is convex, then for all $r, s > 0$,

$$rE + sE = (r + s)E.$$

Definition 6.1.2 (Minkowski Functional)
Let $\mathcal{V}$ be a topological vector space and $E \in \mathcal{U}_0^{\mathcal{V}}$ be convex. The Minkowski (gauge) functional is the function $p_E : \mathcal{V} \to \mathbb{R}$ given by

$$p_E(x) := \inf\{r > 0 : x \in rE\}.$$

The map is clearly not linear since its range is contained in $[0, \infty)$. Moreover, since E is convex, $x \in sE$ for all $s > p_E(x)$.

Definition 6.1.3 (Sublinear Functional)
Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A function $p : \mathcal{V} \to \mathbb{R}$ is a sublinear functional if

(i) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in \mathcal{V}$

(ii) $p(rx) = rp(x)$ for all $0 < r \in \mathbb{R}$

Note that any (semi)norm is a sublinear functional on that space. The converse does not hold, simply consider the identity function.

Proposition 6.1.5
Let $\mathcal{W}$ be a topological vector space and $E \in \mathcal{U}_0$ be convex.

(a) The Minkowski functional p_E is a sublinear functional on $\mathcal{W}$ for E

(b) If E is open, then $E = \{w \in W : p_E(x) < 1\}$

(c) If E is a balanced, then p_E is a seminorm.

Proposition 6.1.6
Let $\mathcal{W}$ be a topological vector space and p a seminorm on W. The following are equivalent

(a) p is continuous on $\mathcal{W}$

(b) There exists a set $U \in \mathcal{U}_0^{\mathcal{W}}$ such that p is bounded above on U.

Example 6.1.7
Take $p_x : \mathcal{C}([0,1], \mathcal{C}) \to \mathbb{R}$ from before and note that any p_x is bounded above on $B_1(0) := \{f \in \mathcal{C}([0,1], \mathcal{C}) : \|f\|_\infty < 1\}$.

Definition 6.1.4 (Locally Convex)
A topology $\mathcal{T}$ on a topological vector space $\mathcal{W}$ is locally convex if it admits a base consisting of convex sets.

We refer to these as locally convex (topological vector) spaces.

Since the topology of a topological vector space is determined by a neighbourhood base of 0, it suffices to check that there is a neighbourhood base of 0 consisting of convex sets.

Proposition 6.1.8
Let $\mathcal{W}$ be a topological vector space and $U \in \mathcal{U}_0$ be convex. Then U contains a balanced, open, convex neighbourhood of 0.

Proof
We know U contains a balanced, open neighbourhood H of 0. Take $\operatorname{conv}(H)$.

Corollary 6.1.8.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. Then $\mathcal{V}$ admits a neighbourhood base at 0 consisting of balanced, open, convex sets.

Proposition 6.1.9
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\mathcal{W} \subseteq \mathcal{V}$ a closed subspace. Then $\mathcal{V}/\mathcal{W}$ is a locally convex space in the quotient topology.

Proof
We know $\mathcal{V}/\mathcal{W}$ is a topological vector space. Since the canonical quotient map is open and linear, it maps a convex neighbourhood of 0 to a convex neighbourhood of $0 + \mathcal{W}$.

6.2 Separating Seminorms

Definition 6.2.1 (Separating)
A family Γ of seminorms on a vector space $\mathcal{W}$ is separating if for all $0 \neq x \in \mathcal{W}$, there exists $p \in \Gamma$ such that $p(x) \neq 0$.

Example 6.2.1
Let $\mathcal{W} = \mathcal{C}([0,1], \mathcal{C})$ and consider $\Gamma = \{p_x : x \in \mathbb{Q} \cap [0,1]\}$. Then Γ is a separating family of seminorms.

Let Γ be a family of seminorms on a vector space $\mathcal{W}$. For $F \subseteq \Gamma$ finite, $x \in \mathcal{W}$, and $\epsilon > 0$, set

$$N(x, F, \epsilon) = \{y \in \mathcal{W} : p(x - y) < \epsilon, p \in F\}.$$

Permitting ourselves a slight abuse notation, we write $N(x, p, \epsilon)$ in the case where $F = \{p\}$.

Theorem 6.2.2
If Γ is a separating family of seminorms on a vector space $\mathcal{W}$, then

$$\mathcal{B} := \{N(x, F, \epsilon) : x \in \mathcal{W}, \epsilon > 0, F \subseteq \Gamma \text{ finite}\}$$

is a base for a locally convex topology $\mathcal{T}$ on $\mathcal{W}$. Moreover, each $p \in \Gamma$ is $\mathcal{T}$-continuous.

Theorem 6.2.3
Suppose that $(\mathcal{V}, \mathcal{T}_\mathcal{V})$ is a locally convex space. Then there exists a separating family γ of seminorms on $\mathcal{V}$ which generate the topology $\mathcal{T}_\mathcal{V}$.

Example 6.2.4
Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space. The norm topology on $\mathfrak{X}$ is the metric topology induced by the norm. Thus it is precisely the locally convex topology generated by $\Gamma := \{\|\cdot\|\}$. Since $\|\cdot\|$ is a norm, it is separating as required.

Note that in any locally convex space $(\mathcal{V}, \mathcal{T})$, each $N(0, F, \epsilon)$ is balanced, open, and convex for all $F \subseteq \Gamma$ finite and $\epsilon > 0$, where Γ is a separating family of seminorms which generate $\mathcal{T}$.

Proposition 6.2.5
Let $\mathcal{V}$ be a vector space and Γ a separating family of seminorms on $\mathcal{V}$. let $\mathcal{T}$ denote the locally convex topology on $\mathcal{V}$ generated by Γ. A net $(x_\lambda)_\lambda$ in $\mathcal{V}$ converges to a point $x \in \mathcal{V}$ if and only if

$$\lim_\lambda p(x - x_\lambda) = 0$$

for all $p \in \Gamma$.

Let $\mathcal{V}$ be a vector space and Γ as separating family of seminorms on $\mathcal{V}$. Recall that the weak topology $\mathcal{T}_w$ is the weakest topology for which each function $p \in \Gamma$ is continuous. This the locally convex topology $\mathcal{T}$ generated by Γ also makes each $p \in \Gamma$ continuous, we see that $T_w \subseteq \mathcal{T}$. Thus if $(x_\lambda)_\lambda$ is a net converging to x in $(\mathcal{V}, \mathcal{T})$, then $(x_\lambda)_\lambda$ converges to x in $(\mathcal{V}, \mathcal{T}_w)$. In other words, $\lim_\lambda p(x_\lambda) = p(x)$ for all $p \in \Gamma$.

However, $\mathcal{T}_w \subsetneq \mathcal{T}$ in general.

Example 6.2.6
Take $\mathcal{V} = \mathbb{K}$ and $\Gamma = \{|\cdot|\}$. The locally convex topology is a topological vector space topology and thus agrees with the usual norm topology. Now, the sequence $x_n := (-1)^n$ converges to 1 in the weak topology but certainly does not converge within the locally convex topology.

Example 6.2.7
Let $\mathcal{V}$ be a vector space and $(\mathfrak{X}_\alpha, \|\cdot\|_\alpha)_{\alpha \in A}$ be a family of normed linear spaces. For each α, let $T_\alpha : \mathcal{V} \to \mathfrak{X}_\alpha$ be a linear map. Suppose $\{T_\alpha\}_\alpha$ is such that for every $0 \neq x \in \mathcal{V}$, there is some $\alpha \in A$ such that $0 \neq T_\alpha x \in \mathfrak{X}_\alpha$ (separating). Then each $p_\alpha : \mathcal{V} \to [0, \infty)$ given by $x \mapsto \|T_\alpha x\|_\alpha$ is easily seen to be a seminorm.

It is not hard to see that $\Gamma := \{p_\alpha : \alpha \in A\}$ is a separating family of seminorms. Let $\mathcal{T}$ denote the locally convex topology on $\mathcal{V}$ generated by Γ. A net $(x_\lambda)_\lambda$ converges to $x \in (\mathcal{V}, \mathcal{T})$ if and only if

$$\lim_\lambda p_\alpha(x - x_\lambda) = \lim_\lambda \|T_\alpha(x - x_\lambda)\|_\alpha = 0$$

for all $\alpha \in A$. In particular, this is the statement that each T_α is continuous.

So $\mathcal{T}$ coincides with the weka topology generated by the family $\{T_\alpha : \alpha \in A\}$, but this is still NOT the same weka topology generated by Γ.

Example 6.2.8
Consider $\mathcal{B}(\ell^2(\mathbb{N}))$ under the operator norm. The norm topology on it admits a neigh-

bourhood base at the operator T

$$\{N(T, \|\cdot\|, \epsilon) : \epsilon > 0\}.$$

This is the locally convex topology generated by $\Gamma = \{\|\cdot\|\}$.

Convergence of a net of operators should be thought of as uniform convergence on the closed unit ball of $\ell^2(\mathbb{N})$.

6.3 Strong and Weak Operator Topologies

Consider a Hilbert space $\mathcal{H}$. For each $x \in \mathcal{H}$, consider $p_x : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ given by

$$T \mapsto \|Tx\|.$$

Then p_x is a seminorm on $\mathcal{B}(\mathcal{H})$ for each $x \in \mathcal{H}$ but it is not a norm in general. It is not hard to see that $\Gamma_{\text{SOT}} := \{p_x : x \in \mathcal{H}\}$ separates the points of $\mathcal{B}(\mathcal{H})$.

Definition 6.3.1 (Strong Operator Topology)
The locally convex topology generated on $\mathcal{B}(\mathcal{H})$ generated by Γ_{SOT} is called the strong operator topology and denoted by SOT.

Note that a net $(T_\lambda)_\lambda$ converges to $T \in \mathcal{B}(\mathcal{H})$ in SOT if and only if

$$\lim_\lambda p_x(T_\lambda - T) = \lim_\lambda \|T_\lambda x - Tx\| = 0$$

for all $x \in \mathcal{H}$. So SOT is the topology of pointwise convergence. That is, it is the weakest topology that makes all evaluation maps $T \mapsto Tx$ continuous.

A neighbourhood base for SOT at the point $T \in \mathcal{B}(\mathcal{H})$ is given by the collection

$$\{N(T, \{x_i : i \in [m]\}, \epsilon) : m \geq 1, x_i \in \mathcal{H}, \epsilon > 0\}$$

where for finite $F \subseteq \mathcal{H}$,

$$N(T, F, \epsilon) = \{R \in \mathcal{B}(\mathcal{H}) : \|Rx_i - Tx_i\| < \epsilon, i \in [m]\}.$$

For each pair $(x, y) \in \mathcal{H} \times \mathcal{H}$, consider the map $q_{x,y} : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ given by $T \mapsto |\langle Tx, y\rangle|$ is a seminorm but not in general a norm on $\mathcal{B}(\mathcal{H})$.

Definition 6.3.2 (Weak Operator Topology)
The locally convex topology on $\mathcal{B}(\mathcal{H})$ generated by $\Gamma_{\text{WOT}} := \{q_{x,y} : (x, y) \in \mathcal{H} \times \mathcal{H}\}$ is called the weak operator topology on $\mathcal{B}(\mathcal{H})$ and denoted WOT.

A net $(T_\lambda)_\lambda \in \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in WOT if and only if

$$\lim_\lambda |\langle (T_\lambda - T)x, y\rangle| = \lim_\lambda |\langle T_\lambda x, y\rangle - \langle Tx, y\rangle| = 0$$

for all $x, y \in \mathcal{H}$. So WOT is the weakest topology that makes functions of the form $T \mapsto \langle Tx, y\rangle$ continuous.

A neighbourhood base for WOT at the point $T \in \mathcal{B}(\mathcal{H})$ is given by

$$\{N(T, \{x_i, y_j : i, j \in [m]\}, \epsilon) : m \geq 1, x_i, y_j \in \mathcal{H}, \epsilon > 0\}$$

where for each $m \geq 1$, $F := \{(x_i, x_i) : i \in [m]\}$ and $\epsilon > 0$, we have

$$N(T, F, \epsilon) = \{R \in \mathcal{B}(\mathcal{H}) : |\langle Rx_i - Tx_i, y_i\rangle| < \epsilon, i \in [m]\}.$$

Proposition 6.3.1
Let $(\mathcal{V}, \mathcal{T}_\mathcal{V})$ and $(\mathcal{W}, \mathcal{T}_\mathcal{W})$ be locally convex spaces. Let $\Gamma_\mathcal{V}$ and $\Gamma_\mathcal{W}$ denote separating families of seminorms which generate the corresponding locally convex topologies on $\mathcal{V}$ and $\mathcal{W}$ respectively.
Let $T : \mathcal{V} \to \mathcal{W}$ be a linear map. The following are equivalent:

(a) T is continuous

(b) For all $q \in \Gamma_\mathcal{W}$, there is some $\kappa > 0$ and $p_1, \ldots, p_m \in \Gamma_\mathcal{V}$ so that $q(Tx) \leq \kappa \max_i p_i(x)$ for all $x \in \mathcal{V}$

Corollary 6.3.1.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. A linear functional f on $\mathcal{V}$ is continuous if and only if there is a continuous seminorm p on $\mathcal{V}$ such that

$$|f(x)| \leq p(x)$$

for all $x \in \mathcal{V}$.

Chapter 7

The Hahn-Banach Theorem

This actually refers to a collection of extension and separation theorems. The importance of these theorems in functional analysis cannot be overstated.

7.1 Linear Functionals

Let $\mathcal{W}$ be a vector space over $\mathbb{K}$. Recall that a *linear functional* on $\mathcal{W}$ is a linear map $f : \mathcal{W} \to \mathbb{K}$. The vector space of all linear functionals on $\mathcal{W}$ is denoted $\mathcal{W}^{\#}$ and referred to as the *algebraic dual* of $\mathcal{W}$. If $\mathcal{W}$ is a topological vector space, the vector space of continuous linear functionals is denoted $\mathcal{W}^*$ and is referred to as the *topological dual* of $\mathcal{W}^*$.

Example 7.1.1
Let $n \geq 1$ and $\mathcal{W} := \mathbb{K}^n$ equipped with the infinity norm. For any $k_1, \dots, k_n \in \mathbb{K}$, the map

$$f(x) := \sum_{k=1}^{n} k_i x_i$$

is a continuous linear functional.

Note that every linear functional on a finite dimensional vector space is continuous.

Example 7.1.2
Consider $c_{00}(\mathbb{K})$, the set of finitely supported sequences in $\mathbb{K}$. This forms a normed linear

space under the infinity norm. The map

$$f(x) := \sum_{n \geq 1} x_n$$

is an unbounded (discontinuous) linear functional.

The following lemma can be useful for translating results where $\mathbb{K} = \mathbb{R}$ to the case $\mathbb{K} = \mathbb{C}$.

Lemma 7.1.3
Let $\mathcal{V}$ be a vector space over $\mathbb{C}$.

(a) If $f : \mathcal{V} \to \mathbb{R}$ is a $\mathbb{R}$-linear functional, then the map $f_{\mathbb{C}}(x) := f(x) - if(ix)$ is a $\mathbb{C}$-linear functional on $\mathcal{V}$ with $f = \operatorname{Re} f_{\mathbb{C}}$.

(b) If $g : \mathcal{V} \to \mathbb{C}$ is $\mathbb{C}$-linear and $f = \operatorname{Re} g$, then $g = f_{\mathbb{C}}$.

(c) If p is a $\mathbb{C}$-seminorm on $\mathcal{V}$ and $f, f_{\mathbb{C}}$ are as in (a), then $|f(x)| \leq p(x)$ for all $x \in \mathcal{V}$ if and only if $|f_{\mathbb{C}}(x)| \leq p(x)$ for all $x \in \mathcal{V}$.

(d) If $\mathcal{V}$ is a normed linear space and $f, f_{\mathbb{C}}$ are as in (a), then $\|f\| = \|f_{\mathbb{C}}\|$.

Proposition 7.1.4
Let $\mathcal{V}$ be a vector space over $\mathbb{K}$ and $f \in \mathcal{V}^{\#}$.

(a) If $g \in \mathcal{V}^{\#}$ and $g\big|_{\ker f} = 0$, then $g = kf$ for some $k \in \mathbb{K}$.

(b) If $g, f_1, \ldots, f_N \in \mathcal{V}^{\#}$ and $g(x) = 0$ for all $\bigcap_{j=1}^{N} \ker f_j$, then $g \in \operatorname{span}\{f_1, \ldots, f_N\}$.

Definition 7.1.1 (Hyperplane)
Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A hyperplane $\mathcal{M} \subseteq \mathcal{V}$ is a linear manifold for which $\dim(\mathcal{V}/\mathcal{M}) = 1$.

If $0 \neq \varphi \in \mathcal{V}^{\#}$, then $\mathcal{M} := \ker \varphi$ is a hyperplane in $\mathcal{V}$ with φ inducing an algebraic isomorphism $\bar{\varphi} : \mathcal{V}/\mathcal{M} \to \mathbb{K}$ via

$$\bar{\varphi}(x + \mathcal{M}) := \varphi(x).$$

Conversely, if $\mathcal{M} \subseteq \mathcal{V}$ is a hyperplane, then $\mathcal{V}/\mathcal{M}$ is algebraically isomorphic to $\mathbb{K}$. Let $\kappa : \mathcal{V}/\mathcal{M} \to \mathbb{K}$ denote such an isomorphism. If $q : \mathcal{V} \to \mathcal{V}/\mathcal{M}$ is the canonical quotient map, then $\kappa \circ q : \mathcal{V} \to \mathbb{K}$ is a linear functional with $\ker(\kappa \circ q) = \mathcal{M}$.

Proposition 7.1.5
If $(\mathcal{V}, \mathcal{T})$ is a topological vector space and $\mathcal{M} \subseteq \mathcal{V}$ is a hyperplane, then either $\mathcal{M}$ is closed in $\mathcal{V}$ or $\mathcal{M}$ is dense.

Proposition 7.1.6
Let $\mathcal{V}$ be a topological vector space and $\rho \in \mathcal{V}^{\#}$. If there is an open neighbourhood $U \in \mathcal{U}_0$ of 0 and a constant $\kappa > 0$ so that $\operatorname{Re} \rho(x) < \kappa$ for all $x \in U$. Then ρ is uniformly continuous on $\mathcal{V}$.

Corollary 7.1.6.1
Let $\mathcal{V}$ be a topological vector space and $\rho \in \mathcal{V}^{\#}$. The following are equivalent:

(a) ρ is continuous on $\mathcal{V}$ so that $\rho \in \mathcal{V}^*$.

(b) $\ker \rho$ is closed.

Let us recall some examples from measure theory which provide interesting examples of classes of linear functionals.

Theorem 7.1.7
Let (X, Ω, μ) be a measure space and $1 < p < \infty$. If $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L^q(X, \Omega, \mu)$, then

$$\beta_g(f) := \int_X fg d\mu$$

defines a continuous linear functional on $L^p(X, \Omega, \mu)$ and the map $g \mapsto \beta_g$ is an isometric linear bijection of $L^q(X, \Omega, \mu) \to L^p(X, \Omega, \mu)^*$
If (X, Ω, μ) is σ-finite, then the same conclusion holds in the case where $p = 1$ and $q = \infty$.

Recall that if X is a locally compact space, then $M_{\mathbb{K}}(X)$ denotes the space of $\mathbb{K}$-valued regular Borel measures on X with the *total variation norm*.

Theorem 7.1.8
If X is locally compact and $\mu \in M_{\mathbb{K}}(X)$, the map $\beta_\mu : \mathcal{C}_0(X, \mathbb{K}) \to \mathbb{K}$ given by $f \mapsto \int_X f d\mu$ defines an element of $\mathcal{C}_0(X, \mathbb{K})$, and the map $\mu \mapsto \beta_\mu$ is an isometric linear isomorphism of $M_{\mathbb{K}}(X)$ onto $\mathcal{C}_0(X, \mathbb{K})^*$.

7.2 The Extension Theorems

These formulations of the Hahn-Banach theorem aim to extend linear functionals from linear submanifolds of a locally convex space to the entire space.

Proposition 7.2.1
Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and $p : \mathcal{V} \to \mathbb{R}$ a sublinear functional. If $\mathcal{M}$ is a proper hyperplane and $f : \mathcal{M} \to \mathbb{R}$ is a linear functional such that $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. There exists a linear functional $g : \mathcal{V} \to \mathbb{R}$ such that $g\big|_{\mathcal{M}} = f$ and $g(x) \leq p(x)$ for all $x \in \mathcal{V}$.

Proof
The fact that some extension exists is trivial. The difficulty comes in showing the existence of a bounded extension. This amounts to checking necessary conditions and then showing they are sufficient.

Theorem 7.2.2 (Hahn-Banach I)
Let $\mathcal{V}$ be a vector space over $\mathbb{R}$ and p a sublinear functional on $\mathcal{V}$. If $\mathcal{M}$ is a linear manifold in $\mathcal{V}$ and $f : \mathcal{M} \to \mathbb{R}$ is a linear functional with $f(m) \leq p(m)$ for all $m \in \mathcal{M}$, then there exists a linear functional $g : \mathcal{V} \to \mathbb{R}$ with $g\big|_{\mathcal{M}} = f$ and $g(x) \leq p(x)$ for all $x \in \mathcal{V}$.

Proof
We use Zorn's lemma to find some g which extends f "as much as possible". Then using the previous proposition, we can argue that g must extend f to all of $\mathcal{V}$.

Theorem 7.2.3 (Hahn-Banach II)
Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. Let $\mathcal{M} \subseteq \mathcal{V}$ be a linear manifold and $p : \mathcal{V} \to \mathbb{R}$ a seminorm on $\mathcal{V}$. If $f : \mathcal{M} \to \mathbb{K}$ is a linear functional and $|f(m)| \leq p(m)$ for all $m \in \mathcal{M}$, there exists a linear functional $g : \mathcal{V} \to \mathbb{K}$ so that $g\big|_{\mathcal{M}} = f$ and $|g(x)| \leq p(x)$ for all $x \in \mathcal{V}$.

Proof
We can simply consider $f_1 := \operatorname{Re} f$ if $\mathbb{K} = \mathbb{C}$ and then complexify the extension given to us by the previous theorem.

Corollary 7.2.3.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\mathcal{W} \subseteq \mathcal{V}$ a linear manifold. If $f \in \mathcal{W}^*$, then there exists $g \in \mathcal{V}^*$ such that $g\big|_{\mathcal{W}} = f$.

Proof
Let Γ be a family of seminorms which generate the locally convex topology on $\mathcal{V}$. Then

$\Gamma_{\mathcal{W}} := \left\{ p\big|_{\mathcal{W}} : p \in \Gamma \right\}$ generates the relative locally convex topology on $\mathcal{W}$. We know that $f \in \mathcal{W}^*$ if and only if there are $p_1, \ldots, p_m \in \Gamma$ and $\kappa > 0$ such that

$$|f(x)| \leq \kappa \max(p_1(x), \ldots, p_m(x)) =: q(x)$$

for all $x \in \mathcal{W}$. Now, q is a seminorm and thus we can extend f to some g such that $|g(x)| \leq q(x)$ for all $x \in \mathcal{V}$, which shows that g is continuous as well.

Theorem 7.2.4 (Hahn-Banach III)
Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space, $\mathcal{M} \subseteq \mathfrak{X}$ a linear manifold, and $f \in \mathcal{M}^*$ a bounded linear functional. There exists $g \in \mathfrak{X}^*$ such that $g\big|_{\mathcal{M}} f$ and $\|g\| = \|f\|$.

Proof
Consider the seminorm $p(x) := \|f\| \cdot \|x\|$.

Corollary 7.2.4.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\{x_j : j \in [m]\}$ a linearly independent set of vectors in $\mathcal{V}$. If $\{k_j : j \in [m]\} \subseteq \mathbb{K}$ are arbitrary, there exists $g \in \mathcal{V}^*$ such that $g(x_j) = k_j$ for all $j \in [m]$.

Corollary 7.2.4.2
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $0 \neq y \in \mathcal{V}$. There exists $g \in \mathcal{V}^*$ so that $g(y) \neq 0$.

Recall that a closed subspace $\mathcal{W}$ of a locally convex space $(\mathcal{V}, \mathcal{T})$ is *topologically complemented* if there is a closed subspace $\mathcal{Y}$ of $\mathcal{V}$ so that $\mathcal{V} = \mathcal{Y} \oplus \mathcal{W}$. The difficult part here is requiring the complement $\mathcal{Y}$ to also be closed.

Proposition 7.2.5
Let $\mathcal{W}$ be a finite-dimensional subspace of a locally convex space $(\mathcal{V}, \mathcal{T})$.. Then $\mathcal{W}$ is topologically complemented in $\mathcal{V}$.

Corollary 7.2.5.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\mathcal{W} \subseteq \mathcal{V}$ be a closed subspace of $\mathcal{V}$. If $x \in \mathcal{V} \backslash \mathcal{W}$, there exists $g \in \mathcal{V}^*$ so that $g\big|_{\mathcal{W}} = 0$ but $g(x) \neq 0$.

Theorem 7.2.6
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\mathcal{W} \subseteq \mathcal{V}$ a linear manifold. Then

$$\bar{\mathcal{W}} = \bigcap_{\ker f : f \in \mathcal{V}^*, \mathcal{W} \subseteq \ker f} .$$

Corollary 7.2.6.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\mathcal{W} \subseteq \mathcal{V}$ a linear manifold. The following are equivalent.

(a) $\mathcal{W}$ is dense in $\mathcal{V}$.

(b) $f \in \mathcal{V}^*$ and $f\big|_{\mathcal{W}} = 0$ implies that $f = 0$.

Corollary 7.2.6.2
Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space and $x \in \mathfrak{X}$. Then

$$\|x\| = \max\{|x^*(x)| : x^* \in \mathfrak{X}^*, \|x^*\| \leq 1\}.$$

Corollary 7.2.6.3
The canonical embedding $\mathfrak{J} : \mathfrak{X} \to \mathfrak{X}^{**}$ is an isometry.

Corollary 7.2.6.4
Let $(\mathfrak{X}, \|\cdot\|)$ be a normed linear space and $\mathfrak{Y} \subseteq \mathfrak{X}$ a closed subspace with $z \in \mathfrak{X}$ but $z \notin \mathfrak{Y}$. Define $d := d(x, \mathfrak{Y}) := \|z + \mathfrak{Y}\|$. Then there is some $x^* \in \mathfrak{X}^*$ so that $\|x^*\| = 1$, $x^*\big|_{\mathfrak{Y}} = 0$ and $x^*(z) = d$.

7.3 The Separation Theorems

Proposition 7.3.1
Let $(\mathcal{V}, \mathbb{T})$ be a locally convex spae over $\mathbb{K}$ and $\varnothing \neq \mathcal{G} \subseteq \mathcal{V}$ an open, convex subset of $\mathcal{V}$ with $0 \notin \mathcal{G}$. There exists a closed hyperplane $\mathcal{M} \in \mathcal{V}$ such that $\mathcal{G} \cap \mathcal{M} = \varnothing$.

Definition 7.3.1 (Affine Manifold)
A affine manifold $\mathcal{M}$ in a topological vector space $(\mathcal{V}, \mathcal{T})$ is a translate of a hyperplane.

An *affine hyperplane* is a translate of a hyperplane.

Corollary 7.3.1.1
Let $(\mathcal{V}, \mathcal{T})$ be a localy convex space and $\varnothing \neq G \subseteq \mathcal{V}$ be open and convex. If $\mathcal{L} \subseteq \mathcal{V}$ is an affine manifold of $\mathcal{V}$ and $\mathcal{L} \cap G = \varnothing$, then there exits a closed, affine hyperplane $\mathcal{Y} \subseteq \mathcal{V}$ so that $\mathcal{L} \subseteq \mathcal{Y}$ and $\mathcal{Y} \cap G = \varnothing$.

Definition 7.3.2
Let $(\mathcal{V}, \mathcal{T})$ be a topological vector space over $\mathbb{R}$. An open (closed) half-space is a subset $\mathcal{S} \subseteq \mathcal{V}$ for which there exists a non-zero continuous linear functional $f : \mathcal{V} \to \mathbb{R}$ and $k \in \mathbb{R}$ so that

$$\mathcal{S} = \{x \in \mathcal{V} : f(x) > k\}$$

($\mathcal{S} = \{x \in \mathcal{V} : f(x) \geq k\}$ respectively).

For $A, B \subseteq \mathcal{V}$, we say they are *separated* if we can find closed half-spaces $\mathcal{S}_A, \mathcal{S}_B$ such that $A \subseteq \mathcal{S}_A, B \subseteq \mathcal{S}_B$ such that $\mathcal{S}_A \cap \mathcal{S}_B$ is a closed affine hyperplane of $\mathcal{V}$. We say that A, B are *strictly separated* if we cna find disjoint open half-spaces $\mathcal{S}_A, \mathcal{S}_B$ with $A \subseteq \mathcal{S}_A, B \subseteq \mathcal{S}_B$.

Theorem 7.3.2 (Hahn-Banach IV - $\mathbb{R}$)
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space over $\mathbb{R}$ and suppose that A, B are non-empty disjoint, open, and convex subsets of $\mathcal{V}$. Then A, B are strictly separated.

Note that if A is open but not B, then $G := \bigcup_{b \in B} A - b$ would still be open. We can conclude that there is a continuus linear functional $g : \mathcal{V} \to \mathbb{R}$ and some $k \in \mathbb{R}$ such that $A \subseteq \{x \in \mathcal{V} : g(x) > k\}$ and $B \subseteq \{x \in \mathcal{V} : g(x) \leq k\}$.

Theorem 7.3.3 (Hahn-Banach IV - $\mathbb{C}$)
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space over $\mathcal{C}$ and $A, B \subseteq \mathcal{V}$ non-empty, disjoint, open, and convex subsets of $\mathcal{V}$. Then there exists a continuous $\mathcal{C}$-linear functional f on $\mathcal{V}$ and $k \in \mathbb{R}$ such that

$$\operatorname{Re} f(a) > k > \operatorname{Re} f(b)$$

for all $a \in A, b \in B$.

Proof
Apply the previous theorem by thinking of $\mathcal{V}$ as a vector space over $\mathbb{R}$. Then complexify the functional obtained from the previous theorem.

Theorem 7.3.4
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $A, B \subseteq \mathcal{V}$ are non-empty, disjoint, closed, and convex subsets of $\mathcal{V}$. Suppose furthermore that B is compact. There are real numbers α, β and a continuous $f \in \mathcal{V}^*$ such that

$$\operatorname{Re} f(a) \geq \alpha > \beta \geq \operatorname{Re} f(b)$$

for all $a \in A, b \in B$. Thus A, B are strictly separated.

Corollary 7.3.4.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space over $\mathbb{R}$ and $\varnothing \neq A \subseteq \mathcal{V}$. The closed convex hull of A is the intersection of closed half-spaces containing A.

Chapter 8

Weak Topologies and Dual Spaces

We now consider some other topologies we can put on Banach spaces which yield interesting insights.

8.1 Weak Topology

Let $\mathcal{V}$ be a vector space over $\mathbb{K}$ and $\Omega \subseteq \mathcal{V}^{\#}$ a separating family of functionals. A base for the $\sigma(\mathcal{V}, \Omega)$ topology on $\mathcal{V}$ is given by

$$\mathcal{B} := \{N(x, F, \epsilon) : x \in \mathcal{V}, \epsilon > 0, F \subseteq \Omega \text{ finite}\}.$$

Recall that a net $(X_\lambda)_{\lambda \in \Lambda}$ converges in $(\mathcal{V}, \sigma(\mathcal{V}, \Omega))$ if and only if

$$\lim_\lambda \rho(x_\lambda) = \rho(x)$$

for all $\rho \in \Omega$.

Definition 8.1.1 (Dual Pair)
Let $\mathcal{V}$ be a vector space over $\mathbb{K}$ and $\mathcal{L} \subseteq \mathcal{V}^{\#}$ both a linear manifold and a separating class of linear functionals. We say that $(\mathcal{V}, \mathcal{L})$ is a dual pair.

Example 8.1.1 (Weak Topology)
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\mathcal{L} := \mathcal{V}^*$. Then the $\sigma(\mathcal{V}, \mathcal{V}^*)$ topology is referred to as the weak topology on $\mathcal{V}$.

If $(x_\lambda)_\lambda$ is a net which converges to x in the weak topology, we say that $(x_\lambda)_\lambda$ converges weakly to x.

Suppose $\lim_\lambda x_\lambda = x$ in the initial topology on $\mathcal{V}$. The fact that f is continuous implies that $\lim_\lambda f(x_\lambda) = f(x)$ for every $f \in \mathcal{V}^*$ so $(x_\lambda)_\lambda$ also converges weakly. It follows that $\sigma(\mathcal{V}, \mathcal{V}^*) \subseteq \mathcal{T}$.

Theorem 8.1.2
Let $(\mathcal{V}, \mathcal{L})$ be a dual pair. Then $\mathcal{L} = (\mathcal{V}, \sigma(\mathcal{V}, \mathcal{L}))$.

Thus the only weakly-continuous linear functionals on a locally convex space $\mathcal{V}$ is $\mathcal{V}^*$.

Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space. For $x \in \mathcal{V}$ define $\hat{x} : \mathcal{V}^* \to \mathbb{K}$ by

$$\rho \mapsto \rho(x).$$

Then $\hat{\mathcal{V}} := \{\hat{x} : x \in \mathcal{V}\}$ is a linear manifold in $(\mathcal{V}^*)^\#$. In fact, it is a separating family of linear functionals so $(\mathcal{V}^*, \hat{\mathcal{V}})$ is a dual pair.

Definition 8.1.2 (Weak*-Topology)
The weak topology on $\mathcal{V}^*$ induced by the family $\hat{\mathcal{V}}$ is referred to as the weak*-topology on $\mathcal{V}^*$.

A base for the weak*-topology on $\mathcal{V}^*$ is given by sets of the form

$$N(\varphi, F, \epsilon) := \{\rho \in \mathcal{V}^* : |\rho(x) - \varphi(x)| < \epsilon, x \in F\}$$

where $F \subseteq \mathcal{V}$ is a finite set. A net $(\rho_\lambda)_\lambda$ converges in the weak*-topology to some $\rho \in \mathcal{V}^*$ if and only if $\lim_\lambda \rho_\lambda(x) = \rho(x)$ for every $x \in \mathcal{V}$. Thus convergence in the weak*-topology is simply point-wise convergence.

Note that a functional φ is weak*-continuous on $\mathcal{V}^*$ if and only if $\varphi = \hat{x}$ for some $x \in \mathcal{V}$.

Proposition 8.1.3
Let $(\mathcal{V}, \mathcal{T}_\mathcal{V}), (\mathcal{W}, \mathcal{T}_\mathcal{W})$ be locally convex spaces and let $T : \mathcal{V} \to \mathcal{W}$ be continuous and linear. Then T is continuous as a linear map between $\mathcal{V}, \mathcal{W}$ when they are equipped with their respective weak topologies as well.

Theorem 8.1.4
Let C be a convex subset of a locally convex space $(\mathcal{V}, \mathcal{T})$. The closure of C in $(\mathcal{V}, \mathcal{T})$ coincides with its weak closure in $(\mathcal{V}, \sigma(\mathcal{V}, \mathcal{V}^*))$.

Let $(\mathfrak{X}, \|\cdot\|)$ be a Banach space. Let $\mathcal{J}$ be the canonical embedding into $\mathcal{V}^{**}$. We have just shown that $\mathcal{J}(\mathfrak{X})$ is exactly the weak*-continuous linear functionals on $\mathfrak{X}^*$.

Proposition 8.1.5
Let $\mathfrak{X}$ be a finite-dimensional Banach space. Then the norm, weak, and weak*-topologies on $\mathfrak{X}$ all coincide.

Theorem 8.1.6 (Uniform Boundedness Principle)
Let (X, d) be a complete metric space and let $H \subseteq \mathcal{C}(X, \mathbb{K})$ be a non-empty family of continuous functions on X such that for each $x \in X$,

$$M_x := \sup_{h \in H} |h(x)| < \infty.$$

There exists an open set $G \subseteq X$ and a constant $M > 0$ such that

$$|h(x)| \leq M$$

for all $h \in H, x \in G$.

Corollary 8.1.6.1 (Uniform Boundedness Principle for Banach Spaces)
Let $(\mathfrak{X}, \|\cdot\|_{\mathfrak{X}})$ and $(\mathfrak{Y}, \|\cdot\|_{\mathfrak{Y}})$ be Banach spaces and $\mathfrak{A} \subseteq \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ denote a family of continuous linear operators from $\mathfrak{X} \to \mathfrak{Y}$. Suppose that for each $x \in \mathfrak{X}$,

$$M_x := \sup_{T \in \mathfrak{A}} \|Tx\| < \infty.$$

Then

$$\sup_{T \in \mathfrak{A}} \|T\| < \infty.$$

Corollary 8.1.6.2
Let $\mathfrak{X}$ be a Banch space and $\mathcal{S} \subseteq \mathfrak{X}$. Then $\mathcal{S}$ is bounded if and only if for all $x^* \in \mathfrak{X}^*$,

$$\sup_{s \in \mathcal{S}} |x^*(s)| < \infty.$$

Corollary 8.1.6.3
Let $\mathfrak{X}$ be a Banach space and $\mathfrak{G} \subseteq \mathfrak{X}^*$. Then $\mathfrak{G}$ is bounded if and only if for all $x \in \mathfrak{X}$,

$$\sup_{s^* \in \mathfrak{G}} |s^*(x)| < \infty.$$

Theorem 8.1.7 (Banach-Steinhaus)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $\{T_n\}_{n\geq 1} \subseteq \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is a sequence which satisfies the property that for every $x \in \mathfrak{X}$, there is some $y_x \in \mathfrak{Y}$ so that

$$\lim_n T_n x = y_x.$$

Then

(a) $\sup_n \|T_n\| < \infty$

(b) $T : \mathfrak{X} \to \mathfrak{Y}$ given by $Tx := y_x$ is a bounded linear map

(c) $\|T\| \leq \liminf_n \|T_n\|$

Corollary 8.1.7.1
Let $\mathfrak{X}$ be a Banach space.
If $(x_n)_n$ is a sequence converging weakly to $x \in \mathfrak{X}$. Then

(i) $\sup_n \|x_n\| < \infty$

(ii) $\|x\| \leq \liminf \|x_n\|$

If $(y_n^*)_n$ is a sequence converging in the weak*-topology to $y^* \in \mathfrak{X}^*$, then

(i) $\sup_n \|y_n^*\| < \infty$

(ii) $\|y^*\| \leq \liminf \|y_n^*\|$

Theorem 8.1.8
Let X be a locally compact, Hausdorff topological space and $\mathcal{M}(X)$ the space of $\mathbb{K}$-valued, finite, regular Borel measures on X, equipped with the total variation norm $\|\mu\| := |\mu|(X)$.
If $\mu \in \mathcal{M}(X)$, then $\beta_\mu : \mathcal{C}_0(X, \mathbb{K}) \to \mathbb{K}$ given by

$$\beta_\mu(f) := \int_X f d\mu$$

is an element of $\mathcal{C}_0(X, \mathbb{K})^*$, and the map $\Theta : \mathcal{M}(X) \to \mathcal{C}_0(X, \mathbb{K})^*$ is an isometric linear isomorphism.

Proposition 8.1.9
Let X be a compact, Hausdorff space. Then a sequence $\{f_n\}_n$ in $\mathcal{C}(X)$ converges weakly to $f \in \mathcal{C}(X)$ if and only if

(i) $\sup_n \|f_n\| < \infty$

(ii) for each $x \in X$, $(f_n(x))_n$ converges to $f(x)$

Theorem 8.1.10 (Tychonoff)
Suppose $(X_\lambda, \mathcal{T}_\lambda)$ is a non-empty collection of compact, topological spaces. Then $X = \prod_\lambda X_\lambda$ is compact in the product topology.

Theorem 8.1.11 (Banach-Alaoglu)
Let $\mathfrak{X}$ be a Banach space. Then the closed unit ball $\mathfrak{X}_1^* := \{x^* \in \mathfrak{X}^* : \|x^*\| \leq 1\}$ of $\mathfrak{X}^*$ is weak*-compact.

Corollary 8.1.11.1
Every Banach space $\mathfrak{X}$ is isometrically isomorphic to a sub-space of $(\mathcal{C}(L, \mathbb{K}), \|\cdot\|_\infty)$ for some compact, Hausdorff space L.

Corollary 8.1.11.2
Let $\mathfrak{X}$ be a Banach space and suppose that $\mathcal{A} \subseteq \mathfrak{X}^*$ is weak*-closed and bounded. Then $\mathcal{A}$ is weak*-compact.

Theorem 8.1.12 (Goldstein)
Let $\mathfrak{X}$ be a Banach space and $\mathcal{J} : \mathfrak{X} \to \mathfrak{X}^{**}$ denote the canonical embedding. Then $\mathcal{J}(\mathfrak{X}_1)$ is weak*-dense in $\mathfrak{X}_1^{**}$. Thus $\mathcal{J}(\mathfrak{X})$ is weak*-dense in $\mathfrak{X}^{**}$.

Proposition 8.1.13
Let $\mathfrak{X}$ be a Banach space. The following are equivalent:

(a) $\mathfrak{X}$ is reflexive

(b) $\mathfrak{X}_1$ is weakly compact

Although weak topologies are not in general metrizable, sometimes their restrictions to bounded sets can be.

Theorem 8.1.14
Let $\mathfrak{X}$ be a Banach sapce. Then $\mathfrak{X}_1^*$ is weak*-metrizable if and only if $\mathfrak{X}$ is separable.

Corollary 8.1.14.1
Let $\mathfrak{X}$ be a separable Banach space. Then $\mathfrak{X}_1^*$ is separable in the weak*-topology.

Theorem 8.1.15
Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X}_1$ is weakly metrizable if and only if $\mathfrak{X}^*$ is separable.

Definition 8.1.3 (Annihilator)
Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M} \subseteq \mathfrak{X}, \mathfrak{N} \subseteq \mathfrak{X}^*$. The annihilator of $\mathfrak{M}$ is the set

$$\mathfrak{M}^{\perp} = \{x^* \in \mathfrak{X}^* : \forall m \in \mathfrak{M}, x^*(m) = 0\}$$

for all $m \in \mathfrak{M}$.
Similarly, the pre-annihilator of $\mathfrak{N}$ is the set

$$^{\perp}\mathbb{N} = \{x \in X : \forall n \in \mathfrak{N}, n^*(x) = 0\}$$

Theorem 8.1.16
Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M} \subseteq \mathfrak{X}$ a closed subspace. Let $q : \mathfrak{X} \to \mathfrak{X}/\mathfrak{M}$ be the canonical quotient map. Then $\Theta : (\mathfrak{X}/\mathfrak{M})^* \to \mathfrak{M}^{\perp}$ given by

$$\xi \mapsto \xi \circ q$$

is an isometric isomorphism of Banach spaces.

Theorem 8.1.17
Let $\mathfrak{X}$ be a Banach space and $\mathfrak{M} \subseteq \mathfrak{X}$ be a closed linear subspace. The map $\Theta : \mathfrak{X}^*/\mathfrak{M}^{\perp} \to \mathfrak{M}^*$ given by

$$x^* + \mathfrak{M}^{\perp} \mapsto x^*\Big|_{\mathfrak{M}}$$

is an isometric isomorphism.

Chapter 9

Extremal Points

The main result of this section is the Krein-Milman theorem.

9.1 Extremal Points

Recall $T \in \mathcal{B}(\mathcal{C}^n)$ is *positive* (semidefinite), denoted $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{C}^n$. A linear functional $\varphi \in \mathcal{B}(\mathcal{C}^n)^*$ is said to be *positive* if $\varphi(T) \geq 0$ whenever $T \geq 0$.

Example 9.1.1
Let $\{e_1, \ldots, e_n\}$ be any orthonormal basis of $\mathcal{C}^n$. Then the normalized trace functional

$$T \mapsto \frac{1}{n}\sum_{k=1}^{n}\langle Te_k, e_k\rangle$$

is a positive linear functional of norm one.

The *state space* $\mathcal{S}(\mathcal{B}(\mathcal{C}^n))$ of $\mathcal{B}(\mathcal{C}^n)$ is the set of all positive, norm-one linear functionals on $\mathcal{B}(\mathcal{C}^n)$, referred to as *states*. This forms a non-empty, compact, convex subset of $\mathcal{B}(\mathcal{C}^n)^*$. The extreme points of the state space are called *pure states*.

Definition 9.1.1 (Extreme Point)
Let V be a vector space and $C \subseteq V$ a convex subset. $e \in C$ is an extreme point of C if whenever $x, y \in C$ satisfy
$$e = tx + (1-t)y$$
for some $t \in (0,1)$, it follows that $x = y = e$.

We write $\mathrm{Ext}(C)$ the set of all extreme points of C.

Definition 9.1.2 (Face)
Let V be a vector space and $\varnothing \neq C \subseteq V$ a convex subset. A convex subset $\varnothing \neq F \subseteq C$ is a face of C if whenever $x, y \in C$ satisfy
$$tx + (1-t)y \in F$$
for some $t \in (0,1)$, it follows that $x, y \in F$.

We emphasize that F being convex is a part of the definition of a face.

Lemma 9.1.2
Let $\mathcal{V}$ be a vector space, $\varnothing \neq C \subseteq \mathcal{V}$ be convex, and $\varnothing \neq F \subseteq C$ a face of C. Suppose that $\{x_j : j \in [n]\} \subseteq C$ and that $x = \sum_{j=1}^{n} t_j x_j$ is a convex combination. If $x \in F$ and $t_j \in (0,1)$ for all $j \in [n]$, then $x_j \in F$ for all $j \in [n]$.

Lemma 9.1.3
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\varnothing \neq K \subseteq \mathcal{V}$ be a compact, convex set. Fix $\rho \in \mathcal{V}^*$, and set
$$r = \sup\{\operatorname{Re} \rho(w) : w \in K\}.$$
Then $F := \{x \in K : \operatorname{Re} \rho(x) = r\}$ is a non-empyt, compact face of K.

Lemma 9.1.4
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\varnothing \neq K \subseteq \mathcal{V}$ be a compact, convex set. Then $\mathrm{Ext}(K) \neq \varnothing$.

Theorem 9.1.5 (Krein-Milman)
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\varnothing \neq K \subseteq \mathcal{V}$ be a compact, convex set. Then $K = \overline{\mathrm{conv}}(\mathrm{Ext}(K))$.

Corollary 9.1.5.1
Let $(\mathcal{V}, \mathcal{T})$ be a locally convex space and $\varnothing \neq K \subseteq \mathcal{V}$ be compact and convex. For $\rho \in \mathcal{V}^*$, there exists $e \in \operatorname{Ext}(K)$ so that

$$\operatorname{Re} \rho(w) \leq \operatorname{Re} \rho(e)$$

for all $w \in K$.

Corollary 9.1.5.2
Let $\mathfrak{X}$ be a Banach space and suppose that $\mathcal{A} \subseteq \mathfrak{X}^*$ is weak*-closed and bounded. Then $\mathcal{A}$ is weak*-compact. If $\mathcal{A}$ is also convex, then $\mathcal{A} = \overline{\operatorname{conv}}^{w^*}$.

Chapter 10

Named Theorems

10.1 Named Theorems

Lemma 10.1.1
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. If $\mathfrak{Y}_2 \subseteq \overline{T\mathfrak{X}_m}$ for some $m \geq 1$, then $\mathfrak{Y}_1 \subseteq T\mathfrak{X}_{2m}$.

Theorem 10.1.2 (Open Mapping)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ be surjective. Then T is an open map.

Corollary 10.1.2.1 (Inverse Mapping Theorem)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ be bijective. Then T^{-1} is continuous and T is thus a homeomorphism.

Corollary 10.1.2.2 (Closed Graph Theorem)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ be linear. If the graph

$$\mathcal{G}(T) := \{(x, Tx) : x \in \mathfrak{X}\}$$

is closed in $\mathfrak{X} \oplus_1 \mathfrak{Y}$, then T is continuous.

The ℓ^1 norm was only chosen as to induce the product topology on $\mathfrak{X} \oplus \mathfrak{Y}$. Any equivalent norm would suffice as well.

Corollary 10.1.2.3
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T : \mathfrak{X} \to \mathfrak{Y}$ be linear. The following are equivalent:

(a) The graph $\mathcal{G}(T)$ is closed

(b) T is continuous

(c) If $\lim_n x_n = 0$ and $\lim_n Tx_n = y$, then $y = 0$

Lemma 10.1.3
Two closed subspaces $\mathfrak{Y}, \mathfrak{Z}$ of a Banach spaces $\mathfrak{X}$ topologically complement each other if and only if the map $\iota : \mathfrak{Y} \oplus_1 \mathfrak{Z} \to \mathfrak{X}$ given by

$$(y, z) \mapsto y + z$$

is a homeomorphism of Banach spaces.

Proposition 10.1.4
Let $\mathfrak{X}$ be a Banach spaces and $\mathfrak{Y}, \mathfrak{Z}$ be topologically complementary subspaces of $\mathfrak{X}$. For each $x \in \mathfrak{X}$, denoted by y_x, z_x such that $x = y_x + z + x$. Define $E : \mathfrak{X} \to \mathfrak{Y}$ via

$$Ex = E(y_x + z_x) = y_x$$

for all $x \in \mathfrak{X}$. Then

(a) E is a continuous linear map. Moreover, $E = E^2$, $\operatorname{Im} E = \mathfrak{Y}$, and $\ker E = \mathfrak{Z}$

(b) Conversely, if $E \in \mathcal{B}(\mathfrak{X})$ and $E = E^2$, then $\mathfrak{M} = \operatorname{Im} E$ and $\mathfrak{N} = \ker E$ are topologically complementary subspaces of $\mathfrak{X}$

Recall that a linear map $E \in \mathcal{B}(\mathfrak{X})$ is *idempotent* if $E = E^2$. This proposition states that a subspace $\mathfrak{Y}$ of a Banach space $\mathfrak{X}$ is complemented if and only if it is the range of a bounded idempotent in $\mathcal{B}(\mathfrak{X})$.

Chapter 11

Operator Theory

11.1 Compact Operators in Banach Spaces

Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. Given $y^* \in \mathfrak{Y}^*$, the map

$$x \mapsto y^*(Tx)$$

is a linear functional on X. Let us denote $T^* y^* \in \mathfrak{X}^\#$ the functional on $\mathfrak{X}$ determined by his formula. Namely,

$$T^* y^*(x) = y^*(Tx)$$

for all $x \in \mathfrak{X}$.

Notice that $\|T^* y^*(x)\| \le \|y^*\| \|T\| \|x\|$, so that $\|T^* y^*\| \le \|T\| \|y^*\| < \infty$. In particular, $T^* y^* \in \mathfrak{X}^*$ so that T maps $\mathfrak{Y}^* \to \mathfrak{X}^*$. Moreover, the estimate above also implies that $\|T^*\| \le \|T\|$. By the Hahn-Banach theorem, we can choose $y^* \in \mathfrak{Y}^*$ such that $\|y^*\| = 1$ and $y^*(Tx) = \|Tx\|$. Thus

$$\begin{aligned}
\|Tx\| &= y^*(Tx) \\
&= T^* y^*(x) \\
&\le \|T^* y^*\| \|x\| \\
&\le \|T^*\| \|x\|
\end{aligned}$$

which shows that $\|T\| \le \|T^*\|$. So

$$\|T^*\| = \|T\|.$$

Definition 11.1.1 (Banach Space Adjoint)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The map T^* is referred to as the Banach space adjoint of T.

Many authors write $< x, x^* >$ to denote $x^*(x)$. In this notation, $T^*y^*(x) = y^*(Tx)$ becomes

$$< x, T^*y^* >=< Tx, y^* > .$$

Proposition 11.1.1
Let $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be Banach spaces, $S, T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$, and $R \in \mathcal{B}(\mathfrak{Y}, \mathfrak{Z})$. Then

(a) for all $k_1, k_2 \in \mathbb{K}$, we have $(k_1 S + k_2 T)^* = k_1 S^* + k_2 T^*$

(b) $(R \circ T)^* = T^* \circ R^*$

Proposition 11.1.2
Let $\mathfrak{X}$ be an n-dimensional Banach space over $\mathbb{K}$, and let $\mathcal{E} := \{e_i : i \in [n]\}$ be a Hamel basis for $\mathfrak{X}$. If $[A] = [a_{ij}]$ is the matrix for A relative to $\mathcal{E}$, then the matrix of the Banach space adjoint A^* of A with respect to the dual basis $\mathcal{F}$ of $\mathcal{E}$ coincides with $[A]^T = [a_{ji}]$, the transpose of $[A]$.

Theorem 11.1.3
Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. There exists a unique operator $T^* \in \mathcal{B}(\mathcal{H})$, called the Hilbert space adjoint of T, satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x, y \in \mathcal{H}$.

Corollary 11.1.3.1
Let $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Then $(T^*)^* = T$. It follows that $\|T\| = \|T*\|$.

Proposition 11.1.4
Let $\mathcal{H}$ be a complex, separable Hilbert space and $\mathcal{E} = \{e_n : n \geq 1\}$ be a countably infinite or finite orthonormal basis for $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$. Then the matrix of T^* with respect to $\mathcal{E}$ is the conjugate transpose of that of T relative to $\mathcal{E}$.

For Hilbert space $\mathcal{H}$ and $A, B \in \mathcal{B}(\mathcal{H})$, it is not hard to see that $(AB)^* = B^*A^*$. Given $x, y \in \mathcal{H}$,

$$\langle x, (AB)^*y\rangle = \langle ABx, y\rangle = \langle Bx, A^*y\rangle = \langle x, B^*A^*y\rangle,$$

from which the result follows. Thus the adjoint operator $* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an *involution* on a Banch algebra. Namely, for all $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathcal{B}(\mathcal{H})$,

(i) $(\alpha A)^* = \bar{\alpha} A^*$

(ii) $(A + B)^* = A^* + B^*$

(iii) $(AB)^* = B^*A^*$

(iv) $(A^*)^* = A$

A norm-closed subalgebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ which is also closed under the adjoint operation is a *(concrete) C^*-algebra.*

Theorem 11.1.5
Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then

$$\|T^*T\| = \|T\|^2.$$

This is known as the C^*-equation. It is a consequence of the *Gelfand-Naimark-Segal (GNS) construction* that if one starts with an involutive Banach algebra $\mathcal{A}$ where each element satisfies the C^*-equation, then there exists an isometric *-embedding of that algebra into the set of bounded linear operators on some Hilbert space.

Proposition 11.1.6
Let $\mathcal{H}$ be a Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Then

$$(\operatorname{Im} T)^\perp = \ker T^*.$$

Therefore,

1. $\overline{\operatorname{Im} T} = (\ker T^*)^\perp$
2. $\operatorname{Im} T$ is not dense in $\mathcal{H}$ if and only if $\ker T^* \neq \{0\}$

Definition 11.1.2 (Compact)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. T is said to be compact if $\overline{T(\mathfrak{X}_1)}$ is compact in $\mathfrak{Y}$.

The set of compact operators from $\mathfrak{X} \to \mathfrak{Y}$ is denoted $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$, or just $\mathcal{K}(\mathfrak{X})$ if $\mathfrak{Y} = \mathfrak{X}$.

Recall that a subset $K \subseteq L$ of a metric space is *totally bounded* if we can find a finite ϵ-cover of K for any $\epsilon > 0$.

Proposition 11.1.7
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$. The following are equivalent:

(a) T is compact

(b) $\overline{T(F)}$ is compact in $\mathfrak{Y}$ for all bounded subsets $F \subseteq \mathfrak{X}$

(c) If $(x_n)_n$ is a bounded sequence in $\mathfrak{X}$, then $(Tx_n)_n$ has a convergent subsequence in $\mathfrak{Y}$

(d) $T(\mathfrak{X}_1)$ is totally bounded

Theorem 11.1.8
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces. Then $\mathcal{K}(\mathfrak{X}, \mathfrak{Y})$ is a closed subspace of $\mathcal{B}(\mathfrak{X}, \mathfrak{Y})$.

Theorem 11.1.9
Let $\mathfrak{M}, \mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ be Banach spaces. Suppose $R \in \mathcal{B}(\mathfrak{M}, \mathfrak{X}), K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$, and $T \in \mathcal{B}(\mathfrak{Y}, \mathfrak{Z})$. Then $TK \in \mathcal{K}(\mathfrak{X}, \mathfrak{Z})$ and $KR \in \mathcal{K}(\mathfrak{M}, \mathfrak{Y})$.

Corollary 11.1.9.1
If $\mathfrak{X}$ is a Banach space, then $\mathcal{K}(\mathfrak{X})$ is a closed, two-sided ideal of $\mathcal{B}(\mathfrak{X})$.

Proposition 11.1.10
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and assume that $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. Then $K(\mathfrak{X})$ is closed in $\mathfrak{Y}$ if and only if $\dim K(\mathfrak{X})$ is finite.

Definition 11.1.3 (Finite Rank)
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces. Then $F \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y})$ is said to be finite rank if $\dim F(\mathfrak{X})$ is finite. The set of finite rank operators from $\mathfrak{X} \to \mathfrak{Y}$ is denoted by $\mathcal{F}(\mathfrak{X}, \mathfrak{Y})$.

Proposition 11.1.11
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces. Then $\mathcal{F}(\mathfrak{X}, \mathfrak{Y}) \subseteq \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$.

Proposition 11.1.12
Let $\mathfrak{X}$ be Banach space. Then $\mathcal{K}(\mathfrak{X}) = \mathcal{B}(\mathfrak{X})$ if and only if $\mathfrak{X}$ is finite-dimensional.

Theorem 11.1.13
Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces and $K \in \mathcal{K}(\mathfrak{X}, \mathfrak{Y})$. Then $K^* \in \mathcal{K}(\mathfrak{Y}^*, \mathfrak{X}^*)$.

Theorem 11.1.14
Let $\mathcal{H}$ be a Hilbert space and $K \in \mathcal{B}(\mathcal{H})$. The following are equivalent:

(i) K is compact

(ii) K^* is compact

(iii) There exists a sequence $\{F_n\}_{n\geq 1} \subseteq \mathcal{F}(\mathcal{H})$ such that $K = \lim_n F_n$

For a subspace $\mathcal{M} \subseteq \mathcal{H}$ of a Hilbert space recall that $\mathcal{M}$ is invariant for some $T \in \mathcal{B}(\mathcal{H})$ provided that $T\mathcal{M} \subseteq \mathcal{M}$.

Definition 11.1.4 (Reducing)
We say that $\mathcal{M}$ is reducing for T if $\mathcal{M}$ is invariant both for T and T^*.

Proposition 11.1.15
Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$, and $\mathcal{M}$ be a reducing subspace of $\mathcal{H}$. Then

$$T = T_1 \oplus T_4 = \begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix}.$$

with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Furthermore, T is compact if and only if both T_1, T_4 are compact, and T is normal if and only if T_1, T_4 are.

Proposition 11.1.16
Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be normal.

(a) For all $x \in \mathcal{H}$, $\|Nx\| = \|N^*x\|$.

(b) Let $p(x, y)$ be any polynomial in two non-commuting variables x, y. Given $\alpha \in \mathbb{C}$,

$$\ker(p(N, N^*) - \alpha I) = \ker(p(N, N^*) - \alpha I)^*$$

is a reducing subspace for N.

(c) If $\alpha \neq \beta \in \mathbb{C}$, then $\ker(N - \alpha I)$ is orthogonal to $\ker(N - \beta I)$.

Proposition 11.1.17
Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. Then $N = 0$ if and only if $\sigma(N) = \{0\}$.

Let $(\mathfrak{X}, \|\cdot\|)$ be a complex Banach space and $T \in \mathcal{B}(\mathfrak{X})$. The *spectrum* of T is the set

$$\sigma(T) := \{\alpha \in \mathbb{K} : (T - \alpha I) \text{ is not invertible in } \mathcal{B}(\mathfrak{X})\}.$$

We denote the set of *eigenvalues* of T by $\sigma_p(T)$ and refer to this as the *point spectrum* of T.

In general, $\sigma_p(T)$ may be empty or non-empty. Clearly $\sigma_p(T) \subseteq \sigma(T)$.

It is known that if T is a compact operator, then

$$\sigma(T) = \sigma_p(T) \cup \{0\},$$

and that for all $\epsilon > 0$,

$$\sigma(T) \cap \{z \in \mathcal{C} : |z| > \epsilon\}$$

is a finite set. Moreover, $\dim \ker(T - \alpha I) < \infty$ for all $\alpha \neq 0$. Thus $\sigma(T) \setminus \{0\}$ is a sequence of eigenvalues of finite-multiplicity, and this sequence converges to 0.

Theorem 11.1.18
Let $\mathcal{H}$ be a complex Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. If $\sigma_p(N) = \{\alpha_n\}_{n\in\Omega}$, then

$$\mathcal{H} = \oplus_{n\in\Omega} \ker(N - \alpha_n I).$$

Theorem 11.1.19 (Spectral Theorem for Compact Normal Operators)
Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ a compact, normal operator. Suppose $\{\alpha_n\}_{n\in\Omega}$ are the distinct eigenvalues of N and that P_n is the orthogonal projection of $\mathcal{H}$ onto $\mathfrak{M}_n := \ker(N - \alpha_n I)$ for each $n \in \Omega$. Then $P_n P_m = 0$ if $n \neq m \in \Omega$ and

$$N = \sum_{n\in\Omega} \alpha_n P_n,$$

where the series converges in the norm topology in $\mathcal{B}(\mathcal{H})$.

Corollary 11.1.19.1
Let $\mathcal{H}$ be a Hilbert space and $N \in \mathcal{B}(\mathcal{H})$ a compact, normal, operator. There exists an orthonormal basis $\{e_\alpha\}_{\alpha\in\Lambda}$ for $\mathcal{H}$ such that each e_α is an eigenvector for N.