PMATH450: Lebesgue Integration & Fourier Analysis

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Contents

Part I

Introduction

Chapter 1

Introduction

1.1 Set Theory

Axiom 1.1.1 (Axiom of Choice)
Let $\mathcal{F}$ be a non-empty collection of non-empty sets

$$\mathcal{F} = \{A_\lambda : \lambda \in \Lambda\}$$

where $\varnothing \neq \Lambda, A_\lambda$.
There is a function

$$f : \Lambda \to \bigcup_{\lambda \in \Lambda} A_\lambda$$

such that

$$f(\lambda) \in A_\lambda$$

for each $\lambda \in \Lambda$.

Definition 1.1.1 (Partial Order)
Let S be a set. A relation $\leq$ on S is a partial order if

(i) $x \leq x$ for all $x \in S$ (reflexive)

(ii) $x \leq y, y \leq x \implies x = y$ for all $x, y \in S$ (anti-symmetry)

(iii) $x \leq y, y \leq z \implies x \leq z$ (transitive)

Definition 1.1.2 (Total Order)
A partial order such that for all $x, y \in S$ either $x \leq y$ or $y \leq x$.

Definition 1.1.3 (Well-Ordered)
A poset $(S, \leq)$ is well-ordered if every non-empty subset has a "smallest" element. This means for $\varnothing \neq T \subseteq S$, there is some $x \in T$ such that

$$y \in T \implies x \leq y$$

Theorem 1.1.1 (Well-Ordering Principle)
A consequence of the Axiom of Choice is that every set can be well-ordered.

Let $(S, \leq)$ be a poset.

Definition 1.1.4 (Upper Bound)
An upper bound for $T \subseteq S$ is some $s \in S$ such that $s \geq t$ for all $t \in T$.

Definition 1.1.5 (Maximal)
An element $s \in S$ is maximal if whenever $x \in S, x \geq s$ then $x = s$.

Remark that maximal elements are not necessarily unique due to the fact that S is only partially ordered.

Definition 1.1.6 (Chain)
A chain is a totally ordered subset of S.

Lemma 1.1.2 (Zorn's Lemma)
Let $(S, \leq)$ be a non-empty poset in which every chain has an upper bound. Then S has a maximal element.

Theorem 1.1.3
TFAE:

1. The Axiom of Choice
2. Zorn's Lemma
3. The Well-Ordering Principle

Theorem 1.1.4
Every vector space has a basis.

Proof
Let V be a vector space and let

$$S := \{I \subseteq V : I \text{ is linearly independent}\}$$

Let C be an arbitrary chain in S. If we show that C has an upper bound with respect to inclusion, then by Zorn's Lemma, S has a maximal element.

Define $Y := \bigcup_{W \in C} W$. Clearly $W \subseteq Y$ for all $W \in C$, so it suffices to show that $Y \in S$.

We claim Y is linearly independent. Let $x_1, \dots, x_n \in Y$ and suppose there are α_i such that $\sum_{i=1}^n \alpha_i x_i = 0$. For every x_i, there is some $x_i \in W_i \in C$ and thus by finiteness, once such W_i is maximal. Call it W^*.

We have $W_i \subseteq W^*$ for all $i \in [n]$. But then $x_i \in W^*$ for all $i \in [n]$ and $\alpha_i = 0$ for all $i \in [n]$ as desired since W^* is linearly independent.

Let M be a maximal element of S. We claim that M is a spanning set. Suppose there is some $x \in V \setminus \operatorname{span} M$. However $M \cup \{x\}$ is then a larger linearly independent set containing M, contradicting the maximality of M.

So M is a linearly independent spanning set and therefore a basis.

1.2 Riemann Integral

Recall the definition of the Riemann integral.

Definition 1.2.1 (Riemann Sum)
Let $F : [a, b] \to \mathbb{R}$ be bounded. Moreover, let $P : a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$.
The upper and lower Riemann sums are:

$$U(f, P) := \sum_{i=1}^{n} \sup f\Big|_{[x_{i-1}, x_i]} (x_i - x_{i-1})$$
$$L(f, P) := \sum_{i=1}^{n} \inf f\Big|_{[x_{i-1}, x_i]} (x_i - x_{i-1})$$

Definition 1.2.2 (Refinement)
We say the partition Q is a refinement of P if Q contains P.

Proposition 1.2.1
If Q is a refinement of P, then

$$U(f,Q) \le U(f,P) \wedge L(f,P) \le L(f,Q)$$

Proposition 1.2.2
Every upper sum for a fixed f dominates every lower sum.

Proof
If P_1, P_2 are any two partitions of $[a,b]$ and Q is their common refinement.

$$U(f,P_1) \ge U(f,Q) \ge L(f,Q) \ge L(f,P_2)$$

Definition 1.2.3 (Riemann Integrable)
If

$$\inf\{U(f,P) : P \text{ is a partition}\} = \inf\{L(f,P) : P \text{ is a partition}\}$$

we say f is Riemann integrable over $[a,b]$ and write

$$R - \int_a^b f := \inf_P U(f,P) = \sup_P L(f,P)$$

We will write $R - \int_a^b f$ to emphasize that this is the Riemann integral.

The nice thing about Riemann integratioh is that a lot of important functions are integrable. Namely, the continuous ones. In fact, even functions which have countably many discontinuities are Riemann integrable. This includes the increasing functions as they only have jump discontinuities and those are necessarily countable since each jump contains one unique rational number.

Definition 1.2.4 (Characteristic Function)
Let $A \subseteq X$. The characteristic function of A is $\chi_A : X \to \{0,1\}$ given by

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

One limit of the Riemann integral is that the characteristic function of rationals is not integrable. Another issue is that it performs poorly under limits. In other words, it is NOT always true that if $R - \int_a^b f_n$ exists and $f_n \to f$ pointwise, then $R - \int_a^b f$ even exists.

Idea of Lebesgue Integration

Lebesgue's idea is to partition the range of the function.

Say $\operatorname{ran} f = [A, B]$. Let $A = y_1 < \cdots < y_n = B$ and define $E_i := f^{-1}(y_{i-1}, y_i]$. If the partition is "fine" enough we would roughly get

$$f \sim \sum_i y_i \chi_{E_i}$$

and so we can define the Lebesgue integral of f over its domain to be

$$\lim \sum_i y_i \text{"length" of } E_i$$

Part II

Lebesgue Integration

Chapter 2

Measure

2.1 Idea Properties

A key element of Lebesgue's integral is an extension of the notion of interval lengths to much more general sets. Ideally, we want

$$m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$$

such that

1. $m(\varnothing) = 0$
2. If I is an interval with endpoints a, b, then $m(I) = b - a$
3. If $\{E_n\}_{n\geq 1}$ are disjoint, then $m \bigcup_{n\geq 1} E_n = \sum_{n\geq 1} mE_n$ (σ-additivity)
4. If $E \subseteq \mathbb{R}$, then $mE = m(E + y)$ for all $y \in \mathbb{R}$ (translation invariance)

Notice that the properties above imply monotonicity. In otherwords, for $A \subseteq B \subseteq \mathbb{R}$

$$m(B) = m(A) + m(B \setminus A) \geq m(A)$$

2.1.1 Proof of Unattainability

Define a relation on R where $x \sim y$ if $x - y \in \mathbb{Q}$.

Lemma 2.1.1
$\sim$ is an equivalence relation.
Moreoever, each equivalence class $[x] \in \mathbb{R}/\sim$ is a translate of $\mathbb{Q}$ and hence is dence in $\mathbb{R}$.

Theorem 2.1.2
No function $m : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ with the desired properties exist.

Proof
Suppose otherwise, and let m be such a function.

Consider $\mathbb{R}/\sim$ and use the axiom of choice to pick a representative of each equivalence class in $\left[0, \frac{1}{2}\right]$. Let E denote this collection of real numbers.

Remark that if x_1, x_2 are distinct rational numbers, $E + x_1, E + x_2$ are disjoint by the definition of $\sim$.

By σ-additivity and translation invariance

$$m\left(\bigcup_{x\in\mathbb{Q}} E + x\right) = \sum_{x\in\mathbb{Q}} m(E + x) = \sum_{x\in\mathbb{Q}} mE \in \{0, \infty\}$$

Clearly, $m(E) = 0$ if and only if the sum is 0.

Now, let y be an arbitrary real number, so $y \sim e$ for some $e \in E$. It follows that $y - e \in \mathbb{Q}$ and hence $y = e + x$ for some $x \in \mathbb{Q}$. This shows that $y \in \bigcup_{x\in\mathbb{Q}} E + x = \mathbb{R}$.

Therefore, the sum diverges and $mE > 0$. Enumerate $\mathbb{Q} \cap \left[0, \frac{1}{2}\right] = \{r_i\}_{i\geq 1}$. Again, appying σ-additivity and using $mE > 0$ gives

$$m\left(\bigcup_{i\geq 1} E + r_i\right) = \sum_{i\geq 1} m(E + r_i) = \sum_{i\geq 1} mE = \infty$$

But $E \subseteq \left[0, \frac{1}{2}\right]$! So monotonicty gives us

$$m\left(\bigcup_{i\geq 1} E + r_i\right) \leq m[0, 1] = 1$$

By contradiction, m cannot exist.

2.1.2 A Compromise?

Which property are we willing to give up to obtain such a function m?

The only two properties which make some sort of sense would be σ-additivity and the choice that m is defined for all subsets.

As it turns out, even giving up countable but not finite additivity does not give the desired function m. Thus, we will only work with nice subsets of $\mathbb{R}$ where a function with all the desired properties exist.

2.2 Outer Lebesgue Measure

We will construct a function which relaxes the sigma additivity property and restrict it to suitable subsets of $\mathbb{R}$.

Denote by $\ell(I)$ for the length of an interval ℓ.

Definition 2.2.1 (Outer Lebesgue Measure)
Let $A \subseteq \mathbb{R}$. Let $\mathcal{C}(A) := \{\{I_n\}_{n\geq 1} : A \subseteq \bigcup I_n, I_n \text{ is an open interval}\}$ be the set of open interval covers of A.
Define

$$m^*(A) := \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : \{I_n\}_{n\geq 1} \in \mathcal{C}(A)\right\}$$

Proposition 2.2.1

1. $m^*\varnothing = 0, m^*\{x\} = 0$
2. monotonicity
3. translation invariance

Proposition 2.2.2
$m^*I = \ell(I)$ for all intervals I.

Proposition 2.2.3 (σ-sub-additivity)
For all $A_k \in \mathbb{R}$ we have

$$m^*\left(\bigcup_{i=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} m^*A$$

Notice that we do not require the A_k's to be disjoint.

Proof
Any union of covers of all A_k's would cover their union.

The sum of lengths would then be at most the outer measures of all A_k's.

Corollary 2.2.3.1
If $A = \{x_n\}_{n\geq 1}$ is a countable set then

$$m^*(A) \leq \sum_{n\geq 1} m^*\{x_n\} = 0$$

Corollary 2.2.3.2
For any $A \subseteq \mathbb{R}, \epsilon > 0$ there is an open set $O \supseteq A$ such that

$$m^*(O) \leq m^*A + \epsilon$$

Proof
Pick an interval cover infinitely close to the outer measure of A in length. Take the union of these intervals to be O.

2.3 Lebesgue Measure

The final step to obtain the Lebesgue measure is to restrict m^* to a suitable class of subsets of $\mathbb{R}$.

Definition 2.3.1 (Lebesgue Measurable)
$A \subseteq \mathbb{R}$ is Lebesgue Measurable if for every $E \subseteq \mathbb{R}$

$$m^*E = m^*(E \cap A) + m^*(E \cap A^c)$$

This is known as the Carathéodory definition of Lebesgue measurability. Notice that by σ-sub-additivity

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap A^c)$$

so to satisfy the definition above we only need to check the $\geq$ case.

Example 2.3.1
The following are Lebesgue measurable

1. $\mathbb{R}, \varnothing$
2. $A \subseteq \mathbb{R}, m^*A = 0$
3. $A \subseteq \mathbb{R}$ such that A^c is Lebesgue measurable

Proposition 2.3.2
(a, ∞) is Lebesgue measurable.

Proof
Fix $E \subseteq \mathbb{R}$. Let $\epsilon > 0$ and choose an open interval cover $\{I_n\}$ such that

$$\sum_{n \geq 1} \ell(I_n) \leq m^*(E) + \epsilon$$

Define $I_n^+ := I_n \cap (a, \infty), I_n^- := I_n \cap (-\infty, a]$. Notice that I_n^+, I_n^- are either intervals or empty.

By the additivity of length

$$\ell(I_n) = \ell(I_n^+) + \ell(I_n^-) = m^*(I_n^+) + m^*(I_n^-)$$

Since $E \cap (a, \infty) \subseteq \bigcup_{n \geq 1} I_n \cap (a, \infty) = \bigcup_{n \geq 1} I_n^+$, monotonicity and σ-sub-additivity gives

$$m^*(E \cap (a, \infty)) \leq m^* \left(\bigcup_{n \geq 1} I_n^+ \right) \leq \sum_{n \geq 1} m^*(I_n^+)$$

Similarly, we get

$$m^*(E \cap (-\infty, a]) \leq \sum_{n \geq 1} m^*(I_n^-)$$

Combining both results

$$\begin{aligned} m^*(E \cap (a, \infty)) + m^*(E \cap (-\infty, a]) &\leq \sum_{n \geq 1} m^*(I_n^+) + m^*(I_n^-) \\ &= \sum_{n \geq 1} \ell(I_n) \\ &\leq m^* E + \epsilon \end{aligned}$$

Since ϵ was arbitrary, this gives the desired inequality and concludes the proof.

2.3.1 Measurable Sets

Definition 2.3.2 (σ-algebra)
$\Omega \in \mathcal{P}(\mathbb{R})$ is a σ-algebra if

(i) $\varnothing \in \Omega$

(ii) $\{A_n\}_{n\geq 1}, A_n \in \Omega$ then $\bigcup_{n\geq 1} A_n \in \Omega$ (closed under countable union)

(iii) $A \in \Omega$ then $A^c \in \Omega$ (closed under complements)

As an immediate corollary of the definition, any intersection of σ-algebras is again a σ-algebra.

Definition 2.3.3 (Borel σ-Algebra)
The intersection of all σ-algebras containing the open sets in $\mathbb{R}$.

We say an element of the Borel σ-algebra is a Borel set.

Since any open set in $\mathbb{R}$ is the countable union of open intervals (take the largest interval containing each rational element of the open set), we can also say that the Borel σ-algebra is the smallest σ-algebra "generated" by the open sets (intervals).

Observe that we can "simply" the generator even further to sets of the form (a, ∞). The intersection operation can be expression using unions and complements. So any interval can be form using the combination of unions and complements of $\{(a, \infty) : a \in \mathbb{R}\}$.

Theorem 2.3.3 (Carathéodory Extension)
The set of Lebesgue Measurable sets $\mathcal{M}$, is a σ-algebra containing the Borel sets and all sets of outter Lebesgue measure zero.
Moreoever, if $A_n \in \mathcal{M}$ are disjoint, then

$$m^*\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} m^* A_n$$

so m^* is σ-additive on $\mathcal{M}$.

Proof
We have already shown that the sets of outer measure 0 are measurable. The fact that $\mathcal{M}$ contains the Borel sets comes for free once we prove $\mathcal{M}$ is a σ-algebra as we have alreay shown that intervals of the form (a, ∞) are Lebesgue Measurable.

We have already shown $\mathbb{R}$ and therefore $\varnothing$ is in $\mathcal{M}$.

We have also shown that $\mathcal{M}$ is closed under complementation.

It remains to check for closedness under countable unions.

Closedness under finite union We first show the result for finite unions by induction. It suffices to show the case for $A, B \in \mathcal{M}$ arbitrary, then $A \cup B \in M$. Choose $X \subseteq \mathbb{R}$ be arbitrary. We have

$$m^*X = m^*(X \cap A) + m^*(X \cap A^c)$$

by definition. Since $B \in \mathcal{M}$

$$m^*(X \cap A^c) = m^*((X \cap A^c) \cap B) + m^*((X \cap A^c) \cap B^c)$$

σ-subadditivity then gives

$$\begin{aligned} m^*X &= [m^*(X \cap A) + m^*((X \cap A^c) \cap B)] + m^*((X \cap A^c) \cap B^c) \\ &\geq m^*(X \cap (A \cup B)) + m^*(X \cap (A \cup B)^c) \end{aligned}$$

which is exactly the inequality desired.

Finite additivity of m^* on $\mathcal{M}$ We now argue for finite disjoint E_i's, $m^*(X \cap \dot\bigcup_{n=1}^N E_n) = \sum_{n=1}^N m^*(X \cap E_n)$. Again, we show the case only for $A, B \in \mathcal{M}$ disjoint since induction solves the rest. Pick $X \subseteq \mathbb{R}$ arbitrarily so by definition

$$\begin{aligned} m^*(X \cap (A \cup B)) &= m^*(X \cap (A \cup B) \cap B) + m^*(X \cap (A \cup B) \cap B^c) \\ &= m^*(X \cap B) + m^*(X \cap A) \end{aligned}$$

Closed under countable unions Now, for any countable $\{E_n\} \subseteq \mathcal{M}$, we write $H_N := \bigcup_{i=1}^N E_n$. Furthermore, define $F_1 := H_1$ and $F_n := H_n \setminus H_{n-1}$ for $n \geq 2$. So each $H_N = \dot\bigcup_{n=1}^N F_n$ disjoint.

Since we have shown $\mathcal{M}$ to be closed under finite union and complements, $F_n \in \mathcal{M}$ for each $n \geq 1$.

Remark that $E = \bigcup_{n\geq 1} E_n = \bigcup_{n\geq 1} H_n = \dot\bigcup_{n\geq 1} F_n$. Pick $X \subseteq \mathbb{R}$.

$$\begin{aligned} m^*X &= m^*(X \cap H_N) + m^*(X \cap H_N^c) \\ &= m^*(X \cap (\dot\bigcup_{n=1}^N F_n)) + m^*(X \cap H_N^c) \\ &\geq m^*(X \cap \dot\bigcup_{n=1}^N F_n) + m^*(X \cap E^c) && X \cap E^c \subseteq X \cap H_N^c \\ &= \sum_{n=1}^N m^*(X \cap F_n) + m^*(X \cap E^c) \end{aligned}$$

Since m^* is σ-sub-additive,

$$\begin{aligned} m^*(X \cap E) &= m^*(X \cap \dot\bigcup_{n\geq 1} F_n) \\ &\leq \sum_{n\geq 1} m^*(X \cap F_n) \end{aligned}$$

Taking limits and combining the last two steps

$$\begin{aligned} m^*X &\geq \sum_{n\geq 1} m^*(X\cap F_n) + m^*(X\cap E^c) \\ &\geq m^*(X\cap E) + m^*(X\cap E^c) \end{aligned}$$

which shows us $E \in \mathcal{M}$.

σ-additivity of m^* on $\mathcal{M}$ Finally, we want to show that

$$m^*E = \sum_{n\geq 1} m^*F_n$$

Simply let $X = E$ in our work above

$$\begin{aligned} m^*E &\geq \sum_{n\geq 1} m^*(E\cap F_n) + m^*(E\cap E^c) \\ &\geq \sum_{n\geq 1} m^*F_n \end{aligned}$$

But the $\leq$ inequality comes directly from σ-sub-additivity so we have equality and conclude the proof. (We were lazy and did not specify an arbitrary set of collection of disjoint F_i's, but the E_i's were arbitrary and that is enough)

2.3.2 Final Result

Definition 2.3.4 (Lebesgue Measure)
The function $m : \mathcal{M} \to [0, \infty]$ defined by

$$E \mapsto m^*E$$

Example 2.3.4
The cantor set C is an uncountable set of Lebesgue measure zero.

Important Properties

The following properties of the Lebesgue measure and $\mathcal{M}$ are important.

Proposition 2.3.5

1. $mI = \ell(I)$ for all intervals I
2. translation invariance
3. σ-additivity for disjoint sets
4. σ-sub-additivity for all sets
5. finite additivity for disjoint sets
6. monotonicity
7. $E \in \mathcal{M}$ has measure zero if and only if for all $\epsilon > 0$ there is an open interval cover of arbitrarily small length
8. compact sets have finite measure

Proposition 2.3.6
$E \in \mathcal{M}, \epsilon > 0$ then there is an open set O such that

$$m^*(O \setminus E) < \epsilon$$

Proof
If $mE < \epsilon$, then simply get an open interval cover of length at most $mE + \epsilon$. Then take O to be the union of all such intervals.

If $mE = \infty$, decompose $E = \bigcup_{n \geq 1} E \cap [-n, n]$ and apply step 1 for each n and $\frac{\epsilon}{2^n}$ (for convergence).

Proposition 2.3.7 (Continuity of Measure)
If $A_1 \subseteq A_2 \subseteq \ldots \subseteq \bigcup_{n \geq 1} A_n =: A$ with $A_n \in \mathcal{M}$, then $A \in \mathcal{M}$ with

$$mA = \lim_{n \to \infty} mA_n$$

Proof
Split A_n's into disjoint sets B_n and apply σ-additivity.

$$mA_n = \sum_{i=1}^{n} mB_i \to mA$$

Proposition 2.3.8 (Downward Continuity of Measure)
If $A_1 \supseteq \cdots \bigcap_{n \geq 1} A_n =: A$ with $A_n \in \mathcal{M}$, and $mA_n < \infty$ for some $n \geq 1$,

$$mA = \lim_{n \to \infty} mA_n$$

Proof
Let $B_n := A_1 \setminus A_n$ and apply the continuity of measure for $B := A_1 \setminus A$.

$$mB_n \to mB \implies mA_1 - mA_n \to mA_1 - mA$$

Notice we use the fact that A_1 has finite measure or else infinity minus infinity makes no sense.

Lebesgue's Criterion of Riemann Integrability

Theorem 2.3.9 (Lebesgue's Criterion)
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable over $[a, b]$ if and only if

$$m\{\text{discontinuities of } f\} = 0$$

Chapter 3

Measurable Functions

3.1 Measurable Functions

Let $X \subseteq \mathbb{R}$ be a Lebesgue measurable set.

Definition 3.1.1 (Lebesgue Mesurable Function)
$f : X \to [-\infty, +\infty]$ is Lebesgue measurable if for every $\alpha \in \mathbb{R}$

$$\{x \in X : f(x) < \alpha\} = f^{-1}[\infty, \alpha)$$

is a Lebesgue measurable set.

Definition 3.1.2 (Lebesgue Measurable Function)
A complex-valued function $f : X \to \mathbb{C}$ is Lebesgue measurable if both its real and imaginary parts are Lebesgue measurable.

Notice that any continuous function is measurable since its pre-image of an open set is open.

Proposition 3.1.1
$f^{-1}[-\infty, \alpha) \in \mathcal{M}$ for all $\alpha \in \mathbb{R}$ if and only if

$$f^{-1}(\beta, \infty]$$

is measurable for all $\beta \in \mathbb{R}$.

Proposition 3.1.2
If f is measurable, then $f^{-1}\{\infty\}$ (and negative infinity) is a measurable set.

Proposition 3.1.3
$f = \chi_E$ is measurable if and only if $E \subseteq \mathbb{R}$ is a measurable set.

Proof
$f^{-1}[-\infty, \alpha) \in \{\mathbb{R}, E^c, \varnothing\}$ depending on if $\alpha \in \{(1, \infty], (0, 1], [-\infty, 0]\}$.

Definition 3.1.3 (Simple Function)
A function of the form

$$f = \sum_{i=1}^{N} a_i \chi_{E_i}$$

for $E_i \in \mathbb{R}$ are measurable sets and $a_i \in \mathbb{R}$.

These are the function which take on finitely many real values. In the case the E_i's are intervals, the simple function is a step function.

Proposition 3.1.4
Every simple function is measurable.

3.2 Properties of Measurable Functions

Proposition 3.2.1
If f, g are real-valued, measurable functions with the same domain, then so are

$$f \pm g, fg, \frac{f}{g}$$

where the last one also requires g does not contain 0 in its range.

Proof
Observe that $\{x : (f + g)(x) < \alpha\} = \{x : f(x) < \alpha - g(x)\}$.

Moreoever, $f(x) < \alpha - g(x)$ if and only if there is some $r \in \mathbb{Q}$ with

$$f(x) < r < \alpha - g(x)$$

Thus

$$\{x : (f + g)(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} \{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\}$$

is the countable union of measurable sets and is therefore measurable.

Similar, we can show f being measurable implies f^2 is measurable. The result for products follow from the fact that

$$fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$$

Finally, we can show if $0 \notin \operatorname{ran} g$ then $\frac{1}{g}$ is measurable if and only if g is measurable. The result for quotients follows from

$$\frac{f}{g} = f \cdot \frac{1}{g}$$

Behavior Under Limits

Proposition 3.2.2
If $\{f_n\}_{n\geq 1}$ are measurable functions, then so are

$$\sup_n f_n, \inf_n f_n$$

If $f_n \to f$ pointwise as $n \to \infty$, then f is measurable.

Proof
The set $\{x : \sup_n f_n > \alpha\} = \bigcup_{n\geq 1}\{x : f_n > \alpha\}$ and is hence a countable union of measurable sets.

Thus $\sup_n f_n$ is a measurable function. The proof that $\inf_n f_n$ is measurable is similar.

If $f_n \to f$, then

$$\begin{aligned} f &= \lim_{n\to\infty} \inf_{k\geq n} f_k \\ &= \sup_n \left(\inf_{k\geq n} f_k\right) && \text{monotonically increasing sequence} \end{aligned}$$

and thus f is measurable.

The following theorem justifies using simple functions as the building blocks of the Lebesgue integral.

Theorem 3.2.3
The positive, real-valued function $f : \mathbb{R} \to [0, \infty)$ is measurable if and only if there are simple functions $\phi_n, n \geq 1$ such that

$$\phi_n \leq \phi_{n+1}$$

and $\phi_n \to f$ pointwise.

Proof
($\implies$) Suppose f is measurable and we will see how to construct the functions ϕ_n.

For $k = 0, 1, \ldots, n2^n - 1$, let

$$\begin{aligned} E_{n,k} &= f^{-1}[k2^{-n}, (k+1)2^{-n}) \\ &= \{x : k2^{-n} \leq f(x) < (k+1)2^{-n}\} \end{aligned}$$

These are all measurable sets. Moreoever, let E_n be the measurable set

$$E_n = \bigcup_{k=0}^{n2^n-1} E_{n,k} = f^{-1}[0, n)$$

Put

$$\begin{aligned} \phi_n(x) &= \begin{cases} k2^{-n}, & x \in E_{n,k} \\ n, & x \notin E_n \end{cases} \\ &= \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \chi_{E_{n,k}} + n\chi_{E_n^c} \end{aligned}$$

The functions ϕ_n are simple and the choice of partitioning the range into subintervals of width 2^{-n} ensures the monotonicity.

It remains to check for point-wise convergence. Fix x and assume $f(x) < N$. Then $x \in E_n$ for all $n \geq N$. So $x \in E_{n,k}$ for some suitable choice of k.

The definitions ensure that

$$f(x) \in [k2^{-n}, (k+1)2^{-n}), \phi_n(x) = k2^{-n}$$

and hence

$$|f(x) - \phi_n(x)| < 2^{-n}$$

for all $n \geq N$ which demonstrates pointwise convergence.

($\impliedby$) This direction is obvious as simple functions are measurable and pointwise limits preserve measurability.

3.3 Almost Everywhere

Definition 3.3.1
We say that $f = g$ almost everywhere, and write $f = h$ a.e. if

$$m\{x : f(x) \neq g(x)\} = 0$$

The idea is that two functions agree on all but a set which is "invisible" as far as the Lebesgue measure is concerned. For example, $\chi_{\mathbb{Q}} = 0$ a.e.

Proposition 3.3.1
If $f = 0$ a.e., then f is measurable.
Moreoever $f = g$ a.e. and f being measurable means g is measurable.

Proof
Remark that $g = (g - f) - f$ where $g - f = 0$ a.e. so it suffices to show the first claim.

Let A be the subset of X with measure 0 where $f \neq g$. Since A is measurable

$$m(X \setminus A) = m(X) - m(X \cap A) = m(X)$$

We claim

$$\begin{aligned} m\left(f^{-1}[-\infty, \alpha)\right) &= m\left(X \setminus f^{-1}[\alpha, \infty]\right) \\ &= \begin{cases} 0, & \alpha \leq 0, X \setminus f^{-1}[\alpha, \infty] \subseteq A \\ mX, & \alpha > 0, \end{cases} \end{aligned} \tag{$*$}$$

$(*)$ This is due to the fact that $f^{-1}[\alpha, \infty] \subseteq A$ and thus is measurable with measure 0. Then

$$f^{-1}[-\infty, \alpha) = X \cap (\mathbb{R} \setminus f^{-1}[\alpha, \infty])$$

can be obtained from measurable sets and is thus measurable with

$$\begin{aligned} mX &= m(f^{-1}[-\infty, \alpha)) + m(f^{-1}[\alpha, \infty]) \\ &= m(f^{-1}[-\infty, \alpha)) \end{aligned}$$

as desired.

Chapter 4

The Lebesgue Integral

4.1 The Lebesgue Integral

Definition 4.1.1 (Standard Representation)
The unique expression of a simple function

$$\sum_{k=1}^{N} a_k \chi_{E_k}$$

where the E_k's are measurable, pairwise disjoint, and

$$\bigcup_{k=1}^{N} E_k = \mathbb{R}$$

.

We will make the convention that

$$0 \cdot \infty = 0$$

so it is NOT and indeterminate.

Recall that for a step function

$$\phi = \sum_{k=1}^{N} a_k \chi_{I_k}$$

where I_k are intervals and $a_j \in \mathbb{R}$

$$R - \int_a^b \phi = \sum_k a_k \ell(I_k \cap [a, b])$$

This motivates the following definition.

Definition 4.1.2 (Lebesgue Integral)
Assume $\phi \geq 0$ is a simple function with standard representation

$$\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$$

The Lebesgue integral of ϕ over the measurable set E is

$$\int_E \phi = \sum_{k=1}^{N} a_k m(E_k \cap E)$$

Proposition 4.1.1
If ϕ also has a representation

$$\phi = \sum_{j=1}^{M} b_j \chi_{F_j}$$

with F_j's measurable, then

$$\int_E \phi = \sum_{j=1}^{M} b_j m(F_j \cap E)$$

Proof
Remark that if A, B are not disjoint, we can write any $\alpha\chi_A + \beta\chi_B$ as

$$\alpha\chi_{A\setminus B} + (\alpha+\beta)\chi_{A\cap B} + \beta\chi_{B\setminus A}$$

with

$$\begin{aligned}\alpha mA + \beta mB &= \alpha m(A\setminus B) + \alpha m(A\cap B) + \beta m(B\setminus A) + \beta m(B\cap A) \\ &= \alpha m(A\setminus B) + (\alpha+\beta) m(A\cap B) + \beta m(B\setminus A)\end{aligned}$$

Thus it does not hurt to assume the F_j's are disjoint.

Write

$$F_{M+1} := \mathbb{R} \setminus \left(\bigcup_{j=1}^{M} F\right), b_{M+1} = 0$$

and notice $\phi = \sum_{j=1}^{M+1} b_j \chi_{F_j}$.

We have

$$
\begin{aligned}
\sum_{j=1}^{M} b_j m(E \cap F_j) &= \sum_{j=1}^{M+1} b_j m(E \cap F_j) \\
&= \sum_{j=1}^{M+1} b_j \sum_{k=1}^{N} m(E \cap F_j \cap E_k) && \sigma\text{-additivity} \\
&= \sum_{j=1}^{M+1} \sum_{k=1}^{N} a_k m(E \cap F_j \cap E_k) && \phi\Big|_{F_j \cap E_k} = a_k = b_j \\
&= \sum_{k=1}^{N} a_k m(E \cap E_k) && \sigma\text{-additivity} \\
&= \int_E \phi
\end{aligned}
$$

as desired.

Notice that the Riemann and Lebesgue integrals of step functions are identical.

Proposition 4.1.2
If $\phi = 0$ a.e. then $\int_E \phi = 0$ for any measurable set E.

Proof
Write

$$\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$$

for the standard representation.

Then

$$
\begin{aligned}
\int_E &= \sum_{k=1}^{N} a_k m(E \cap E_k) \\
&= \sum_{k: mE_k = 0} a_k m(E \cap E_k) + \sum_{k: mE_k > 0} a_k m(E \cap E_k) \\
&= \sum_{k: mE_k = 0} a_k \cdot 0 + \sum_{k: mE_k > 0} 0 \cdot m(E \cap E_k) \\
&= 0
\end{aligned}
$$

In particular

$$\int_{[a,b]} \chi_{\mathbb{Q}} = 0$$

for any $[a, b] \subseteq \mathbb{R}$.

Next, we define the Lebesgue integral of a non-negative measurable function.

Definition 4.1.3 (Lebesgue Integral)
Suppose $f : E \subseteq \mathbb{R} \to [0, \infty]$ is measurable and E is a measurable set. The Lebesgue integral of f over E is

$$\int_E f := \sup\left\{\int_E \phi : 0 \leq \phi \leq f, \phi \text{ is simple}\right\}$$

This gives us two competing definitions for the Lebesgue integral of a positive simple function, but it can be checked that they are equivalent.

Notice that if $0 \leq f \leq g$ are measurable functions, then

$$\int_E f \leq \int_E g$$

as any $\phi \leq f$ also satisfies $\phi \leq g$.

For $f : E \to \mathbb{R}$ let

$$f^+(x) = \max(f, 0)$$
$$f^-(x) = max(-f, 0)$$

Remark that

$$f = f^+ - f^-$$

Since we want the Lebesgue integral to be linear, we are forced to define

Definition 4.1.4 (Lebesgue Integral)
For measurable E and $f : E \to \mathbb{R}$, we let the Lesbesgue integral of f over E as

$$\int_E f = \int_E f^+ - \int_E f^-$$

(provided this is not a $\infty - \infty$ form)

Remark that f being measurable implies f^+, f^- are both measurable, it follows that $|f| = f^+ + f^-$ is also measurable.

Definition 4.1.5 (Integrable)
We say the measurable function f is integrable on E if

$$\int_E |f| < \infty$$

Since $f^+, f^- \le |f|$, by the monotonicity of property of non-negative functions, $\int_E f^+, \int_E f^-$ are finite for any integrable function f and hence $\int_E f$ is well-defined.

Definition 4.1.6 (Lebesgue Integral)
For $f : E \subseteq \mathbb{R} \to \mathbb{C}$, we say f is integrable if both $\operatorname{Re} f, \operatorname{Im} f$ are integrable functions and define its Lebesgue integral over E to be

$$\int_E f = \int_E \operatorname{Re} f + i \int_E \operatorname{Im} f$$

4.2 Properties of the Lebesgue Integral

Proposition 4.2.1 (Monotonicity)
If $0 \le f \le g$ then

$$\int_E f \le \int_E g$$

So if $|f| \le M$, then $\int_E |f| \le \int_E M = Mm(E)$.

Proposition 4.2.2
Assume f is measurable, non-negative or integrable, and E is a measurable set.

$$\int_E f = \int_{\mathbb{R}} f \cdot \chi_E$$

This means there is no loss with assuming $E = \mathbb{R}$ and we can simply write $\int f$.

Proof
First observe that the measurability of E, f ensures $f\chi_E$ is measurable.

Suppose $f = \sum_{k=1}^{N} a_k \chi_{E_k}$. Then

$$\begin{aligned}\int_E f &= \sum_{k=1}^{N} a_k m(E_k \cap E) \\ &= \int_{\mathbb{R}} \sum_{k=1}^{N} a_k \chi_{E_k \cap E} \\ &= \int_{\mathbb{R}} \left(\sum_{k=1}^{N} a_k \chi_{E_k} \right) \chi_E \\ &= \int_{\mathbb{R}} f \chi_E\end{aligned}$$

Observe it suffices to show the result for non-negative real-valued measurable functions as the rest simply builds from this. Assume $f : \mathbb{R} \to [0, \infty]$ and ϕ is a simple function with $0 \leq \phi \leq f$.

By inspection, $\phi \chi_E$ is a simple function with $0 \leq \phi \chi_E \leq f \chi_E$. Thus

$$\begin{aligned}\int_E \phi &= \int_R \phi \chi_E \\ &\leq \sup \left\{ \int_{\mathbb{R}} \psi : 0 \leq \psi \leq f \chi_E \right\} \\ &= \int_{\mathbb{R}} f \chi_E && \text{definition}\end{aligned}$$

and thus

$$\int_E f = \sup \left\{ \int_E \phi : 0 \leq \phi \leq f \right\} \leq \int_{\mathbb{R}} f \chi_E$$

On the other hand, let ψ be a simple function with $0 \leq \psi \leq f \chi_E$. We also have $0 \leq \psi \leq f$ and $\psi = \psi \chi_E$. Thus

$$\begin{aligned}\int_{\mathbb{R}} \psi &= \int_{\mathbb{R}} \psi \chi_E \\ &= \int_E \psi \\ &\leq \sup \left\{ \int_E \phi : 0 \leq \phi \leq f \right\} \\ &= \int_E f && \text{definition}\end{aligned}$$

and

$$\int_{\mathbb{R}} f \chi_E = \sup \left\{ \int_{\mathbb{R}} \psi : 0 \leq \psi \leq f \chi_E \right\} \leq \int_E f$$

We conclude $\int_{\mathbb{R}} f\chi_E = \int_E f$ and our proof by the initial remark.

Proposition 4.2.3
If $m0 = 0$, then $\int_E f = 0$.

Proof
The integral of any simple function is 0.

Proposition 4.2.4
For all $\alpha \in \mathbb{C}$

$$\int \alpha f = \alpha \int f$$

Proof
It is true for $\alpha \geq 0$ by definition.

We can then iteratively show this for $\alpha = -1, \alpha \in \mathbb{R}, \alpha = i, \alpha \in \mathbb{C}$.

$$\begin{aligned} -f &= -f^+ + f^- \\ if &= -\operatorname{Im} f + iRef \end{aligned}$$

Proposition 4.2.5 (Triangle Inequality)
$|\int f| \leq \int |f|$.

Proof
Pick $|\alpha| = 1$ such that

$$\left|\int f\right| = \alpha \int f$$

Then

$$\begin{aligned}\left|\int f\right| &= \int \alpha f \\ &= \int \operatorname{Re} \alpha f + i \int \operatorname{Im} \alpha f \\ &= \int \operatorname{Re} \alpha f && \text{components must match} \\ &\leq \int |\alpha f| \\ &= \int |f|\end{aligned}$$

Proposition 4.2.6
For all ϕ, ψ simple

$$\int \phi + \psi = \int \chi + \int \psi$$

Proof
Using the respective standard representations of ϕ, ψ, re-express them both using possibly different scalars and the SAME finite set of characteristic functions.

The result follows by the fact that the integral does not depend on the choice of representation.

Proposition 4.2.7 (Translation Invariance)
$\int_{\mathbb{R}} f(x+y)dx = \int_{\mathbb{R}} f(x)dx$

Proof
First check that translation preserves measurability.

Then let $\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$ be any simple function. Using the translation invariance of the Lebesgue measure

$$\begin{aligned}\int_{\mathbb{R}} \phi(x+y)dx &= \sum_{k=1}^{N} a_k m(E_k - y) \\ &= \sum_{k=1}^{N} a_k m E_k \\ &= \int_{\mathbb{R}} \phi\end{aligned}$$

The results of non-negative, real-valued, complex-valued f then follow directly from this result.

Chapter 5

Convergence Theorems

5.1 Monotone Convergence Theorem

Recall one of the motivations for moving past the Riemann integral was that

$$\lim_{n\to\infty R} - \int f_n = R - \int \lim f_n$$

does NOT hold in general.

Unfortunately, the equation above does not hold in general for Lebesgue integrals either.

Example 5.1.1
Let $f_n : [0,1] \to \mathbb{R}$ be such that

$$f_n(x) = \begin{cases} n, & x \in \left(0, \frac{1}{n}\right) \\ 0, & \text{else} \end{cases}$$

Then $f_n \to 0$ but

$$\int_{[0,1]} f_n = 1$$

for all n but

$$\int_{[0,1]} \lim f_n = 0$$

Lemma 5.1.2
Let $\phi \geq 0$ be a simple function and assume $A_n, n \geq 1$ are measurable sets with $A_n \subseteq A_{n+1}$ for each n. Put $A = \bigcup_{n\geq 1} A_n$. Then

$$\lim_{n\to\infty} \int_{A_n} \phi = \int_A \phi$$

Proof
Assume $\phi = \sum_{i=1}^N a_i \chi_{E_i}$ where the sets E_i are measurable. Then

$$\int_{A_n} \phi = \sum_{i=1}^N a_i m(E_i \cap A_n)$$

For each i

$$E_i \cap A_n \subseteq E_i \cap A_{n+1}$$

and also

$$\bigcup_{n\geq 1} (E_i \cap A_n) = E_i \cap A$$

By the continuity of measure

$$m(E_i \cap A_n) \to m(E_i \cap A)$$

hence

$$\int_{A_n} \phi \to \sum_{i=1}^N a_i m(E_i \cap A) \int_A \phi$$

This lemma is the special case of the Monotone Convergence theorem with

$$f_n := \phi\chi_{A_n}, f = \phi\chi_A$$

Theorem 5.1.3 (Monotone Convergence)
Suppose $f_n \geq 0$ are measurable and $f_n \leq f_{n+1}$ for $n \geq 1$. Moreoever, assume $f_n \to f$ pointwise. Then

$$\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f$$

Remark that we allow the range of f_n, f to contain ∞. The extra hypothesis here is that the sequence f_n converges up to f.

Proof
Since each f_n is measurable and f is the pointwise limit, f is also measurable.

By assumption, $f_n \leq f$ for all $n \geq 1$ and thus by monotonicity

$$\int_E f_n \leq \int_E f$$

Since the sequence of integrals is also increasing, it must have a limit (allowing for ∞). Hence

$$\lim_{n\to\infty} \int_E f_n \leq \int_E f$$

It remains to prove the other inequality. We will check that any simple function ϕ with $0 \leq \phi \leq f$ we have

$$\int_E \phi \leq \lim_{n\to\infty} \int_E f_n$$

and then appealing to the definition of the integral of a non-negative function.

Fix such a ϕ and take any $\alpha \in (0,1)$. Let

$$\begin{aligned} A_n &= \{x \in E : f_n(x) \geq \alpha\phi(x)\} \\ &= \big((f_n - \alpha\phi)^{-1}[0,\infty]\big) \cap E \end{aligned}$$

and remark it is a combination of measurable sets and thus measurable. Since $f_n \leq f_{n+1}$, we have $A_n \subseteq A_{n+1}, n \geq 1$.

Suppose $\phi(x) \neq 0$, then

$$\alpha\phi(x) < \phi(x) \leq f(x)$$

as $\alpha < 1$. Since $f_n \to f$, we must have $f_n(x) \geq \alpha\phi(x)$ eventually for all $n \geq N_x$.

In any case $x \in \bigcup_{n\geq 1} A_n$. Since all $A_n \subseteq E$ we have

$$E = \bigcup_{n\geq 1} A_n$$

By monotonicity

$$\begin{aligned}
\alpha \int_{A_n} \phi = \int_{A_n} \alpha\phi & \\
\leq \int_{A_n} f_n & \qquad \text{choice of } A_n \\
= \int f_n \chi_{A_n} & \\
\leq \int f_n \chi_E & \qquad \text{monotonicity} \\
= \int_E f_n & \\
\leq \lim_{n\to\infty} \int_E f_n & \qquad \text{monotonicity}
\end{aligned}$$

Appealing to our lemma, we have

$$\alpha \int_E \phi = \alpha \lim_{n\to\infty} \int_{A_n} \phi \leq \lim_{n\to\infty} \int_E f_n$$

But

$$\alpha \int_E f := \sup_{0\leq\phi\leq f} \alpha \int_E \phi$$

thus

$$\alpha \int_E f \leq \lim_{n\to\infty} \int_E f_n$$

for all $\alpha < 1$.

Letting $\alpha \to 1$ completes the proof.

We can actually weaken the conditions to that $f_n \to f$ a.e. and the result still holds.

5.1.1 Consequences

Proposition 5.1.4
If $f, g \geq 0$ are measurable,

$$\int_E f + g = \int_E f + \int_E g$$

Proof
This holds for simple functions by our work prior. We have also seen that there are

positive simple functions $\phi_n \uparrow f, \psi_n \uparrow g$. Hence

$$\phi_n + \psi_n \uparrow f + g$$

Apply MCT to get

$$\begin{aligned}\int_E f + g \\ &= \lim_{n\to\infty} \int \phi_n + \psi_n \\ &= \lim_{n\to\infty} \left(\int_E \phi_n + \int_E \psi_n \right) \\ &= \int_E f + \int_E g\end{aligned}$$

Proposition 5.1.5
If $f \geq 0$ and is measurable, then for any measurable set E

$$\int f = \int_E f + \int_{E^c} f$$

Proof
Use the previous proposition and

$$\int_E f = \int f\chi_E, \int_{E^c} f = \int f\chi_{E^c}$$

Proposition 5.1.6
If f, g are integrable, then $\int f + g = \int f + \int g$.

Proof
First show that if $f_1, f_2 \geq 0$ are integrable, and $f = f_1 - f_2$, then f is integrable with

$$\int f = \int f_1 - \int f_2$$

This allows us to split the complex case into the real case into the non-negative case and apply our results from before.

Example 5.1.7
For $f \geq 0$ measurable

$$\int_{[-n,n]} f \to \int_{\mathbb{R}} f$$

Fatou's Lemma

Lemma 5.1.8 (Fatou)
For $f_n \geq 0$ measurable, we have

$$\int \lim_{n\to\infty} \inf_{k\geq n} f_k \leq \lim_{n\to\infty} \inf_{k\geq n} \int f_k$$

Proof
Let $g_n = \inf_{k\geq n} f_n$. Then g_n is measurable and $g_n \leq f_k$ for all $n \leq k$.

Thus $\int g_n \leq \int f_k$ for all $n \leq k$, implying

$$\int g_n \leq \inf_{k\geq n} \int f_k$$

Moreoever, the sequence $(g_n)_{n\geq 1}$ is increasing in n. This means $\lim_{n\to\infty} g_n$ exists. Similarly, the sequence $\left(\inf_{k\geq n} \int f_k\right)_{n\geq 1}$ is increasing in n and hence has a limit.

$$\lim_{n\to\infty} \int g_n \leq \lim_{n\to\infty} \inf_{k\geq n} \int f_k$$

Put

$$F(x) := \lim_{n\to\infty} \inf_{k\geq n} f_k(x)$$

Since $g_n \uparrow F$, then MCT gives

$$\lim_n \int g_n = \int F = \int \lim_{n\to\infty} \inf_{k\geq n} f_k$$

Example 5.1.9
Suppose $f_n \to f$ pointwise, where f_n are integrable with $\int |f_n| \leq 1$ for all n.

By Fatou's lema,

$$\begin{aligned}\int |f| &= \int \lim_{n\to\infty} \inf_{k\geq n} |f_k| \\ &\leq \lim_{n\to\infty} \inf_{k\geq n} \int |f_n| \\ &\leq 1\end{aligned}$$

so f is integrable.

5.2 Dominated Convergence Theorem

With Fatou's lemma, we can prove another very important convergence theorem.

Theorem 5.2.1 (Dominated Convergence)
Suppose f_n are measurable function with $f_n \to f$. Suppose there is an integrable function g with

$$|f_n(x)| \leq g(x)$$

for all x and n.
Then

$$\int f = \lim_{n\to\infty} \int f_n$$

In fact,

$$\int |f_n - f| \to 0$$

The extra and critical hypothesis is the existence of the single function g which MUST be integrable and dominates all the functions $|f_n|$.

Proof
We will apply Fatou's lemma to the sequence of functions

$$2g - |f - f_n|$$

Since $|f_n| \leq g$ we also have $|f| \leq g$ and thus

$$2g - |f - f_n| \geq 0$$

It is clearly measurable thus

$$
\begin{aligned}
\int 2g &= \int \liminf_n (2g - |f - f_n|) \\
&\leq \liminf_n \int 2g - |f - f_n| \\
&= \limsup_n \int 2g - \int |f - f_n| && \text{linearity, previous corollary} \\
&= \int 2g - \limsup_n \int |f - f_n| && \text{switch inf to sup due to sign}
\end{aligned}
$$

Since g is integrable, $|\int 2g| < \infty$, so we can subtract off $\int 2g$ from both sides to obtain

$$\limsup_n \int |f - f_n| \leq 0$$

But then

$$0 \leq \liminf_n \int |f - f_n| \leq \limsup_n \int |f - f_n| \leq 0$$

and we have equality throughout.

In particular

$$\lim_n \int |f_n - f| = 0$$

Since $\int |f_n(x)| \leq \int g(x) \leq \infty$, each f_n is integrable so $\int f_n$ is well defined. Furthermore, another application of Fatou's lemma shows that

$$
\begin{aligned}
\int |f(x)| &\leq \liminf_n \int |f_n(x)| \\
&\leq \int g(x) \\
&< \infty
\end{aligned}
$$

So also f is integrable and $\int f$ is well defined. Finally, we note that

$$
\begin{aligned}
\left| \int f_n - \int f \right| &= \left| \int f_n - f \right| \\
&\leq \int |f_n - f| \\
&\to 0
\end{aligned}
$$

thus

$$\int f_n \to \int f$$

5.3 Lebesgue & Riemann Integral

We have already observed the equivalence of the Lebesgue and Riemann integrals for step function. We now see it for all cases.

Theorem 5.3.1
If f is Riemann integrable over $[a, b]$, then f is Lebesgue integrable over $[a, b]$ and their Riemann and Lebesgue integrals coincide.

Proof
Recall that a Riemann integrable function is bounded $|f| \leq C$ and the measure of the set of discontinuities is zero. Let E be the set discontinuities so

$$f = f\chi_E + f\chi_{E^c}$$

The function $f\chi_E$ is measurable being equal to 0 a.e. The function is measurable being continuous. Hence f is measurable.

Since

$$\int_{[a,b]} |f| \leq \int_{[a,b]} C \leq C(b-a)$$

f is Lebesgue integrable.

Take any partition of $[a, b]$,

$$P : a = x_0 < x_1 < \cdots < x_n = b$$

and let

$$\begin{aligned} M_i &= \sup\{f(x) : x \in [x_{i-1}, x_i]\} \\ m_i &= \inf\{f(x) : x \in [x_{i-1}, x_i]\} \end{aligned}$$

Notice that the upper/lower Riemann sums are the Lebesgue integrals of related simple functions

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= \int_{[a,b]} \sum_{i=1}^{n} M_i \chi_{[x_{i-1},x_i)} \\ L(f,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \\ &= \int_{[a,b]} \sum_{i=1}^{n} m_i \chi_{[x_{i-1},x_i)} \end{aligned}$$

We have

$$\sum_{i=1}^{n} M_i \chi_{[x_{i-1},x_i)} \geq f \geq \sum_{i=1}^{n} m_i \chi_{[x_{i-1},x_i)}$$

and by the monotonicity of the Lebesgue integral

$$\int_{[a,b]} \sum_{i=1}^{n} M_i \chi_{[x_{i-1},x_i)} \geq \int_{[a,b]} f \geq \int_{[a,b]} \sum_{i=1}^{n} m_i \chi_{[x_{i-1},x_i)}$$

Consequently, for all partitions P we have

$$U(f,P) \geq \int_{[a,b]} f \geq L(f,P)$$

By the definition of the Riemann integral

$$\inf_P U(f,P) = \sup_P L(f,P) = R - \int_a^b f$$

which gives

$$R - \int_a^b f = \int_{[a,b]} f$$

Our brand new shiny Lebesgue integral generalizes the Riemann integral. It follows that any Lebesgue integrable function can be integrated using previous techniques given it is also Riemann integrable. However, the Lebesgue integral can integrate a wider class of functions, as well as having very useful limit theorems. These theorems can be applied to the Riemann integral if all relevant functions are Riemann integrable.

Chapter 6

L^p Spaces

6.1 Definitions

Throughout, f will denote a measurable function defined on a measurable set E. Let $1 \leq p < \infty$ and observe $|f|^p$ is measurable.

Definition 6.1.1
Define

$$\|f\|_{L^p(E)} := \left(\int_E |f|^p\right)^{\frac{1}{p}}$$

If E is clear we will write

$$\|f\|_p$$

We want the above to be a norm. However, if $\|\cdot\|_p$ has any chance of being a norm, we need it to be zero if and only if $f = 0$.

Define an equivalence relation $\sim$ on the set of mesurable functions defined on E

$$f \sim g \iff f - g = 0 \text{ a.e on } E$$

Definition 6.1.2
Put

$$L^P(E) := \{\text{equivalence classes of measurable functions } f : E \to \mathbb{C} : \|f\|_{L^p(E)} < \infty\}$$

Proposition 6.1.1
$L^p(E)$ is a vector space.

Proof
Given $\alpha \in \mathbb{C}, f, g \in L^p(E)$, we know $\alpha f + g$ is measurable with

$$\int |\alpha f + g|^p \leq \int (|\alpha f| + |g|)^p \leq 2^{p-1} \int |\alpha|^p \cdot |f|^p + |g|^p < \infty$$

by the convexity of $y = x^p$.

We will show that $\|\cdot\|_p$ is a norm on $L^p(E)$ but the triangle inequality is quite involved.

Definition 6.1.3 (Essential Supremum)
Write

$$\|f\|_{L^\infty(E)} := \inf_{A \in \mathcal{M}: m(E \setminus A) = 0} \{\sup|f(x)| : x \in A\}$$

Notice that if $E = \mathbb{R}$

$$\|f\|_{L^\infty(\mathbb{R})} = \inf_{m(A^c)=0} \{\sup|f(x)| : x \in A\}$$

Observe that

$$\|f\|_\infty \leq \sup_{x \in E} |f(x)|$$

but the inequality can be strict.

The idea is to "forgive" what happens on a set of measure 0.

Proposition 6.1.2
If $f = 0$ a.e. then

$$\|f\|_\infty = 0$$

Proposition 6.1.3
If $f = g$ a.e. then

$$\|f\|_\infty = \|g\|_\infty$$

Proof
Let $A := \{x : f(x) = g(x)\}$ so that $m(E \setminus A) = 0$.

Pick B arbitrary such that $m(E \setminus B) = 0$. But then

$$\begin{aligned} m(E \setminus (A \cap B)) &= m(E \cap (A \cap B)^c) \\ &= m(E \cap (A^c \cup B^c)) \\ &= m((E \cap A^c) \cup (E \cap B^c)) \\ &= 0 \end{aligned}$$

Since $A \cap B \subseteq B$

$$\begin{aligned} \sup_{x \in A \cap B} f(x) &\leq \sup_{x \in B} f(x) \\ \inf_{B \in \mathcal{M}: m(E \setminus B) = 0} \sup_{x \in A \cap B} f(x) &\leq \inf_{B \in \mathcal{M}: m(E \setminus B) = 0} \sup_{x \in B} f(x) \\ &=: \|f\|_\infty \end{aligned}$$

But since $A \cap B \in \mathcal{M}$ and $m(E \setminus (A \cap B)) = 0$, we are taking the infimum over a bigger set in the essential norm and

$$\|f\|_\infty \leq \inf_{B \in \mathcal{M}: m(E \setminus B) = 0} \sup_{x \in A \cap B} f(x)$$

thus we have equality.

The same logic applies to g so

$$\begin{aligned} \|f\|_\infty &:= \inf_{B \in \mathcal{M}: m(E \setminus B) = 0} \sup_{x \in A \cap B} f(x) \\ &= \inf_{B \in \mathcal{M}: m(E \setminus B) = 0} \sup_{x \in A \cap B} g(x) \qquad\qquad f - g\Big|_A = 0 \\ &=: \|g\| \end{aligned}$$

as desired.

Proposition 6.1.4
If f is continuous on $\mathbb{R}$, then

$$\|f\|_\infty = \sup_x |f(x)|$$

Proposition 6.1.5
We have

$$\|f\|_\infty = \inf\{\alpha \in \mathbb{R} : m|f|^{-1}(\alpha, \infty] = 0\}$$

Proof
The statement is trivial if $mE = 0$. We proceed assuming $mE > 0$.

($\geq$) Suppose that $m(E \setminus A) = 0$ and $\sup_{x \in A} |f(x)| = \alpha$. Then

$$B_\alpha := f^{-1}(\alpha, \infty] \subseteq E \setminus A$$

so $mB_\alpha = 0$.

It follows that

$$\inf\{\alpha \in \mathbb{R} : m|f|^{-1}(\alpha, \infty] = 0\} \leq \inf_{A \in \mathcal{M} : m(E \setminus A) = 0} \sup_{x \in A} |f(x)| =: \|f\|_\infty$$

($\leq$) Suppose now that for $C_\alpha := |f|^{-1}[-\infty, \alpha]$ we have $m(E \setminus C_\alpha) = 0$.

Then

$$\sup_{x \in C_\alpha} |f(x)| \leq \alpha$$

It follows that

$$\|f\|_\infty := \inf_{A \in \mathcal{M} : m(E \setminus A) = 0} \sup_{x \in A} |f(x)| \leq \inf\{\alpha \in \mathbb{R} : m|f|^{-1}(\alpha, \infty] = 0\}$$

Having shown both inequalities, we conclude equality.

Definition 6.1.4
Define

$$L^\infty(E) := \{\text{equivalence classes of measurable } f : E \to \mathbb{C} : \|f\|_\infty < \infty\}$$

Definition 6.1.5 (Essentially Bounded)
$f : E \to \mathbb{C}$ where it is bounded on all except a set of measure 0.

Proposition 6.1.6
$\|f\|_\infty = 0$ implies $f = 0$ a.e.

Proof
By the previous proposition.

Proposition 6.1.7
If $mE < \infty$ and $f \in L^\infty(E)$ then

$$\|f\|_{L^p(E)} \to \|f\|_{L^\infty(E)}$$

as $p \to \infty$.

6.2 Triangle Inequality

Our goal is to prove that $\|\cdot\|_p$ for $1 \leq p \leq \infty$ satisfies the triangle inequality. Combined with the positive-definite property given by the equivalence relation and the easy to see absolute homogeneity, it would show $L^p(E)$ is a normed vector space.

6.2.1 Hölder's Inequality

Definition 6.2.1 (Conjugate Indices)
$1 \leq p, q \leq \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 0$$

We understand that $\frac{1}{\infty} = 0$.

Theorem 6.2.1 (Hölder's Inequality)
Suppose p, q are conjugate indices. If f, g are measurable functions

$$\int |fg| \leq \|f\|_p \|g\|_q = \left(\int |f|^p\right)^{\frac{1}{p}} \left(\int |g|^q\right)^{\frac{1}{q}}$$

In particular, if $f \in L^p$ and $g \in L^q$ then

$$fg \in L^1, \|fg\|_1 \leq \|f\|_p \|g\|_q$$

Observe that the case $p = q = 2$, Holder's inequality gives the Cauchy Schwartz inequality.

Proof
Case I: $p = 1, q = \infty$ Let $A := \{x : |g(x)| > \|g\|_\infty\}$. Then $mA = 0$ and

$$\begin{aligned}
\int |fg| &= \int_A |fg| + \int_{A^c} |fg| \\
&= \int_{A^c} |fg| \\
&\leq \|g\|_\infty \int_{A^c} |f| \\
&\leq \|g\|_\infty \|f\|_1
\end{aligned}$$

Case I: $1 < p, q < \infty$ We first show that for all $a, b \geq 0$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Observe the statement trivially holds if either $a = 0$ or $b = 0$. So assume otherwise.

To see this, choose $s, t \in \mathbb{R}$ such that

$$a = e^{\frac{s}{p}}, b = e^{\frac{t}{q}}$$

The convexity of the exponential gives

$$\begin{aligned}
ab &= \exp\left(\frac{s}{p}\right) \exp\left(\frac{t}{q}\right) \\
&= \exp\left(\frac{1}{p}s + \frac{1}{q}t\right) \\
&\leq \frac{1}{p}e^s + \frac{1}{q}e^t \\
&= \frac{1}{p}a^p + \frac{1}{q}b^q
\end{aligned}$$

Now, if $\|f\|_p = 0$ or $\|g\|_q = 0$, then $fg = 0$ a.e. and the result is clear.

Similarly, if $\|f\|_p = \infty$ or $\|g\|_q = \infty$, the result is clear.

Assume otherwise.

We will apply our previous result with $a = \frac{|f(x)|}{\|f\|_p}$ and $b = \frac{|g(s)|}{\|g\|_q}$. It follows that

$$\frac{|f||g|}{\|f\|_p \|g\|_q} \leq \frac{1}{p}\left(\frac{|f|}{\|f\|_p}\right)^p + \frac{1}{q}\left(\frac{|g|}{\|g\|_q}\right)^q$$

Since $\|f\|_p, \|g\|_q$ are constants

$$\int \left(\frac{|f|}{\|f\|_p}\right)^p = \int \left(\frac{|g|}{\|g\|_q}\right)^q = 1$$

But then

$$\int \frac{|f||g|}{\|f\|_p\|g\|_q} \leq \frac{1}{p} + \frac{1}{q} = 1$$

which gives the result.

Of course, if $f \in L^p, g \in L^q$ the integral above is finite and thus

$$fg \in L^1$$

as claimed.

Proposition 6.2.2
Pick $p > 1$.

(a) $\|f\|_{L^p[0,1]} \geq \|f\|_{L^1[0,1]}$

(b) $L^p[0,1] \subset L^1[0,1]$

(c) replace $[0,1]$ with $[a,b]$

Theorem 6.2.3 (Riesz Representation)
Let $1 \leq p < \infty$ and suppose q is the conjugate index. For any $f \in L^p$

$$\|f\|_p = \sup\left\{\left|\int fg\right| : \|f\|_q \leq 1\right\}$$

Proof
PMath 451.

6.2.2 Minkowski's Inequality

This is the triangle inequality for L^p spaces and will allow us to show that $\|\cdot\|_p$ is indeed a norm.

Theorem 6.2.4 (Minkowski's Inequality)
For f, g measurable

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

for all $1 \le p \le \infty$.

Proof
Case I: $p = \infty$ Triangle inequality for the absolute value and the definition.

Case II: $p = 1$ Triangle inequality for the absolute value.

Case III: $1 < p < \infty$ Without loss of generality, $f, g \in L^p$ or else the both sides of the inequality is ∞ and the statement is trivial. Similarly, let us assume $\|f+g\|_p > 0$ or the statement always holds.

Let q be the conjugate index of p and observe that

$$\begin{aligned} 1 + \frac{p}{q} &= p \\ p + q &= pq \\ p &= q(p-1) \end{aligned}$$

Now, remark that

$$\begin{aligned} (\|f+g\|_p)^p &= \int |f+g|^p \\ &= \int |f+g|^{p-1}|f+g| \\ &\le \int |f+g|^{p-1}(|f|+|g|) \end{aligned}$$

By Hölder's inequality

$$\int |f+g|^{p-1}|f| \le \||f+g|^{p-1}\|_q \|f\|_p$$

Moreoever

$$\begin{aligned} \left(\||f+g|^{p-1}\|_q\right)^q &= \int |f+g|^{(p-1)q} && \text{definition} \\ &= \int |f+g|^p && (p-1)q = p \\ &= (\|f+g\|_p)^p \\ &< \infty && L^p \text{ is a vector space} \end{aligned}$$

Taking q-th root on both sides gives

$$\||f+g|^{p-1}\|_q \leq (\|f+g\|_p)^{\frac{p}{q}}$$

Substituting back into our result from Hölder's inequality, we get

$$\int |f+g|^{p-1}|f| \leq \|f+g\|_p^{\frac{p}{q}} \|f\|_p$$

and similarly

$$\int |f+g|^{p-1}|g| \leq \|f+g\|_p^{\frac{p}{q}} \|g\|_p$$

thus

$$\begin{aligned}
\|f+g\|_p^p &\leq \|f+g\|_p^{\frac{p}{q}} (\|f\|_p + \|g\|_p) \\
\|f+g\|_p^{p-\frac{p}{q}} &\leq \|f\|_p + \|g\|_p \\
\|f+g\|_p &\leq \|f\|_p + \|g\|_p && p - \frac{p}{q} = 1
\end{aligned}$$

as required.

Corollary 6.2.4.1
For $1 \leq p \leq \infty$, L^p is a normed linear space and hence a metric space.

Proof
We have already seen that L^p is a linear (vector) space.

Minkowski's inequality shows that $\|\cdot\|_p$ satisfies the triangle inequality. By taking equivalence classes of function equal a.e. we obtain the absolute homogeneity. The scalar multiplication property of norms is easily seen.

L^p is a metric space since every norm gives rise to a metric

$$d(f,g) = \|f-g\|_p$$

where f, g are any representative of their respective equivalence classes.

6.3 Completeness

Definition 6.3.1 (Banach Space)
A NLS which is complete with respect to the induced metric.

6.3.1 Riesz-Fisher Theorem

Definition 6.3.2 (Uniformly Cauchy)
A sequence of functions $f_n : S \to (X, d)$ from the set S to a metric space is uniformly Cauchy if for all $\epsilon > 0$ there is $N_0 \in \mathbb{N}$ where

$$d(f_n(x), f_m(x)) < \epsilon$$

for all $x \in S$ and $n, m \geq N$.

Proposition 6.3.1
A uniformly Cauchy sequence converges uniformly to a function $S \to (X, d)$.

Theorem 6.3.2 (Riesz-Fisher)
L^p is Banach space for all $1 \leq p \leq \infty$.

Proof
Case I: $p = \infty$ For this choice of p, the proof relies upon properties of uniformly Cauchy sequences.

Let (f_n) be a Cauchy sequence in L^∞ and for $k, n, m \in \mathbb{N}$ set

$$A_k := \{x : |f_k(x)| > \|f_k\|_\infty\}$$
$$B_{n,m} := \{x : |f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty\}$$

and observe they are all sets of measure zero. Let E be their countable union, which again has measure 0.

Thus the sequence (f_n) is uniformly Cauchy on E^c.

$$\sup_{x \in E^c} |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \to 0$$

as $n, m \to \infty$.

We know uniformly sequences of real or complex-valued functions always converge uniformly by the completeness of $\mathbb{C}, \mathbb{R}$. So there is some f defined on E^c such that

$$\sup_{x \in E^c} |f_n(x) - f(x)| \to 0$$

Define $f(x) = 0$ on E so that

$$\|f_n - f\|_\infty \leq \sup_{x \in E^c} |f_n(x) - f(x)| \to 0$$

as $mE = 0$.

This shows $f_n \to f$ under the norm of L^∞. To see that $f \in L^\infty$, pick N such that

$$\sup_{x \in E^c} |f_N(x) - f(x)| \leq 1$$

Since $A_N \subseteq E$, the usual triangle inequality gives

$$\begin{aligned} \sup_{x \in E^c} |f(x)| &\leq \sup_{x \in E^c} (|f_N(x) - f(x)| + |f_N(x)|) \\ &\leq 1 + \|f_N\|_\infty \end{aligned}$$

and as $mE = 0$, this suffices to show that

$$\|f\|_\infty \leq 1 + \|f_N\|_\infty < \infty$$

which gives $f \in L^\infty$.

Case I: $p < \infty$ Let (f_n) be a Cauchy sequence in L^p, meaning that for every $\epsilon > 0$ there is some N such that if $n, m \geq N$

$$\|f_n - f_m\| < \epsilon$$

We will a subsequence (f_{n_k}) converges, which suffices to show the claim as the original Cauchy sequence necessarily converges to the same limit.

Choose a subsequence (f_{n_i}) such that

$$\|f_{n_{i+1}} - f_{n_i}\| < 2^{-i}$$

and set

$$\begin{aligned} g_k &:= \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}| \\ g &:= \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}| \end{aligned}$$

The triangle inequality gives

$$\begin{aligned} \|g_k\|_p &\leq \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \\ &\leq \sum_{i=1}^{k} 2^i \\ &\leq 1 \end{aligned}$$

for all k.

Now, the sequence of non-negative functions g_k increases to g. Thus for each p, $g_k^p \uparrow g^p$. The monotone convergence theorem hence implies

$$\begin{aligned} \|g\|_p^p &= \int g^p \\ &= \lim_k \int g_k^p \\ &= \lim_k \|g_k\|_p^p \\ &\leq 1 \end{aligned}$$

and $g \in L^p$.

Since $\|g\|_p < \infty$, $g(x) < \infty$ for almost every x, say for all except $x \in A$ where $mA = 0$. Thus for all $x \in A^c$

$$\sum_{i=1}^{\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

is absolutely convergent and thus $\sum_{i=1}^{\infty} f_{n_{i+1}(x)} - f_{n_i}(x)$ converges.

Define the function f on A^c as the pointwise limit

$$\begin{aligned} f(x) &:= \lim_k \left(f_{n_1}(x) + \sum_{i=1}^{k} (f_{n_{i+1}}(x) - f_{n_i}(x)) \right) \\ &= \lim_k f_{n_k}(x) \end{aligned}$$

For $x \in A$, we simply define $f(x) = 0$. The function f is measurable since $f\big|_{A^c}$ is measurable being the pointwise limit of measurable functions, and $f\big|_A$ is measurable since this function equals 0 a.e.

Moreoever

$$\begin{aligned} |f(x)| &\leq |f_{n_1}(x)| + \sum_{i=1}^{\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)| \\ &= |f_{n-1}(x)| + g(x) \end{aligned}$$

and as both f_{n_1} and g belong to L^p, so does f.

It remains to show that $\|f_{n_k} - f\|_p \to 0$. We have already seen that this is the case for all $x \in A^c$. This means $|f_{n_k} - f| \to 0$ a.e. Also on A^c we have

$$|f_{n_k} - f| \leq \sum_{i=k}^{\infty} |f_{n_{i+1}} - f_{n_i}| \leq g$$

so $|f_{n_k} - f|^p \leq g^p \in L^1(A^c)$ since $g \in L^p$.

Appealing to the dominated convergence theorem, we have

$$\begin{aligned}\lim_k \int_{A^c} |f_{n_k} - f|^p &= \int_{A^c} \lim_k |f_{n_k} - f|^p \\ &= 0\end{aligned}$$

Since $mA = 0$, we deduce that $\|f_{n_k} - f\|_p \to 0$ as desired.

As any convergent sequence is Cauchy, we get the following corollary.

Corollary 6.3.2.1
If $f_n \to f$ in L^p for $1 \leq p < \infty$, there is a subsequence f_{n_k} that converges to f pointwise almost everywhere.

Proof
The proof of the theorem shows that there is a subseqeuence $f_{n_k} \to f'$ pointwise a.e. and $\|f_{n_k} - f'\|_p \to 0$.

But

$$\lim_k \|f_{n_k} - f\|_p = \lim_n \|f_n - f\|_p = 0 = \lim_k \|f_{n_k} - f'\|_p$$

so by the uniqueness of limits

$$f = f' = \lim_k f_{n_k}$$

almost everywhere.

Chapter 7

Lusin's Theorem & Fubini's Theorem

7.1 Lusin's Theorem

We saw that $L^\infty[a,b] \subseteq L^p[a,b]$ for all $p < \infty$. We want to explore the relation of continuous functions within L^p.

Specifically, we wish to formalize the second of Littlewood's 3 principles.

1. Every measurable set is "nearly" a finite union of intervals
2. Every measurable function is "nearly" continuous
3. Every pointwise convergence sequence of measurable functions is "nearly" uniformly convergent.

7.1.1 Egoroff's Theorem

This is a formal version of the third principle.

Theorem 7.1.1 (Egoroff)
Assume $mE < \infty$. Let (f_n) be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f.
For each $\epsilon > 0$, there is a closed set $F \subseteq E$ for which

$$f_n \to f$$

uniformly on F and $m(E \setminus F) < \epsilon$.

7.1.2 Lusin's Theorem

Lemma 7.1.2
$C[a,b]$ is not dense in $L^\infty[a,b]$.

Proof
Consider the function

$$f(x) := \begin{cases} -1, & x \in \left[a, \frac{a+b}{2}\right] \\ 1, & x \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

We claim that if $g = f$ a.e., then g is not continuous. To see this, suppose the opposite holds. Let E be the set upon which f, g do not disagree. If $E = \varnothing$ there is nothing to prove, so assume otherwise.

Write $x_0 := \frac{a+b}{2}$. For all $k \geq 1$ both

$$\left(x_0 - \frac{1}{k}, x_0\right), \left(x_0, x_0 + \frac{1}{k}\right) \not\subseteq E$$

so there is some $x_k \in \left(x_0 - \frac{1}{k}, x_0\right), z_k \in \left(x_0, x_0 + \frac{1}{k}\right)$ such that

$$g(x_k) = f(x_k) = -1, g(z_k) = f(z_k) = 1$$

Clearly $x_k, z_k \to x_0$ so by sequential continuity

$$-1 = \lim g(x_k) = g(\lim x_k), 1 = \lim g(z_k) = g(\lim z_k)$$

which is a contradiction as $g(\lim z_k) = g(x_0) = 1 \neq -1$.

Thus the equivalence class in L^∞ which contains f does not contain ANY continuous functions!

Suppose there is some sequence $f_n \to f$ where $f_n \in C[a,b]$. Then f_n is cauchy with respect to the essential supremum, which by continuity means they are cauchy in the supremum norm.

But $C[a,b]$ under the supremum norm is continuous, which means f_n converges to a continuous function g. Since the supremum and essential supremum norm coincide on continuous functions

$$\|f_n - g\|_{L^\infty[a,b]} = \sup_{x \in [a,b]} |f_n(x) - g(x)| \to 0$$

So $f_n \to g$ under the supremum norm as well.

But limits in metric spaces are unique! Hence $f = g$ a.e. This is a clear contradiction by our initial remark.

Proposition 7.1.3
$C[a, b]$ is a closed subset of $L^\infty[a, b]$.

Proof
All convergent sequence converge to continuous functions.

The following is a formal version of Littlewood's first principle.

Lemma 7.1.4
Suppose $mE < \infty$. Then for every $\delta > 0$, there is a finite union of open intervals U such that

$$m(U \setminus E) + m(E \setminus U) < \delta$$

Proof
Choose $\{I_n\}_{n\geq 1}$ open intervals whose union contains E and for which

$$\sum_n \ell(I_n) < mE + \frac{\delta}{2} < \infty$$

Thus

$$m\left(\bigcup I_n \setminus E\right) < \frac{\delta}{2}$$

Choose N such that

$$\sum_{n=N+1}^{\infty} \ell(I_n) < \frac{\delta}{2}$$

and put

$$U := \bigcup_{n=1}^{N} I_n$$

It is clear that

$$m(U \setminus E) \leq m\left(\bigcup I_n \setminus E\right) < \frac{\delta}{2}$$

But

$$E \setminus U \subseteq \bigcup_{n\geq N+1} I_n$$

so

$$m(E \setminus U) \leq \frac{\delta}{2}$$

as well.

This suffices to prove the claim.

Lemma 7.1.5
Let $f : [a, b] \to \mathbb{R}$ be measurable. Fix $\epsilon > 0$.
There is a continuous functions h such that

$$m\{x \in [a, b] : |f(x) - h(x)| \geq \epsilon\} < \epsilon$$

Furthermore, if $m \leq f \leq M$, then we can choose h with $m \leq h \leq M$.

Proof
Step I: We argue there is N such that

$$m\{x \in [a, b] : |f(x)| \geq N\} < \frac{\epsilon}{3}$$

Indeed, for each $n \in \mathbb{N}$, let

$$A_n := \{x \in [a, b] : |f(x)| \geq n\}$$

The sets A_n are decreasing and

$$mA_1 \leq b - a < \infty$$

Thus by the downward continuity of measure

$$0 = m\varnothing = m\left(\bigcap_n A_n\right) = \lim_n mA_n$$

Step II: Now we show that for $\epsilon > 0$ and $N \in \mathbb{N}$, there is a simple function ϕ such that

$$|f(x) - \phi(x)| < \epsilon$$

assuming $|f(x)| < N$.

Furthermore, if $m \leq f \leq M$, then we can choose ϕ with

$$m \leq \phi \leq M$$

To see this first pick $k \in \mathbb{N}$ with $\frac{1}{k} < \epsilon$. Partition $[-N, N)$ into subintervals of width $\frac{1}{k}$. Call these

$$I_j := [a_j, b_j)$$

for $j = 1, 2, \ldots, 2Nk$.

Let

$$E_j := f^{-1}(I_j) \in \mathcal{M}$$

and define $\phi(x) = a_j$ for $a \in E_j$. Then

$$|f(x) - \phi(x)| \leq \frac{1}{k} < \epsilon$$

To see the second claim holds, we need only partition $[m, M)$ instead of $[-N, N)$.

Step III: We claim that given any simple function ϕ defined on $[a, b]$, there is a step function g on $[a, b]$ such that

$$g(x) = \phi(x)$$

except on a set of measure at most $\frac{\epsilon}{3}$. Again, if $m \leq \phi \leq M$, we can choose g such that $m \leq g \leq M$.

Assume that ϕ has the standard representation

$$\phi = \sum_{i=1}^{K} a_k \chi_{E_k}$$

where $E_k \subseteq [a, b]$ are measurable sets. For each k, appeal to our previous lemma with E_k and $\delta = \frac{\epsilon}{k}$, to obtain sets U_k where each of which is a finite union of open intervals.

Put

$$g := \sum_{k=1}^{K} a_k \chi_{U_k}$$

As the U_k's are finite unions of intervals, g is a step function. By construction $\phi = g$ except on

$$\bigcup_{i=1}^{K} (U_k \setminus E_k) \cup (E_k \setminus U_k)$$

This shows the first claim. To see the second, notice the only issue may arise on sets $U_k \setminus E_k$. Express

$$g = \sum_{\ell=1}^{L} \alpha_\ell \chi_{V_\ell}$$

in standard representation.

By construction α_ℓ can only exceed M or fail to exceed m if

$$V_\ell \subseteq \bigcup_{k=1}^{n} U_k \setminus E_k$$

But that set has measure at most $\frac{\epsilon}{3}$. So we can take

$$g' := \sum_{\ell}^{L} \min(\max(\alpha_\ell, m), M)\chi_{V_\ell}$$

without violating the result of the first statement while satisfying the second statement.

Step IV: Now we show that given any step function g and $\epsilon > 0$, there is a continuous function h such that $g = h$ except on a set of measure at most $\frac{\epsilon}{3}$. Furthermore, if $m \leq g \leq M$, we can choose $m \leq h \leq M$.

To see this express

$$g = \sum_{k=1}^{K} a_k \chi_{I_k}$$

where I_k's are consecutive intervals which partition $[a, b]$.

Suppose I_k has endpoints ℓ_k, r_k, we ignore the endpoints purposely here.

Set $\delta = \epsilon 6K$. Consider the function h. For each $1 < k < K$ let

$$h\Big|_{(\ell_k+\delta, r_k-\delta)} = a_k$$

The exception is $k = 1, K$ where we apply the same construction except on

$$[\ell_1, \ell_1 - \delta), (\ell_K + \delta, \ell_K]$$

Then on $(r_k - \delta, \ell_{k+1} + \delta)$, let h be linear.

Clearly h is continuous. Moreoever, except on at most $2K$ sets each with measure $\delta = \frac{\epsilon}{6K}$, $h = g$.

This shows the claim.

Step V: Now we bring everything together.

Pick N as in step I. Then choose ϕ, g, h.

By construction $g = h$ exception on the union of the sets

$$\{x : \phi(x) \neq g(x)\}, \{x : g(x) \neq h(x)\}, \{x : |f(x)| \geq N\}$$

where union of these sets have measure at most ϵ.

Theorem 7.1.6 (Lusin)
The continuous functions are dense in $L^p[a,b]$ for $1 \leq p < \infty$, but NOT dense in $L^\infty[a,b]$.

Proof
By our previous lemma, we need only show the density of $C[a,b]$ in $L^p[a,b]$ for $p < \infty$.

Clearly $|f| < \infty$ except on a set of measure zero. Thus by choosing a member of the same equivalence class, we can assume f is complex-valued.

It suffices to approximate each of $\operatorname{Re} f, \operatorname{Im} f$, so without loss of generality, we assum $f : [a,b] \to \mathbb{R}$.

Case I: f is bounded Say $|f| \leq N$.

Pick $\epsilon > 0$ and put

$$\delta := \min\left(\frac{\epsilon}{(2N)^p}, \frac{\epsilon}{b-a}, 1\right)$$

Obtain a continuous function h as in our previous lemma with δ as the approximation error. Let

$$A := \{x : |f-h| \geq \delta\}$$

so that

$$mA < \delta. -N \leq h \leq N$$

Now consider $\|f-h\|_p$. We will write $A^c = [a,b] \setminus A$.

$$\begin{aligned}
\int_{[a,b]} |f-h|^p &= \int_A |f-h|^p + \int_{A^c} |f-h|^p \\
&\leq \int_A (|f| - |h|)^p + \delta^p \int_{A^c} 1 \\
&\leq (2N)^p mA + \delta^p (b-a) \\
&\leq \delta((2N)^p + b - a) \\
&\leq 2\epsilon
\end{aligned}$$

Case II: f is unbounded Put

$$f_n(x) := \begin{cases} f(x), & |f(x)| \leq n \\ 0, & \text{else} \end{cases}$$

Observe that $f_n \to f$ a.e. and

$$|f_n - f|^p \leq |f|^p \in L^1$$

Hence by the dominated convergence theorem

$$\int |f_n - f|^p \to 0$$

Choose N such that $\|f_N - f\|_p < \epsilon$. As $f_N \in L^p$ is bounded, the previous paragraphs show there is a continuous function h such that

$$\|f_N - h\|_p < 2\epsilon$$

Minkowski's inequality gives

$$\|f - h\|_p \leq \|f - f_N\|_p + \|f_N - h\|_p < 2\epsilon$$

Corollary 7.1.6.1
The polynomials are dense in $L^p[0,1]$ for $1 \leq p < \infty$.

Proof
Approximate $f \in L^p[0,1]$ with a continuous functions g. Then apply the Stone-Weirstrauss theorem to obtain a polynomial which approximates g.

The supremum norm upper bounds the essential supremum upper bounds the $L^p[0,1]$ norm. An essential ingredient is that $m[0,1] = 1$.

Proposition 7.1.7
$L^p[0,1]$ is separable for $1 \leq p < \infty$, but not $L^\infty[0,1]$.

7.2 Fubini's Theorem

Definition 7.2.1 (Borel Set)
A Borel set in $\mathbb{R}^2$ is any set in the σ-algebra of subset of $\mathbb{R}^2$ generated by the open sets in $\mathbb{R}^2$.

Theorem 7.2.1 (Fubini)
Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and

$$f^{-1}(U)$$

is a Borel set in $\mathbb{R}^2$ for all open sets $U \subseteq \mathbb{R}$.
Assume that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x,y)| dx \right) dy < \infty$$

Then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) dx \right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y) dy \right) dx$$

Part III

Fourier Analysis

Chapter 8

Hilbert Spaces

8.1 Inner Products

Definition 8.1.1 (Inner Product)
Let V be a (complex) vector space. An inner product on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that

(i) $\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$

(ii) $\langle f + g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$

(iii) $\langle f, g \rangle = \overline{\langle g, f \rangle}$

(iv) $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0 \iff f = 0$

Proposition 8.1.1
(i), (ii), (iii) implies

$$\langle f, \beta g \rangle = \bar{\beta} \langle f, g \rangle$$

and

$$\langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$$

Example 8.1.2
The most important example for this course is

$$L^2[0, 1]$$

with

$$\langle f, g \rangle = \int_0^1 f \bar{g}$$

Theorem 8.1.3 (Cauchy-Schwartz Inequality)
For all $f, g \in V$

$$|\langle f, g\rangle| \leq \|f\|\|g\|$$

Proof
If $g = 0$ then the result trivially holds. Suppose now that $g \neq 0$ and let

$$\lambda := \frac{\langle f, g\rangle}{\|g\|^2}$$

We have

$$\begin{aligned}
0 &\leq \|f - \lambda g\|^2 \\
&= \langle f - \lambda g, f - \lambda g\rangle \\
&= \langle f, f\rangle - \lambda\langle g, f\rangle - \bar{\lambda}\langle f, g\rangle + \lambda\bar{\lambda}\langle g, g\rangle \\
&= \|f\|^2 - \lambda\overline{\langle f, g\rangle} - \bar{\lambda}\langle f, g\rangle + \lambda\bar{\lambda}\|g\|^2 \\
&= \|f\|^2 - \frac{|\langle f, g\rangle|^2}{\|g\|^2} - \frac{|\langle f, g\rangle|^2}{\|g\|^2} + \frac{|\langle f, g\rangle|^2}{\|g\|^2} \\
&= \|f\|^2 - \frac{\langle f, g\rangle^2}{\|g\|^2}
\end{aligned}$$

Proposition 8.1.4
Every inner product gives rise to a norm

$$\|f\| = \sqrt{\langle f, f\rangle}$$

Proof
Positive-definiteness and absolute scalability comes from the inner product.

The triangle inequality follows from the Cauchy Schwartz inequality

$$\begin{aligned}
\|f + g\|^2 &= \langle f + g, f + g\rangle \\
&= \langle f, g\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle \\
&= \|f\|^2 + 2\,\mathrm{Re}\langle f, g\rangle + \|g\|^2 \\
&\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 && \text{Cauchy-Schwartz} \\
&= (\|f\| + \|g\|)^2
\end{aligned}$$

Definition 8.1.2 (Hilbert Space)
A Hilbert space is Banach space where the norm comes from an inner product.

Lemma 8.1.5
Fix $z \in \mathcal{H}$. Then map $T : \mathcal{H} \to \mathbb{C}$ given by

$$T(y) = \langle y, z \rangle$$

is a continuous linear map.

Proof
Linearity comes from the inner product. To see continuity, let $y_n \to y$ in $\mathcal{H}$.

$$\begin{aligned} |T(y_n) - T(y)| &= |\langle y_n - y, z \rangle| \\ &\leq \|y_n - y\| \|z\| \\ &\to 0 \end{aligned}$$

8.2 Orthogonality

An important property of Hilber spaces is that they have a "geometry" in the same spirit as Euclidean geometry. In particular, there is a sense of orthogonality.

Definition 8.2.1 (Orthogonal Vectors)
We say $x, y \in \mathcal{H}$ are orthogonal if

$$\langle x, y \rangle = 0$$

We write

$$x \perp y$$

to indicate orthogonality.

Proposition 8.2.1
In any Hilbert space

$$x \perp x \iff x = 0$$

Definition 8.2.2 (Orthogonal Set)
We say $S \subseteq \mathcal{H}$ is orthogonal if

$$\forall x \neq y \in S, x \perp y$$

Definition 8.2.3 (Orthonormal Set)
We say $S \subseteq \mathcal{H}$ is orthonormal if it is orthogonal and in addition

$$\|x\| = 1$$

for all $x \in S$.

Example 8.2.2
Suppose $\mathcal{H} = \ell^2$

$$\{e_n : n \geq 1\}$$

is an orthonormal set.

They are referred to as the standard coordinate vectors for ℓ^2.

Theorem 8.2.3 (Pythagorean)
If $\{x_n\}_{n=1}^N$ is orthogonal

$$\left\| \sum_{n=1}^N x_n \right\|^2 = \sum_{n=1}^N \|x_n\|^2$$

Proof
Inspect the proposition which constructed a norm from the inner product. In particular, the term $2\operatorname{Re}\langle x_i, x_j \rangle$ is 0 for all $i \neq j$. The rest follows by induction.

Corollary 8.2.3.1
Orthogonal sets are linearly independent.

Proof
If $\{x_n\}_{n=1}^N$ are orthogonal and non-zero vectors with

$$\sum_{i=1}^N \alpha_i x_i = 0$$

Then

$$0 = \left\| \sum_{n=1}^{N} \alpha_i x_i \right\|^2 = \sum_{i=1}^{N} |\alpha_i|^2 \|x_i\|^2$$

But the non-negative sum is zero if and only if all α_i are 0.

So $\{x_n\}_{n=1}^N$ is a linearly independent set!

8.2.1 Bessel's Inequality

Here are some important facts about orthogonal sets.

Proposition 8.2.4
Let $\{x_k\}_{k\geq 1} \subseteq \mathcal{H}$ be orthogonal vectors. Let S_N be the partial sum up to N. Then $(S_N)_{N\geq 1}$ converges if and only if

$$\sum_{k\geq 1} \|x_k\|^2 < \infty$$

Proof
Let $N > M$ and consider

$$S_N - S_M = \sum_{k=M+1}^{N} x_k$$

The Pythagorean theorem implies

$$\|S_N - S_M\|^2 = \left\| \sum_{k=M+1}^{N} x_k \right\|^2 = \sum_{k=M+1}^{N} \|x_k\|^2$$

Thus S_N is Cauchy if and only if the partial sum of norms is Cauchy.

Proposition 8.2.5
Let $\{e_k\}_{k\geq 1} \subseteq \mathcal{H}$ be an orthonormal set and let $\beta_k \in \mathbb{C}$ with

$$\sum_{k\geq 1} |\beta_k|^2 < \infty$$

Then there is some $x \in \mathcal{H}$ such that

$$\langle x, e_k \rangle = \beta_k$$

for all k and

$$\|x\| = \|(\beta_k)_{k\geq 1}\|_{\ell^2}$$

Proof
By orthonormality, we have

$$\sum_{k\geq 1} \|\beta_k e_k\|^2 = \sum_{k\geq 1} |\beta_k|^2 < \infty$$

and hence by the previous proposition

$$\sum_{k\geq 1} \beta_k e_k \to x \in \mathcal{H}$$

Thus

$$\begin{aligned} \|x\|^2 &= \lim_N \left\| \sum_{k=1}^N \beta_k e_k \right\|^2 \\ &= \sum_{k=1}^\infty |\beta_k|^2 \\ &= \|(\beta_k)_{k\geq 1}\|_{\ell^2}^2 \end{aligned}$$

It remains to check that $\langle x, e_k \rangle = \beta_k$. For this define

$$x_N := \sum_{k=1}^N \beta_k e_k$$

The orthonormality of the vectors $\{e_k\}$ means that

$$\beta_j = \langle x_N, e_j \rangle$$

whenever $N \geq j$. Thus this in particular holds under the limit by the continuity of the map $\langle \cdot, e_j \rangle$.

Theorem 8.2.6 (Bessel's Inequality)
If $\{e_k\}_{k\geq 1}$ is an orthonormal set and $x \in \mathcal{H}$, then

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$$

In particular. the sequence

$$(\langle x, e_k \rangle)_{k\geq 1} \in \ell^2$$

Proof
Define

$$x_N := \sum_{j=1}^{N} \langle x, e_j \rangle e_j$$

If $N \geq k$, then

$$\begin{aligned} \langle x - x_N, e_k \rangle &= \langle x, e_k \rangle - \langle x_N, e_k \rangle \\ &= \langle x, e_k \rangle - \langle x, e_k \rangle \langle e_k, e_k \rangle \\ &= 0 \end{aligned}$$

and $x - x_N \perp e_k$ for all $k \leq N$.

But then means $x - x_N$ is orthogonal to x_N, since we can expand the inner product

$$\langle x - x_N, x_N \rangle = \sum_{j=1}^{N} \langle x - x_N, \langle x, e_j \rangle e_j \rangle = 0.$$

Thus

$$\begin{aligned} \|x\|^2 &= \|x - x_N\|^2 + \|x_N\|^2 \\ &\geq \|x_N\|^2 \\ &= \sum_{k=1}^{N} |\langle x, e_k \rangle|^2 \end{aligned}$$

Passing to limits completes the proof.

8.3 Bases

8.3.1 Algebraic Bases

Definition 8.3.1 (Algebraic Basis)
Linear independent spanning set.

Proposition 8.3.1
An orthonormal set cannot span an infinite dimensional Hilbert space.

Proof
Suppose $\{e_n\}_{n\geq 1}$ is an orthonormal set. Then

$$x := \sum_{n\geq 1} \frac{1}{n} e_n \in \mathcal{H}$$

since its norm is the p-series with $p > 1$.

Suppose $x \in \operatorname{span}\{e_n\}$, say

$$x = \sum_{k=1}^{n} \alpha_k e_k$$

We have

$$\left\langle \sum_{k=1}^{N} \alpha_k e_k, e_{N+1} \right\rangle = 0$$

but

$$\langle x, e_{N+1} \rangle = \left\langle \sum_{n\geq 1} \frac{1}{n} e_n, e_{N+1} \right\rangle = \frac{1}{N+1}$$

which is a contradiction. Notice that we subtly used the continuity of the inner product in the first argument here.

Now consider a non necessarily countably infinite orthonormal set Λ and select a countable subset $\{e_n\}_{n\geq 1}$. Form x as above and suppose $x \in \operatorname{span}\Lambda$. Say

$$x = \sum_{j=1}^{M} \beta_j f_j$$

for $f_j \in \Lambda$.

Choose $e_N \in \{e_n\}_{n\geq 1}$ such that

$$e_N \notin \{f_1, \ldots, f_M\}$$

Arguing as above, we again obtain a contradiction

Thus Λ cannot span $\mathcal{H}$.

The proof of the proposition essentially constructs a countable linear combination of orthonormal vectors, but not a FINITE linear combination.

8.3.2 Complete Orthonormal Sets

Since orthonormal sets are so easy to work with, we modify the definition of a basis for Hilbert spaces.

Definition 8.3.2 (Complete)
An orthonormal set $S \subseteq \mathcal{H}$ is complete if whenever $\langle x, s\rangle = 0$ for every $s \in S$, then

$$x = 0$$

Definition 8.3.3 (Basis)
A complete orthonormal set.

A complete orthonormal set is a maximal orthonormal set as we cannot adjoin any additional elements to the set and maintain orthonormality.

Proposition 8.3.2
If $\mathcal{H}$ is a finite dimensional Hilbert space, then any complete orthonormal set is an algebraic basis.

Proof
First notice that any orthonormal set is linearly independent.

If it does not span, we can grab a vector outside of its span and use the Gram-Schmidt process to augment the orthonormal set.

<++>

Theorem 8.3.3
If $\{e_k\}_{k\geq 1}$ is a complete orthonormal set and $x \in \mathcal{H}$, then

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$$

and therefore

$$H = \overline{\text{span}\{e_k\}_{k\geq 1}}$$

Moreoever if $S \subseteq \mathcal{H}$ is orthonormal and $\mathcal{H} = \overline{\text{span}(S)}$, then S is complete.

Proof
Part I: Let $x \in \mathcal{H}$. By Bessel's inequality

$$\sum_{k\geq 1} |\langle x, e_k \rangle|^2 \leq \|x\|^2 < \infty$$

Thus

$$y = \sum_{k\geq 1} \langle x, e_k \rangle e_k \in \mathcal{H}$$

Note that

$$\langle y, e_k \rangle = \langle x, e_k \rangle$$

for all k, hence

$$y - x \perp e_k$$

for all k.

As $\{e_k\}_{k\geq 1}$ is complete

$$y = x$$

as desired.

Part II: Suppose that $x \in \mathcal{H}$ is orthogonal to all vectors in S. By linearity

$$\langle x, y \rangle = 0$$

for all $y \in \text{span}(S)$.

But $x \in \overline{\text{span}(S)}$, hence

$$x = \lim_n x_n$$

where each $x_n \in \text{span}(S)$.

Thus

$$0 = \langle x_n, x \rangle = 0 \to \langle x, x \rangle$$

and thus $x = 0$.

By definition S is complete.

Corollary 8.3.3.1
Let $\{e_n\}_{n\geq 1}$ be orthonormal. Then $\{e_n\}$ is complete if and only if

$$\|x\|^2 = \sum_{k\geq 1} |\langle x, e_k\rangle|^2$$

for all $x \in \mathcal{H}$.
In other words, Bessel's inequality is an equality.

Proof
$\underline{(\implies)}$ Assume $\{e_n\}$ is complete. By part I of the previuos theorem

$$x = \sum_{k\geq 1} \langle x, e_k\rangle e_k$$

and by the Pythgorean theorem

$$\|x\|^2 = \sum_{k\geq 1} \|\langle x, e_k\rangle e_k\|^2 = \sum_{k\geq 1} |\langle x, e_k\rangle|^2$$

$\underline{(\neg \implies \neg)}$ Suppose now that $\{e_n\}$ is not complete, there must be some $x \neq 0$ with

$$\langle x, e_k\rangle = 0$$

for all k.

If we assume

$$\|x\|^2 = \sum_{k\geq 1} |\langle x, e_k\rangle|^2$$

then we must have $x = 0$, a contradiction.

Corollary 8.3.3.2
The orthonormal set $\{e_n\}_{n\geq 1}$ is complete if and only if the linear map $T : \mathcal{H} \to \ell^2$ given by

$$T(x) := (\langle x, e_k\rangle)_{k\geq 1}$$

is injective.

Proof
Assume $\{e_n\}$ is complete and suppose $Tx = Ty$. Then

$$\langle x, e_k \rangle = \langle y, e_k \rangle$$

for all k, so

$$(x - y) \perp e_k$$

for all k.

This implies that $x = y$.

The converse is similar.

Suppose that $\{e_n\}$ from above is complete. It can be shown that T is in fact isometric and surjective. Thus $\mathcal{H} \cong \ell^2$.

Theorem 8.3.4
Every Hilbert space has a basis.
If $\mathcal{H}$ is separable, then any orthonormal set is countable or finite.

Proof
Let $\mathcal{S}$ be the collection of all orthonormal sets from $\mathcal{H}$.

Define the partial order by inclusion on $\mathcal{S}$. Let $\mathcal{C} \subseteq \mathcal{S}$ be an arbitrary chain. Then consider

$$X := \bigcup_{C \in \mathcal{C}} C$$

To see it is in $\mathcal{S}$, take any $x \neq y \in X$ and remark that since $\mathcal{C}$ is a chain, there must be some $C \in \mathcal{C}$ which contains both x, y. Thus

$$\|x\| = \|y\| = 1, \langle x, y \rangle = 0$$

so X is an orthonormal set.

By Zorn's lemma, $\mathcal{S}$ has a maximal element A.

If A is not complete, then there is some $x \in \mathcal{H}$ such that $\langle x, a \rangle = 0$ for all $a \in A$. But then

$$A \subset A \cup \left\{ \frac{x}{\|x\|} \right\}$$

is orthonormal which violates the maximality of A.

Hence A is a complete orthonormal set and a basis by definition.

Now suppose that $\mathcal{H}$ is a separable Hilbert space and that $\{e_\alpha\}_{\alpha \in \Lambda}$ is an uncountable orthonormal set. Pick a countably dense subset

$$\{f_n\}_{n\geq 1} \subseteq \mathcal{H}$$

For each $\alpha \in \Lambda$, there is some $n \geq 1$ with

$$e_\alpha \in B\left(f_n, \frac{1}{2}\right)$$

Since there are only countably many such balls and uncountably many e_α's, some ball $B\left(f_N, \frac{1}{2}\right)$ must contain uncountably many e_α's.

In particular, there is a pair $e_1, e_2 \in B\left(f_n, \frac{1}{2}\right)$. By the Pythagorean theorem, we have

$$\|e_1 - e_2\|^2 = \|e_1\|^2 + \|e_2\|^2 = 2$$

But

$$\begin{aligned}\sqrt{2} &= \|e_1 - e_2\| \\ &\leq \|e_1 - f_n\| + \|f_n - e_2\| \\ &\leq \frac{1}{2} + \frac{1}{2} \\ &= 1\end{aligned}$$

which is a contradiction.

Corollary 8.3.4.1
Every separable infinite dimensional Hilbert space is isometrically isomorphic to ℓ^2.

The isometry T can in fact be chosen to have the property that

$$\langle x, y\rangle = \langle Tx, Ty\rangle$$

for all $x, y \in \mathcal{H}$ where the first inner product is the inner product on $\mathcal{H}$ and the second inner product is the inner product on ℓ^2. We call such a map is inner product preserving.

Chapter 9

Fourier Analysis on the Circle

9.1 Notation

Fourier analysis can be studied in a variety of settings. We will work with the circle

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

By identifying z with e^{it} we can identify $\mathbb{T}$ with

$$\mathbb{T} = [0, 2\pi]$$

or alternatively by identifying $z = e^{2\pi i t}$, we can view $\mathbb{T}$ as

$$\mathbb{T} = [0, 1]$$

We give $\mathbb{T}$ the topology it inherits from $\mathbb{C}$ and hence it is compact.

$\mathbb{T}$ is a group under addition mod 2π when viewed as $[0, 2\pi]$ or under multiplication when viewed as a subset of $\mathbb{C}$.

Functions defined on $[0, 2\pi]$ must satisfy $f(0) = f(2\pi)$ and hence have a unique 2π-periodic extension to all of $\mathbb{R}$.

We denote the continuous function defined on $\mathbb{T}$ by

$$C(\mathbb{T})$$

We write m to denote the Lebesgue measure restricted to $[0, 2\pi)$ and normalized so that $m(\mathbb{T}) = 1$. Notice that m is still translation invariant where addition is understood to be mod 2π.

With this notation

$$\int_{\mathbb{T}} f dm = \frac{1}{2\pi} \int_{[0,2\pi]} f(x) dx$$

For $1 \le p < \infty$, we will put

$$\|f\|_{L^p(\mathbb{T})} = \|f\|_p := \left(\frac{1}{2\pi} \int_{[0,2\pi]} |f(x)|^p dx \right)^{\frac{1}{p}}$$

and $\|f\|_\infty$ the L^∞ norm on $\mathbb{T}$.

The notation $L^p(\mathbb{T})$ will mean the space of equivalence classes of measurable functions on $\mathbb{T}$ with $\|f\|_p < \infty$. We have

$$C(\mathbb{T}) \subseteq L^\infty(\mathbb{T}) \subseteq L^P(\mathbb{T}) \subseteq L^1(\mathbb{T})$$

for $1 < p < \infty$.

It can be showed that Hölder's inequality gives

$$\left| \frac{1}{2\pi} \int_{[0,2\pi]} fg \right| \le \|f\|_p \|g\|_q$$

where p, q are conjugate indices. In the special case of $p = q = 2$, we have that $L^2(\mathbb{T})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{[0,2\pi]} f\bar{g}$$

Proposition 9.1.1
$C(\mathbb{T})$ is dense in L^p for $p < \infty$.
L^2 is dense in L^1.

9.2 Fourier Series & Fourier Coefficients

Proposition 9.2.1
The set of functions

$$\{\exp(inx) : n \in \mathbb{Z}\}$$

is an orthonormal family of continuous functions on $\mathbb{T}$.

Proof
To see normality

$$\frac{1}{2\pi}\int_{[0,2\pi]} |e^{inx}|^2 dx = \frac{1}{2\pi}\int_{[0,2\pi]} \cos^2 nx + \sin^2 nx dx$$
$$= 1$$

for all $n \in \mathbb{Z}$.

For orthogonality

$$\frac{1}{2\pi}\int_{[0,2\pi]} e^{inx}\overline{e^{imx}} dx = \frac{1}{2\pi}\int_{[0,2\pi]} e^{inx}e^{-imx} dx$$
$$= 0$$

given that $n \neq m$.

Definition 9.2.1 (Trigonometric Polynomial)
A continuous function of the form

$$p(x) := \sum_{n=-M}^{N} a_n e^{inx}$$

Definition 9.2.2 (Frequencies)
The set of integers n where $a_n \neq 0$ are known as the frequencies of p.

Definition 9.2.3 (Degree)
The maximum $|n|$ such that $a_n \neq 0$.

Write write

$$\mathrm{Trig}(\mathbb{T}) := \mathrm{span}\{e^{inx} : n \in \mathbb{Z}\}$$

for the set of all trigonometric polynomials.

Theorem 9.2.2
$\mathrm{Trig}(\mathbb{T})$ is dense in $C(\mathbb{T})$ in the supremum norm and hence in $L^p(\mathbb{T})$ in the L^p norm for $p < \infty$.

Proof
Density in $C(\mathbb{T})$ is a direct consequence of the Stone-Weirstrauss theorem. $\mathrm{Trig}(\mathbb{T})$ is an

algebra in $C(\mathbb{T})$ that contains the constant functions and so does not vanish. It separates points since $s \neq t$ implies

$$e^{is} \neq e^{it}.$$

Finally, it is is closed under conjugation.

Now, $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ for $p < \infty$ and

$$\|\cdot\|_p \leq \|\cdot\|_\infty$$

thus the trigonometric polynomials are also dense in $L^p(\mathbb{T})$.

Corollary 9.2.2.1
$\mathrm{Trig}(\mathbb{T})$ is a basis of L^2.

Proof
The span of $\{e^{inx} : n \in \mathbb{Z}\}$ is dense in $L^2(\mathbb{T})$. By our previous work with general Hilbert spaces, it is consequently a basis for the Hilbert space $L^2(\mathbb{T})$.

Remark that if the choice $\mathbb{T} = [0, 1]$ is made, then

$$\{\exp(i2\pi nx) : n \in \mathbb{Z}\}$$

is a complete orthonormal set.

Definition 9.2.4 (Fourier Coefficient)
The n-th Fourier Coefficient of $f \in L^2(\mathbb{T})$ is

$$\hat{f}(n) := \langle f, e^{inx} \rangle = \frac{1}{2\pi} \int_{[0,2\pi]} f(x) e^{-inx} dx$$

Definition 9.2.5 (Fourier Series)
The formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

is known as the Fourier series of f.

9.2.1 Parseval's Theorem

Proposition 9.2.3
Let $f \in L^2$ and put

$$S_N f(x) := \sum_{n=-N}^{N} \langle f, e^{inx} \rangle e^{inx} = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

This is a trigonometric polynomial of degree at most N.
We have

$$\|f - S_n f\|_2 \to 0$$

Proof
$\text{Trig}(\mathbb{T})$ is a basis of $\mathbb{T}$.

Proposition 9.2.4 (Uniqueness of Fourier Coefficients)
If $f, g \in L^2$ and $\hat{f}(n) = \hat{g}(n)$, then $f = g$ a.e.

Proof
If $h \in L^2$ and $\hat{h}(n) = 0$ for all $n \in \mathbb{Z}$ then $h = 0$ a.e. from the definition of a basis.

Thus if f, g have the same Fourier coefficients

$$\widehat{f - g}(n) = \hat{f}(n) - \hat{g}(n) = 0$$

for all n thus $f - g = 0$ a.e. and $f = g$ a.e.

Theorem 9.2.5 (Parseval I)
For all $f \in L^2$

$$\begin{aligned}\|f\|_2 &= \left(\frac{1}{2\pi}\int_{[0,2\pi]} |f|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=-\infty}^{\infty} \left|\langle f, e^{inx}\rangle\right|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2\right)^{\frac{1}{2}} \\ &= \left\|\left(\hat{f}(n)\right)_n\right\|_{\ell^2}\end{aligned}$$

Thus the L^2 norm of f and the ℓ^2 norm of the sequence of Fourier coefficients of f coincide.

Proof
Bessel's inequality is an equality from the completeness of $\text{Trig}(\mathbb{T})$ as an orthonormal set.

Another way to state this is that the map $T : L^2(\mathbb{T}) \to \ell^2$ given by

$$T(f) := (\hat{f}(n))_{n\in\mathbb{Z}}$$

is an isometric isometry.

From now on ℓ^2 indicates the square summable sequences indexed by $\mathbb{Z}$ rather than by $\mathbb{N}$.

Theorem 9.2.6 (Parseval II)
T is inner product preserving.
So

$$\langle f, g\rangle = \langle T(f), T(g)\rangle_{\ell^2}$$

and

$$\frac{1}{2\pi}\int_{[0,2\pi]} f\bar{g} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$$

for all $f, g \in L^2$.

Observe that taking $g = f$ proves the first statement of Parseval's Theorem.

9.3 Fourier Coefficients in $L^1(\mathbb{T})$

We have previously discussed the fact that $L^2 \subseteq L^1$. However, if $f \in L^1$ we can still define

$$\hat{f}(n) := \frac{1}{2\pi}\int_{[0,2\pi]} f(x)e^{-inx}dx$$

for $n \in \mathbb{Z}$. We continue to call this the n-th Fourier Coefficient of f.

Similarly, we define the Fourier series of $f \in L^1$ as the formal sum

$$S(f) := \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

9.3.1 Alternative Form of Fourier Series

Using the relation

$$e^{inx} = \cos nx + i\sin nx$$

together with

$$\cos(-nx) = \cos nx, -\sin(-nx) = \sin nx$$

we can rewrite

$$\sum_{n=-\infty}^{\infty} a_n e^{inx} = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nx + B_n \sin nx$$

where

$$A_n := a_n + a_{-n} = \frac{1}{\pi}\int_{[0,2\pi]} f(x)\cos nx dx$$
$$B_n := i(a_n - a_{-n}) = \frac{1}{\pi}\int_{[0,2\pi]} f(x)\sin nx dx$$

Observe that if f is real-valued, then the A_n, B_n's are real-valued. However, this is not necessarily the case if f takes on imaginary values.

9.3.2 Basic Properties

Let $f, g \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$

Proposition 9.3.1
We have

$$\widehat{f+g}(n) = \hat{f}(n) + \hat{g}(n)$$

and

$$\widehat{\alpha f}(n) = \alpha \hat{f}(n)$$

for $\alpha \in \mathbb{C}$.

Proof
Linearity of the integral.

Proposition 9.3.2

$$\widehat{\bar{f}(n)} = \overline{\hat{f}(-n)}$$

Proof
Prove it iteratively for simple functions, real-valued functions, etc.

Proposition 9.3.3
Let $t \in \mathbb{T}$ and define

$$f_t(x) := f(x-t)$$

Then

$$\hat{f}_t(n) = e^{-int}\hat{f}(n)$$

Proof
Since $e^{inx} = e^{-in(x-t)}e^{-int}$, we have

$$\begin{aligned}
\hat{f}_t(n) &= \frac{1}{2\pi}\int_{[0,2\pi]} f_t(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} f(x-t)e^{-inx}dx \\
&= \frac{e^{-int}}{2\pi}\int_{[0,2\pi]} f(x-t)e^{-in(x-t)}dx \\
&= \frac{e^{-int}}{2\pi}\int_{[0,2\pi]} f(x)e^{-inx}dx \\
&= e^{-int}\hat{f}(n)
\end{aligned}$$

Proposition 9.3.4
We have

$$(\hat{f}(n))_{n\in\mathbb{Z}} \in \ell^\infty$$

and

$$\left\|(\hat{f}(n))\right\|_{\ell^\infty} \le \|f\|_1$$

Proof
$|\hat{f}(n)| \le \|f\|_1$ since

$$\begin{aligned}
|\hat{f}(n)| &= \left|\frac{1}{2\pi}\int_{[0,2\pi]} f(x)e^{-inx}\right| \\
&\le \frac{1}{2\pi}\int_{[0,2\pi]} |f(x)e^{-inx}| \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} |f(x)|\cdot|e^{-inx}| \\
&= \frac{1}{2\pi}\int_{[0,2\pi]} |f(x)| \\
&=: \|f\|_1
\end{aligned}$$

Compare this with the $L^1(\mathbb{T})$ case when we knew the sequence is square-summable and even converges to 0! At least we still know the sequence is bounded.

Proposition 9.3.5
If $f_k \to f$ in L^1 then $\hat{f}_k(n) \to \hat{f}(n)$ for all $n \in \mathbb{Z}$ (even uniformly in n).

Proof
We have

$$\begin{aligned}
|\hat{f}_k(n) - \hat{f}(n)| &= |\widehat{f_k - f}(n)| \\
&\le |f_k - f|_1 \\
&\to 0
\end{aligned}$$

Proposition 9.3.6
$\|f_t\|_p = \|f\|_p$ for all $t \in \mathbb{T}$ and $p \ge 1$.

Proposition 9.3.7
$\|f(x+t) - f(x)\|_p \to 0$ as $t \to 0$ for $1 \le p < \infty$.

Proof
Continuous case first. Then use the density of continuous functions in L^p.

9.3.3 Riemann-Lebesgue Lemma

Lemma 9.3.8 (Riemann-Lebesgue)
Let $f \in L^1(\mathbb{T})$

$$\lim_{n\to\pm\infty} \hat{f}(n) = 0$$

Proof
Pick $\epsilon > 0$. Since $\mathrm{Trig}(\mathbb{T})$ is dense in all $L^p, p < \infty$, we can choose $p \in \mathrm{Trig}(\mathbb{T})$ such that

$$\|f - p\|_1 < \epsilon$$

Let $N = \deg p$. Then $\hat{p}(n) = 0$ for all $|n| > N$. Consequently for all $|n| > N$

$$\begin{aligned} |\hat{f}(n)| &\le |\hat{f}(n) - \hat{p}(n)| + |\hat{p}(n)| \\ &\le \|f - p\|_1 \\ &< \epsilon \end{aligned}$$

Corollary 9.3.8.1
The sequence $(\hat{f}(n))_n$ actually belongs to the subspace of ℓ^∞

$$c_0 := \{(x_n)_n : |x_n| \to 0\}$$

Corollary 9.3.8.2
If $f \in L^1(\mathbb{T})$ then

$$\lim_{n\to\infty} \int_{[0,2\pi]} f(x) \sin nx dx = 0 = \lim_{n\to\infty} \int_{[0,2\pi]} f(x) \cos nx dx$$

Proposition 9.3.9
If $f \in L^1(\mathbb{T})$ then for all $\beta \in \mathbb{R}$

$$\lim_{n\to\infty} \int_{[0,2\pi]} f(x) \sin(nx + \beta) dx = 0$$

Chapter 10

Fourier Series

10.1 Dirichlet Kernel

Our goal here is to "rebuild" a function given a Fourier series.

Definition 10.1.1 (Dirichlet Kernel)
The N-th Dirichlet kernel is the function

$$D_N(t) := \sum_{n=-N}^{N} e^{int} = 1 + 2\sum_{n=1}^{N} \cos nt \in \mathrm{Trig}(\mathbb{T})$$

The N-th Dirichlet kernel is an even, real-valued degree N trigonometric polynomial.

Proposition 10.1.1
$D_N(0) = 2N + 1 = \sup_t |D_N(t)|$.

Proposition 10.1.2
$\widehat{D_N}(n) = 1$ if $|n| \leq N$ and 0 otherwise.

Lemma 10.1.3
Let $t \in \mathbb{T} \setminus \{0\}$

$$D_N(t) = \frac{\sin\left[\left(N + \frac{1}{2}\right) t\right]}{\sin \frac{t}{2}}$$

This also holds for $\lim D_N(t)$ as $t \to 0$.

Proof
A telescoping sum argument gives

$$\begin{aligned}
\left(\sin\frac{t}{2}\right)D_N(t) &= \left(\frac{e^{it/2}-e^{-it/2}}{2i}\right)\sum_{n=-N}^{N} e^{int} \\
&= \sum_{n=-N}^{N}\left(\frac{e^{i(n+1/2)t}-e^{i(n-1/2)t}}{2i}\right) \\
&= \left(\frac{e^{i(N+1/2)t}-e^{i(-N-1/2)t}}{2i}\right) \\
&= \sin\left(N+\frac{1}{2}\right)t
\end{aligned}$$

Corollary 10.1.3.1
For $t \in (\delta, 2\pi-\delta)$

$$|D_N(t)| \leq \frac{1}{|\sin\frac{\delta}{2}|}.$$

Remark that this bound is independent of the choice of N.

To see the motivation behind the Dirichlet kernel, observe that for $f \in L^1$

$$\begin{aligned}
S_N f(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\
&= \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int}dt\right)e^{inx} \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\sum_{n=-N}^{N} e^{in(x-t)}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(t)D_N(x-t)dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(t)D_N(t-x)dt && D_N \text{ is even} \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(t+x)D_N(t)dt && \text{translation invariance}
\end{aligned}$$

10.2 Functional Analysis

Our goal in this section is to show the existence of a continuous function whose Fourier series diverges at a point and an L^1 function whose Fourier series diverges in the L^1 norm.

Definition 10.2.1 (Linear Map)
Let X, Y be Banach spaces.
A map $F : X \to Y$ is linear if

$$F(\alpha x + y) = \alpha F(x) + F(y)$$

for all $x, y \in X$ and scalars α.

Definition 10.2.2 (Operator Norm)
Given a linear map F, we define

$$\|F\|_{\text{op}} = \sup_{\|x\|_X \leq 1} \|F(x)\|_Y$$

Definition 10.2.3 (Bounded Linear Map)
We say $F : X \to Y$ linear is bounded if

$$\|F\|_{\text{op}} < \infty$$

Proposition 10.2.1
For any linear map F, $F(0) = 0$.
Furthermore

$$\|F\|_{\text{op}} = \sup_{x \neq 0} \frac{\|F(x)\|_Y}{\|x\|_X}.$$

Proof
If $x \neq 0$, normalize it so

$$\|F\|_{\text{op}} \geq \left\| F\left(\frac{1}{\|x\|_X} x\right) \right\|_Y$$

On the other hand for all $\|x\|_X \leq 1$

$$\frac{\|F(x)\|}{\|x\|_X} \geq \|F(x)\|$$

But then

$$\sup_{x\neq 0}\frac{\|F(x)\|}{\|x\|_X} \geq \sup_{\|x\|_X\leq 1}\frac{\|F(x)\|_Y}{\|x\|_X} \geq \sup_{\|x\|_X\leq 1}\|F(x)\|_Y$$

Thus

$$\|F\|_{\text{op}}\|x\|_X \geq \|F(x)\|_Y$$

for all x.

Proposition 10.2.2
A linear map is bounded if and only if it is continuous.

Proof
$\underline{\implies}$ Suppose F is bounded and let $x_n \to x$. Then

$$\begin{aligned}\|F(x_n) - F(x)\| &= \|F(x_n - x)\| \\ &\leq \|F\|_{\text{op}}\|x_n - x\| \\ &\to 0\end{aligned}$$

$\underline{\impliedby}$ Suppose that F is continuous at 0. Then for every $\epsilon > 0$, there is some $\delta > 0$ such that

$$\|x\| \leq \delta \implies \|F(x) - F(0)\| < \epsilon.$$

Apply this with $\epsilon = 1$ and the corresponding δ. If $\|x\| \leq 1$, then $\|\delta x\| \leq \delta$ so

$$\delta\|F(x)\| = \|F(\delta x)\| < 1.$$

Hence $\|F(x)\| \leq \frac{1}{\delta}$ whenever $\|x\| \leq 1$ which implies

$$\|F\|_{\text{op}} \leq \frac{1}{\delta}.$$

Example 10.2.3
The linear map $F : L^1(\mathbb{T}) \to \mathbb{C}$ given by

$$f \mapsto \hat{f}(n)$$

satisfies

$$\|F\|_{\text{op}} = 1.$$

Example 10.2.4
The isometry $F : L^2 \to \ell^2$ given by

$$f \mapsto (\hat{f}(n))_{n\geq 1}$$

satisfies

$$\|F\|_{\text{op}} = 1.$$

Proposition 10.2.5
Take $S_N : L^1 \to L^1$ given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx} = \frac{1}{2\pi} \int_{\mathbb{T}} f(t+x) D_N(t) dt.$$

Then

$$\|S_N\|_{\text{op}} = \|D_N\|_1 < \infty$$

Proof (Sketch)
By Fubini's Theorem and some computation

$$\|S_N f\|_1 \leq \|D_N\|_1 \|f\|_1.$$

Take

$$f_k = \frac{\chi_{(-\frac{1}{k}, \frac{1}{k})}}{m\left(-\frac{1}{k}, \frac{1}{k}\right)}$$

and observe that $\|f_k\|_1 = 1$.

It suffices to show that $\frac{\|S_N f_k\|_1}{\|f_k\|_1} \to \|D_N\|_1$ so

$$\|S_N\|_{\text{op}} \geq \frac{\|S_N f_k\|_1}{\|f_k\|_1} \to \|D_N\|_1.$$

The key point of attack is the fact that D_N is uniformly continuous as it is continuous on a compact set and that multiplying $1 \cdot \|f_k\|_1 = \frac{1}{2\pi} \int_{\mathbb{T}} f_k(t) dt$ anywhere does not change identities.

10.2.1 Uniform Boundedness Principle

Theorem 10.2.6 (Uniform Boundedness Principle)
Suppose X, Y are Banach spaces. Let $\mathcal{F}$ be a family of bounded linear maps from $X \to Y$.
If for every $x \in X$

$$\sup_{F \in \mathcal{F}} \|F(x)\|_Y < \infty$$

then

$$\sup_{F \in \mathcal{F}} \|F\|_{\text{op}} < \infty.$$

Consequently, if $\sup_{F \in \mathcal{F}} \|F\|_{\text{op}} = \infty$, then there is some $x \in X$ such that

$$\sup_{F \in \mathcal{F}} \|F(x)\|_Y = \infty.$$

Divergent Fourier Series

Proposition 10.2.7
There is some $C > 0$ such that

$$\|D_N\|_1 \geq C \log N$$

for all N.

Theorem 10.2.8
There is some $f \in L^1$ such that

$$\left\| \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \right\|_1 \to \infty.$$

In other words, f has a divergent Fourier series.

Proof
Take $X = Y = L^1(\mathbb{T})$ and consider the family

$$\mathcal{F} := \{S_N : L^1 \to L^1, N \in \mathbb{N}\}.$$

We already know that

$$\|S_N\|_{\text{op}} = \|D_N\|_1.$$

By the previous proposition

$$\sup_{S_N \in \mathcal{F}} \|S_N\|_{\text{op}} \geq \|D_N\|_1 \geq C \log N \to \infty.$$

Hence by the uniform boundedness principle there is some $f \in L^1$ such that $\sup_N \|S_N f\|_1 = \infty$.

In particular

$$\left\| \sum_{n=-N}^{N} \hat{f}(n) e^{inx} \right\|_1 \to \infty.$$

Divergent Fourier Series of a Continuous Function

Theorem 10.2.9
There is a continuous function f with

$$(S_n(f)(0))_{n \geq 1}$$

being a divergent sequence.

Proof (Sketch)
We wish to apply the uniform boundedness principle with $X = C(\mathbb{T})$ and $Y = \mathbb{C}$. Define $T_N : C(\mathbb{T}) \to \mathbb{C}$ by

$$f \mapsto S_N(f)(0)$$

and put

$$\mathcal{F} := \{T_N : N \in \mathbb{N}\}.$$

Clearly each T_N is linear. Pick any $f \in C(\mathbb{T})$ such that

$$\|f\|_\infty \leq 1.$$

Then

$$\begin{aligned} |T_N(f)| &= |S_N(f)(0)| \\ &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(d) D_N(t) dt \right| \\ &\leq \|f\|_\infty \frac{1}{2\pi} \int_0^{2\pi} |D_N(t)| dt \\ &\leq \|f\|_\infty \|D_N\|_1 \end{aligned}$$

so $\|T_N\|_{\text{op}} \le \|D_N\|_1$ and these are bounded maps.

We claim that $\|T_N\|_{\text{op}} \ge \frac{1}{2}\|D_N\|_1$. This claim combined with an application of the uniform boundedness principle suffices to prove the theorem.

Our plan of attack is to construct a sequence of continuous functions g_n with $\|g_n\|_\infty = 1$ and $|S_n(g_n)(0)| \ge \frac{1}{2}\|D_N\|_1$. Since

$$\|T_n\|_{\text{op}} \ge |T_n(g_n)| = |S_n(g_n)(0)|$$

this will complete the proof.

The idea is to choose

$$g_n \approx \operatorname{sgn} D_n$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(t)D_n(t) \approx \frac{1}{2\pi}\int_0^{2\pi} |D_n(t)|dt = \|D_n\|_1.$$

We cannot make g_n exactly D_n since we want g_n to be continuous. However, D_n has finitely many zeroes being a non-zero trigonometric polynomial. If we keep $|g| \le 1$ everywhere and arrange for the sum of the lengths of intervals to be "small enough", then

$$|S_n(g_n)(0)| \approx \|D_n\|_1.$$

It is in fact possible to make these theorems constructive. However, we will need to develop more Fourier analysis.

10.3 A Divergent Construction

We now give a construction of a continuous function whose Fourier series diverges at 0.

10.3.1 Background

Recall that we constructed a sequence of continuous functions g_n such that $\|g_n\|_\infty = 1$ and

$$|S_n(g_n)(0)| \ge \frac{1}{2}\|D_n\|_1 \ge C\log n$$

for some $C > 0$.

10.3.2 Building Blocks

Put

$$f_n := \sigma_{2n^2}(g_n) = F_{2n^2} * g_n$$

where F_k is the k-th Fejér kernel.

This is a trigonometric polynomial of degree at most $2n^2$ satisfying

$$\|f_n\|_\infty = \|F_{2n^2} g_n\|_\infty \leq \|F_{2n^2}\|_1 \|g_n\|_\infty \leq 1.$$

10.3.3 The Construction

Define $n_k := 2^{3^k}$ and put

$$f(t) := \sum_{k=1}^{\infty} \frac{1}{k^2} f_{n_k}(n_k t).$$

The choice of n_k is not critical. We require only that the sequence diverges rapidly.

10.3.4 Proof of Divergence

Since $\|f_n\|_\infty \leq 1$

$$|G_k(0)| = |f_{n_k}(0)| \leq \|f_{n_k}\|_\infty \leq 1$$

and

$$|\widehat{f_{n_k}(0)}| \leq \|f_{n_k}\|_\infty \leq 1.$$

Chapter 11

Summability Kernels

11.1 Convolution

L^1 is not closed under pointwise multiplication. However, there is a binary operation $*$ on L^1 which makes it an algebra.

Definition 11.1.1 (Convolution)
For $f, g \in L^1$
$$f * g := \frac{1}{2\pi} \int_{\mathbb{T}} f(t) g(x-t) dt.$$

Remark that

$$\begin{aligned} f * e^{inx} &:= \frac{1}{2\pi} \int_{\mathbb{T}} f(t) e^{in(x-t)} \\ &= e^{inx} \hat{f}(n). \end{aligned}$$

Thus convolution with a trigonometric polynomial yields a trigonometric polynomial.

Proposition 11.1.1
$*$ is commutative.

Proof
Use the change of variables $t \mapsto -t, t \mapsto t - x$ and we have

$$\begin{aligned} f * g(x) &= \frac{1}{2\pi} \int_{\mathbb{T}} f(-t) g(x+t) dt \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} f(x-t) g(t) \\ &=: g * f(x) \end{aligned}$$

as desired.

Proposition 11.1.2
For $f, g \in L^1$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1 < \infty.$$

Proof
We first need to show that $t \mapsto f(x)g(x-t)$ is measurable.

Then an application of Fubini's theorem yields the result.

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{T}} |f * g(x)| dx &= \frac{1}{2\pi} \int_{\mathbb{T}} \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(x) g(x-t) dt \right| dx \\ &\leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)| \left(\frac{1}{2\pi} \int_{\mathbb{T}} |g(x-t)| dt \right) dx \\ &= \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)| \cdot \|g\|_1 dx \\ &= \|f\|_1 \|g\|_1 & < \infty. \end{aligned}$$

Proposition 11.1.3
If $f \in L^\infty, g \in L^1$

$$f * g \in L^\infty.$$

Proof
We have

$$\begin{aligned} |f * g(x)| &\leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t) g(x-t)| dt \\ &\leq \frac{\|f\|_\infty}{2\pi} \int_{\mathbb{T}} |f(x-t)| dt \\ &= \|f\|_\infty \|g\|_1 \\ &< \infty. \end{aligned}$$

Proposition 11.1.4
$S_N(f)(x) = f * D_N(x)$.

Proposition 11.1.5
For all $f, g \in L^1$ and $n \in \mathbb{Z}$

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n).$$

Proof
By definition

$$\begin{aligned}
\widehat{f * g}(n) &= \frac{1}{2\pi}\int_{\mathbb{T}} f * g(x) e^{-inx} dx \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(x-t)\right) e^{-inx} dt dx \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(t)e^{-int} \left(\frac{1}{2\pi}\int_{\mathbb{T}} g(x-t)e^{-in(x-t)} dx\right) dt && \text{Fubini's theorem} \\
&= \hat{f}(n)\hat{g}(n).
\end{aligned}$$

Observe that this means there is no identity element for this binary operation since any candidate $g \in L^1$ necessarily satisfies $\hat{g}(n) = 1$ for all n. this contradicts the Riemann-Lebesgue lemma.

Proposition 11.1.6
If $f \in L^p, g \in L^q$ for conjugate indices p, q, then

$$f * g \in L^\infty.$$

11.2 Summability Kernels

There are "approximate" identities under $*$.

Definition 11.2.1 (Summability Kernel)
A summability kernel (bounded approximate identity) is a sequence $(K_n)_{n\geq 1} \subseteq L^1$ which satisfies

(i) $\widehat{K}_n(0) = 1 = \frac{1}{2\pi}\int_{\mathbb{T}} K_n(t) dt$

(ii) There is some M such that $\frac{1}{2\pi}\int_{\mathbb{T}} |K_n(t)| dt \leq M$ for all n

(iii) For every $0 < \delta < \pi$ we have $\lim_{n\to\infty} \int_\delta^{2\pi-\delta} |K_n(t)| dt = 0$

Definition 11.2.2 (Positive Summability Kernel)
A summability kernel where

$$K_n(t) \geq 0$$

for all t, n.

Observe that the second condition is now redundant as the first condition implies the second with $M = 1$.

11.3 Uniform Convergence

Summability kernels have excellent convergence properties.

Theorem 11.3.1
Let $f \in C(\mathbb{T})$ and assume that (K_n) is a summability kernel.
Then

$$K_n * f \to f$$

uniformly.

Proof
We want

$$\sup_{x\in\mathbb{T}} |K_n * f(x) - f(x)| \to 0$$

as $n \to \infty$.

Using the definition of convolution and the fact that $\int K_n = 1$, we can write

$$\begin{aligned}
|K_n * f(x) - f(x)| &= \left| \frac{1}{2\pi} \int_{\mathbb{T}} K_n(t) f(x-t)dt - f(x) \frac{1}{2\pi} \int_{\mathbb{T}} K_n(t)dt \right| \\
&= \left| \frac{1}{2\pi} \int_{\mathbb{T}} K_n(t)[f(x-t) - f(x)]dt \right| \\
&\leq \frac{1}{2\pi} \int_{\mathbb{T}} |K_n(t)| \cdot |f(x-t) - f(x)|dt
\end{aligned}$$

Fix $\epsilon > 0$ and choose $\delta > 0$ from the uniform continuity of f to give

$$|x - y| \leq \delta \implies |f(x) - f(y)| < \frac{\epsilon}{M}.$$

Pick N such that

$$n \geq N \implies \frac{1}{2\pi} \int_{\delta}^{2\pi-\delta} |K_n(t)| \leq \frac{\epsilon}{2\|f\|_\infty}.$$

This exists by the definition of a summability kernel.

Split the integral in question into the ranges

$$[-\delta, \delta], [\delta, 2\pi - \delta].$$

If $t \in [-\delta, \delta]$, then

$$|(x-t)-x| \le \delta \implies |f(x-t)-f(x)| \le \frac{\epsilon}{M}.$$

So

$$\begin{aligned}\frac{1}{2\pi}\int_{-\delta}^{\delta}|K_n(t)||f(x-t)-f(x)| &\le \frac{\epsilon}{2\pi M}\int_{-\delta}^{\delta}|K_n(t)| \\ &\le \frac{\epsilon}{2\pi M}\int_{\mathbb{T}}|K_n(t)| \\ &\le \epsilon.\end{aligned}$$

Now for $n \ge N$

$$\begin{aligned}\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|K_n(t)||f(x-t)-f(x)| &\le \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|K_n(t)|(\|f\|_\infty + \|f\|_\infty) \\ &= 2\|f\|_\infty \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|K_n(t)| \\ &\le \epsilon.\end{aligned}$$

Hence for all $n \ge N$ and $x \in \mathbb{T}$

$$|K_n * f(x) - f(x)| \le 2\epsilon.$$

11.4 Fejér Kernel

The Dirichlet kernel is not a summability kernel since it is not bounded over all N.

Definition 11.4.1 (Fejér Kernel)
The N-th Fejér kernel is the degree N trigonometric polynomial

$$F_N(t) = \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)e^{int}.$$

By Euler's formula

$$\begin{aligned} F_N(t) &= 1 + 2\sum_{j=1}^{N}\left(1 - \frac{j}{N+1}\right)\left(\frac{e^{ijt}+e^{-ijt}}{2}\right) \\ &= 1 + 2\sum_{j=1}^{N}\left(1 - \frac{j}{N+1}\right)\cos jt. \end{aligned}$$

Observe that F_N is an even function and satisfies $\widehat{F_N}(0) = 1$. Moreover

$$\int_{\mathbb{T}} e^{inx} = \begin{cases} 2\pi & n = 0 \\ 0, & n \neq 0 \end{cases}$$

so property (i) of positive summability kernels is satisfied.

Proposition 11.4.1
$F_N = \frac{1}{N+1}\sum_{n=0}^{N} D_n$.

Lemma 11.4.2
For all $t \in [0, 2\pi]$

$$F_N(t) = \frac{1}{N+1}\left(\frac{\sin\frac{(N+1)t}{2}}{\sin\frac{t}{2}}\right)^2$$

where the case $t = 0$ should be interpreted as the limit.

Proof
Observe that

$$\sin^2\frac{t}{2} = -\frac{1}{4}(e^{it} - 2 + e^{-it}).$$

We can then write

$$\begin{aligned} &\sin^2\frac{t}{2}\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)e^{int} \\ &= -\frac{1}{4}(e^{it} - 2 + e^{-it})\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)e^{int} \\ &= \frac{1}{N+1}\left(-\frac{1}{4}e^{-i(N+1)t} + \frac{1}{2} + -\frac{1}{4}e^{i(N+1)t}\right) && \text{telescoping sum} \\ &= \frac{1}{N+1}\sin^2\frac{(N+1)t}{2}. \end{aligned}$$

Observe that

$$F_N(0) = N + 1.$$

Moreover

$$|F_N(t)| \leq F_N(0).$$

So

$$\|F_N\|_\infty = N + 1.$$

Corollary 11.4.2.1
(F_N) is a positive summability kernel.

Proof
The lemma shows that $F_N \geq 0$.

We have already noted the first condition.

Property (iii) is satisfied since if $\delta > 0$ and $t \in [\delta, 2\pi - \delta]$

$$\left|\sin \frac{t}{2}\right| \geq \sin \frac{\delta}{2}.$$

Thus

$$\begin{aligned}
\int_\delta^{2\pi-\delta} |F_N(t)| dt &\leq \frac{1}{N+1} \int_\delta^{2\pi-\delta} \frac{\sin^2(N+1)t/2}{\sin^2 \delta/2} \\
&\leq \frac{1}{(N+1)\sin^2 \delta/2} \int_0^{2\pi} \sin^2 \xi + \cos^2 \xi && \xi := \frac{(N+1)t}{2} \\
&= \frac{1}{(N+1)\sin^2 \delta/2} \int_0^{2\pi} 1 \\
&= \frac{2\pi}{(N+1)\sin^2 \delta/2} \\
&\to 0 && N \to \infty
\end{aligned}$$

and (F_N) is indeed a positive summability kernel.

From now on, we write

$$\sigma_N(f) := F_N * f = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n) e^{int}.$$

Proposition 11.4.3
$(\sigma_N(f)) \to f$ uniformly for all $f \in C(\mathbb{T})$.

Proof
The Fejér kernel is a positive summability kernel.

Observe that this gives a constructive proof that $\mathrm{Trig}(\mathbb{T})$ is dense in $C(\mathbb{T})$ with respect to the sup norm.

Proposition 11.4.4
If

$$(S_N(f)(x_0)) \to L$$

then so does

$$(\sigma_N(f)(x_0)).$$

Consequently. if $f \in C(\mathbb{T})$ and $(S_N(f))$ converges pointwise, then

$$S_N(f)(x) \to f(x)$$

for all x.

11.5 L^1 Convergence

Theorem 11.5.1
If $f \in L^1(\mathbb{T})$ and (K_N) is a summability kernel, then

$$\|K_N * f - f\|_1 \to 0.$$

Proof
Get $g \in C(\mathbb{T})$ such that $\|f - g\|_1 < \frac{\epsilon}{M}$ where $M \geq 1$ satisfies property (ii) of the summability kernel.

We can choose N such that $\|K_n * g - g\|_1 \leq \|K_n * g - g\|_\infty < \epsilon$ for all $n \geq N$.

Then

$$\begin{aligned}
\|K_n * f - f\|_1 &\leq \|K_n * f - K_n * g\|_1 + \|K_n * g - g\|_1 + \|g - f\|_1 \\
&\leq \|K_n * (f - g)\|_1 + \epsilon + \frac{\epsilon}{M} \\
&\leq \|K_n\|_1 \|f - g\|_1 + \epsilon + \frac{\epsilon}{M} \\
&\leq M\frac{\epsilon}{M} + \epsilon + \frac{\epsilon}{M} \\
&\leq 3\epsilon
\end{aligned}$$

for all $n \geq N$.

Corollary 11.5.1.1
For all $f \in L^1$

$$\|\sigma_n(f) - f\|_1 \to 0.$$

Moreover, a subsequence $\sigma_{n_j}(f) \to f$ pointwise a.e.

Theorem 11.5.2 (Uniqueness)
If $f \in L^1$ and $\hat{f}(n) = 0$ for all integers n, then

$$f = 0$$

almost everywhere.

Proof
$\sigma_n(f) \to 0$ for all n. But

$$\sigma_n(f) \to f \in L^1$$

thus $f = 0$ in the L^1 sense (a.e.).

Proposition 11.5.3
Suppose $f \in L^p(\mathbb{T})$ for $1 \leq p < \infty$ and (K_n) is a summability kernel, then

$$\|K_n * f - f\|_p \to 0.$$

Chapter 12

More Fourier Series

12.1 A Divergent Construction

We now give a construction of a continuous function whose Fourier series diverges at 0.

12.1.1 Background

Recall that we constructed a sequence of continuous functions g_n such that $\|g_n\|_\infty = 1$ and

$$|S_n(g_n)(0)| \geq \frac{1}{2}\|D_n\|_1 \geq C \log n$$

for some $C > 0$.

12.1.2 Building Blocks

Put

$$f_n := \sigma_{2n^2}(g_n) = F_{2n^2} * g_n$$

where F_k is the k-th Fejér kernel.

This is a trigonometric polynomial of degree at most $2n^2$ satisfying

$$\|f_n\|_\infty = \|F_{2n^2} g_n\|_\infty \leq \|F_{2n^2}\|_1 \|g_n\|_\infty \leq 1.$$

By computation

$$
\begin{aligned}
|(S_n(f_n)-S_n(g_n))(t)| &= \left|\sum_{j=-n}^{n}(\hat{f}_n(j)-\hat{g}_n(j))e^{ijt}\right| \\
&= \left|\sum_{j=-n}^{n}(\widehat{F_{2n^2}}(j)\hat{g}_n(j)-\hat{g}_n(j))e^{ijt}\right| \\
&= \left|\sum_{j=-n}^{n}\left(\frac{|j|}{2n^2+1}\hat{g}_n(j)-\hat{g}_n(j)\right)e^{ijt}\right| \\
&\le \sum_{j=-n}^{n}\frac{|j||\hat{g}_n(j)|}{2n^2+1} \\
&\le \frac{2\|g_n\|_1}{2n^2+1}\sum_{j=1}^{n}|j| \\
&\le \frac{2}{2n^2+1}\cdot\frac{n(n+1)}{2} \\
&\le 1.
\end{aligned}
$$

Thus

$$|S_n(f_n)(0)| \ge |S_n(g_n)(0)| - 1 \ge C' \log n$$

for $C' > 0$ and n sufficiently large.

12.1.3 The Construction

Define $n_k := 2^{3^k}$ and put

$$f(t) := \sum_{k=1}^{\infty}\frac{1}{k^2}f_{n_k}(n_k t).$$

The choice of n_k is not critical. We require only that the sequence diverges rapidly.

Each f_{n_k} is continuous. Moreover, the sup norm of each term is at most $\frac{1}{k^2}$. This is a converging series and thus by the Weierstrass M-test,

$$f \in C(\mathbb{T}).$$

We have

$$f_{n_k}(t) = \sum_{j=-2n_k^2}^{2n_k^2}\widehat{f_{n_k}}(j)e^{ijt}$$

thus putting $G_k(t) := f_{n_k}(n_k t)$ gives

$$G_k(t) = \sum_{j=-2n_k^2}^{2n_k^2} \widehat{f_{n_k}}(j) e^{ijn_k t}$$

and

$$f(t) = \sum_{k \geq 1} G_k(t).$$

Observe that

$$\widehat{G_k}(jn_k) = \widehat{f_{n_k}}(j)$$

and $\widehat{G_k}(m) = 0$ if $m \notin n_k \mathbb{Z}$.

In particular it is zero for all $m \in (-n_k, n_k) \setminus \{0\}$. This means that if $N < n_k$, then $S_N(G_k)$ is the constant function

$$\widehat{G_k}(0) = \widehat{f_{n_k}}(0).$$

Also, G_k is trigonometric polynomial of degree at most $2n_k^3$.

If $k \geq M+1$, then

$$n_M^2 = 2^{2 \cdot 3^M} < 2^{3^{M+1}} \leq 2^{3^k} = n_k$$

hence

$$S_{n_M^2}(G_k)(t) = \widehat{G_k}(0) = \widehat{f_{n_k}}(0).$$

It follows by linearity that

$$\begin{aligned} S_{n_M^2}(f)(t) &= \sum_{k \geq 1} \frac{1}{k^2} S_{n_M^2}(G_k)(t) \\ &= \sum_{k=1}^{M} \frac{1}{k^2} D_{n_M^2} * G_k(t) + \sum_{k=M+1}^{\infty} \frac{1}{k^2} \widehat{f_{n_k}}(0). \end{aligned}$$

For any $k \leq M-1$, we have

$$n_M^2 = 2^{2 \cdot 3^M} > 2^{3^M + 1} \geq 2^{3^{k+1}+1} = 2n_k^3.$$

As G_k is of degree at most $2n_k^3$, we know that

$$D_{n_M^2} * G_k = G_k$$

for all $k \leq M-1$.

For the case when $k = M$, we have

$$\begin{aligned} D_{n_M^2} * G_k(t) &= D_{n_M^2} * G_M(t) \\ &= \sum_{j=-n_M^2}^{n_M^2} \widehat{G_M}(j) e^{ijt}. \end{aligned}$$

But $\widehat{G_M}(j) \neq 0$ only when $j \in n_M\mathbb{Z}$, thus evaluating at $t = 0$ gives

$$\begin{aligned} D_{n_M^2} * G_M(0) &= \sum_{j=-n_M}^{n_M} \widehat{G_M}(jn_M) \\ &= \sum_{j=-n_M}^{n_M} \widehat{f_{n_M}}(j) \\ &= D_{n_M} * f_{n_M}(0) \\ &= S_{n_M}(f_{n_M})(0). \end{aligned}$$

Putting everything together yields

$$\begin{aligned} S_{n_M^2}(f)(0) &= \sum_{k=1}^{M-1} \frac{1}{k^2} G_k(0) + \frac{D_{n_M^2} * G_M(0)}{M^2} + \sum_{k=M+1}^{\infty} \frac{1}{k^2} \widehat{f_{n_k}}(0) \\ &= \sum_{k=1}^{M-1} \frac{1}{k^2} G_k(0) + \frac{S_{n_M}(f_{n_M})(0)}{M^2} + \sum_{k=M+1}^{\infty} \frac{1}{k^2} \widehat{f_{n_k}}(0). \end{aligned}$$

12.1.4 Proof of Divergence

Since $\|f_n\|_\infty \leq 1$

$$|G_k(0)| = |f_{n_k}(0)| \leq \|f_{n_k}\|_\infty \leq 1$$

and

$$|\widehat{f_{n_k}(0)}| \leq \|f_{n_k}\|_\infty \leq 1.$$

But then

$$\begin{aligned} |S_{n_M^2}(f)(0)| &\geq \frac{|S_{n_M}(f_{n_M})(0)|}{M^2} - \sum_{k=1}^{M-1} \frac{1}{k^2} |G_k(0)| - \sum_{k=M+1}^{\infty} \frac{1}{k^2} \left| \widehat{f_{n_k}}(0) \right| \\ &\geq \frac{|S_{n_M}(f_{n_M})(0)|}{M^2} - \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &\geq \frac{C_1 \log n_M}{M^2} - C_2. \end{aligned}$$

But

$$\log n_M = \log 2^{3^M} = 3^M \log 2 >> M^2.$$

Consequently

$$\left| S_{n_M^2}(f)(0) \right| \to \infty$$

as $M \to \infty$.

This completes our construction.

12.2 Pointwise Convergence

Despite the fact that the Fourier series of even continuous functions need not converge, there are many circumstances under which the Fourier series is well behaved.

12.2.1 Differentiable-Like Functions

Theorem 12.2.1
Suppose $f \in L^1(\mathbb{T})$ is differentiable at $a \in \mathbb{T}$.
Then

$$S_n(f)(a) \to f(a).$$

Proof
Use the trick that $\int D_n = 1$ to write

$$\begin{aligned}
|S_n(f)(a) - f(a)| &= \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(a-t) D_n(t) dt - f(a) \right| \\
&= \left| \frac{1}{2\pi} \int_{\mathbb{T}} [f(a-t) - f(a)] D_n(t) dt \right| \\
&= \left| \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(a-t) - f(a)}{\sin t/2} \sin(n + 1/2) t dt \right|
\end{aligned}$$

Observe that $t \to 0$ gives

$$g(t) := \frac{f(a-t) - f(a)}{\sin t/2} = \frac{f(a-t) - f(a)}{t} \cdot \frac{t}{\sin t/2} \to -f'(a) \cdot 2.$$

Thus we can pick $\delta > 0$ sufficiently so that

$$|t| \leq \delta \implies |g(t)| \leq 2|f'(a)| + 1.$$

But then

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}}|g| &= \frac{1}{2\pi}\int_{[-\delta,\delta]}|g| + \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}|g| \\
&\le \frac{2\delta(2|f'(a)|+1)}{2\pi} + \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}\left|\frac{f(a-t)-f(a)}{\sin t/2}\right| \\
&\le C_1 + \frac{1}{2\pi}\cdot\frac{1}{\sin\delta/2}\int_{\delta}^{2\pi-\delta}|f(a-t)|+|f(a)|dt \\
&\le C_1 + C_2(\|f\|_1 + |f(a)|) \\
&< \infty.
\end{aligned}$$

Thus $g \in L^1$ and the Riemann-Lebesgue lemma implies

$$\int_{\mathbb{T}} g(t)\sin(n+1/2)tdt \to 0$$

as $n \to \infty$.

It follows that

$$|S_n(f)(a) - f(a)| \to 0.$$

Corollary 12.2.1.1
Suppose $f \in L^1(\mathbb{T})$ is constant on some open interval I, say $f\big|_I = c$.
Then

$$S_n(f)(x) \to c$$

for all $x \in I$.

Definition 12.2.1 (Lipschitz Condition)
We say f satisfies a Lipschitz condition of order $\alpha \ge 0$ at the point a if there is a constant c and $\delta > 0$ such that if $|t| \le \delta$, then

$$|f(a-t) - f(a)| \le c|t|^{\alpha}.$$

We write

$$f \in \operatorname{Lip}\alpha$$

for short.

For $\alpha = 0$, it is only required that f is bounded near a.

If $f \in \operatorname{Lip}\alpha$ for $\alpha > 0$, then f is continuous at α.

Proposition 12.2.2
If f is differentiable at a, then

$$f \in \operatorname{Lip} 1$$

at a.

Proof
By definition

$$\lim_{t \to 0} \frac{|f(a-t) - f(a)|}{|t|} = |f'(a)|.$$

Thus there is some $\delta > 0$ such that for all $|t| \leq \delta$

$$|f(a-t) - f(a)| \leq |t|(|f'(a)| + 1).$$

Proposition 12.2.3
If I is an open interval and $f : I \to \mathbb{R}$ such that

$$f \in \operatorname{Lip} \alpha$$

for $\alpha > 1$ at each $a \in I$, then f is constant on I.

Proof
$f \in \operatorname{Lip} \alpha$ for $\alpha > 0$ implies differentiability.

Observe that if $f \in \operatorname{Lip} \alpha$ for some $\alpha > 0$

$$\begin{aligned} |g(t)| &= \left| \frac{f(a-t) - f(a)}{t} \cdot \frac{t}{\sin t/2} \right| \\ &\leq C \frac{|t|^\alpha}{|t|} \left| \frac{t}{\sin t/2} \right| \\ &\leq C_1 |t|^{\alpha - 1}. \end{aligned}$$

Thus choosing a suitable t allows us to split the integral and show $g \in L^1$.

Proposition 12.2.4
If $f \in L^1$ and $f \in \operatorname{Lip} \alpha$ for some $\alpha > 0$,

$$S_n(f)(a) \to f(a).$$

Definition 12.2.2 (Right/Left Hand Derivative)
We say f has a right hand derivative at a if

$$\lim_{t\to 0^+} \frac{f(a+t)-f(a^+)}{t}$$

exists.
Here

$$f(a^+) := \lim_{t\to 0^+} f(a+t).$$

A piecewise differentiable function has left and right hand derivatives.

Proposition 12.2.5
If $f \in L^1$ is continuous at a and has a right and left hand derivative at a, then

$$S_n(f)(a) \to f(a).$$

Proof
We have

$$f \in \operatorname{Lip} 1$$

at a.

12.2.2 Discontinuous Functions

Theorem 12.2.6 (Fejér)
Let f be integrable. Fix $a \in \mathbb{T}$ and assume

$$\lim_{t\to 0^+} \frac{f(a+t)+f(a-t)}{2} =: L$$

exists.
Then

$$\sigma_n(f)(a) \to L.$$

Proof
Remark that F_n is an even function, thus we have

$$\begin{aligned}\sigma_n(f)(a) &= \frac{1}{2\pi}\int_0^\pi f(a-t)F_n(t)dt + \frac{1}{2\pi}\int_{-\pi}^0 f(a-t)F_n(t)dt \\ &= \frac{1}{\pi}\int_0^\pi f(a-t)F_n(t)ddt.\end{aligned}$$

Similarly

$$1 = \frac{1}{2\pi}\int_{-\pi}^\pi F_n(t)dt = \frac{1}{\pi}\int_0^\pi F_n(t)dt.$$

Fix $\epsilon > 0$, there is some $\delta > 0$ such that $0 < t < \delta$ implies

$$\left|\frac{f(a-t)+f(a+t)}{2} - L\right| < \epsilon.$$

Write

$$\begin{aligned}|\sigma_n(f)(a) - L| &= \left|\frac{1}{2\pi}\int_0^\pi (f(a-t)+f(a+t))F_n(t)dt - \frac{L}{\pi}\int_0^\pi F_n(t)dt\right| \\ &\leq \frac{1}{\pi}\left|\int_0^\pi \left(\frac{f(a-t)+f(a+t)}{2} - L\right) F_n(t)dt\right| \\ &\leq \int_0^\delta \cdots + \int_\delta^\pi \cdots \\ &:= I + J.\end{aligned}$$

It is clear that

$$I \leq \frac{1}{\pi}\int_0^\delta \epsilon F_n(t)dt \leq \epsilon.$$

To bound J observe that for $\delta \leq t \leq \pi$

$$F_n(t) = \frac{1}{n+1}\left(\frac{\sin(n+1)t/2}{\sin t/2}\right)^2 \leq \frac{1}{n+1}\cdot\frac{1}{\sin^2\delta/2}.$$

Thus there is some constant C such that

$$\begin{aligned}J &\leq \frac{1}{\pi(n+1)\sin^2\delta/2}\int_\delta^\pi \left|\frac{f(a-t)+f(a+t)}{2} - L\right| \\ &\leq \frac{C}{n+1}(\|f\|_1 + |L|) \\ &< \epsilon\end{aligned}$$

for n sufficiently large.

The result follows by the arbitrary choice of ϵ.

Corollary 12.2.6.1
If f is continuous at a, then

$$\sigma_n(f)(a) \to f(a).$$

Corollary 12.2.6.2
If

$$\lim_{t\to 0^+} f(a+t) = f(a^+)$$

and

$$\lim_{t\to 0^-} f(a+t) = f(a^-)$$

exist, then

$$L := \frac{f(a^+) + f(a^-)}{2}$$

and

$$\sigma_n(f)(a) \to L.$$

Observe that if f has a jump discontinuity at a, then $\sigma_n(f)(a)$ converges to the "average of the jump discontinuity".

Corollary 12.2.6.3
If f is increasing, then

$$\sigma_n(f)(a) \to \frac{f(a^+) + f(a^-)}{2}.$$

Corollary 12.2.6.4
Suppose $f \in L^1(\mathbb{T})$ is either continuous at a or has a jump discontinuity at a.
If $(S_n(f)(a))_n$ converges, then

$$S_n(f)(a) \to \frac{f(a^+) + f(a^-)}{2}.$$

Here are some results ensuring the convergence of $(S_n(f)(a))_n$.

Theorem 12.2.7 (Jordan)
Suppose f is integrable, has a jump discontinuity at a, and there are constants c and $\delta > 0$ such that

$$|f(a+t) - f(a^+)|, |f(a-t) - f(a^-)| \leq c|t|$$

for all $0 < t < \delta$.
Then

$$S_n(f)(a) \to \frac{f(a^+) + f(a^-)}{2}.$$

This can be shown using the Riemann-Lebesgue lemma in the same spirit as previous proofs.

Theorem 12.2.8 (Hardy Tauberian)
Let f be integrable and assume there is a constant c such that

$$\left|\hat{f}(n)\right| \leq \frac{c}{|n|}$$

for all $n \neq 0$.
The sequence

$$(S_n(f)(a))_n$$

converges if and only if the sequence

$$(\sigma_n(f)(a))_n$$

converges.
Furthermore, if $(\sigma_n(f))_n$ converges on some interval, so does $(S_n(f))_n$.

12.3 Pointwise Convergence within L^p

We now prove the remarkable fact that the Fourier series of an L^p function converges in L^p norm for every $1 < p < \infty$.

12.3.1 The Hilbert Transform

Definition 12.3.1 (Hilbert Transform)
The Hilbert transform $H : \mathrm{Trig}(\mathbb{T}) \to \mathrm{Trig}(\mathbb{T})$ acts on the trigonometric polynomial f such that

$$\widehat{H(f)}(n) = (-i)\,\mathrm{sgn}(n)\hat{f}(n).$$

Observe that Hf is the trig polynomial of the same degree as f given by

$$Hf(t) := \sum_{j=-\infty}^{-1} i\hat{f}(j)e^{ijt} + \sum_{i=1}^{\infty} -i\hat{f}(j)e^{ijt}.$$

Since computing Fourier coefficients is a linear operation, so is the Hilbert transform.

Proposition 12.3.1
We have

$$f + iHf = \hat{f}(0) + 2\sum_{j=1}^{\infty} \hat{f}(j)e^{ijt}.$$

There is a natural extension of H to a linear operator on L^2 with Parseval's theorem implying

$$\|Hf\|_2 \leq \|f\|$$

with equality if $\hat{f}(0) = 0$. This us because the Fourier series of an L^2 function is square-summable. Thus $H : L^2 \to L^2$ is a bounded map.

Remark that

$$\begin{aligned} e^{iNt}f(t) &= e^{iNt}\sum_{j=-\infty}^{\infty} \hat{f}(j)e^{ijt} \\ &= \sum_{j=-\infty}^{\infty} \hat{f}(j)e^{i(j+N)t} \\ &= \sum_{j=-\infty}^{\infty} \hat{f}(j-N)e^{ijt}. \end{aligned}$$

This yields the equality

$$H(e^{iNt}f) = \sum_{j=-\infty}^{-1} i\hat{f}(j-N)e^{ijt} + \sum_{j=1}^{\infty} -i\hat{f}(j-N)e^{ijt}.$$

Similarly,

$$\begin{aligned} e^{-iNt}H(e^{iNt}f) &= \sum_{j=-\infty}^{-N-1} i\hat{f}(j)e^{ijt} + \sum_{j=-N+1}^{\infty} -i\hat{f}(j)e^{ijt} \\ e^{iNt}H(e^{-iNt}f) &= \sum_{j=-\infty}^{N-1} i\hat{f}(j)e^{ijt} + \sum_{j=N+1}^{\infty} -i\hat{f}(j)e^{ijt}. \end{aligned}$$

Hence

$$\begin{aligned} S_{N-1}(f) &= \frac{-1}{2i}\left(-\sum_{-N}^{N-1} i\hat{f}(j)e^{ijt} - \sum_{j=-N+1}^{N} i\hat{f}(j)e^{ijt} + i\hat{f}(N)e^{iNt} + i\hat{f}(-N)e^{-iNt}\right) \\ &= \frac{-1}{2i}\left(e^{-iNt}H(e^{iNt}f) - e^{iNt}H(e^{-iNt}f)\right) + \frac{-1}{2i}\left(i\hat{f}(N)e^{iNt} + i\hat{f}(-N)e^{-iNt}\right). \end{aligned}$$

For all functions F, $\|e^{iNt}F\|_p = \|F\|_p$. Thus assuming $H : L^p \to L^p$ is a bounded operator,

$$\begin{aligned}
\|S_{N-1}(f)\|_p &\leq \frac{1}{2}\left(\|e^{iNt}H(e^{iNt}f)\|_p + \|e^{iNt}H(e^{-iNt}f)\|_p\right) + \left|\hat{f}(N)\right| + \left|\hat{f}(-N)\right| \\
&\leq \frac{1}{2}\left(2\|H\|_{\text{op}}\|f\|_p + 2\|f\|_p\right) \\
&\leq (1 + \|H\|_{\text{op}})\|f\|_p.
\end{aligned}$$

This is relevant due to the following lemma.

Lemma 12.3.2
$\|S_N(f) - f\|_p \to 0$ for all $f \in L^p$ if and only if there is some C such that

$$\|S_N(f)\|_p \leq C\|f\|_p$$

for all N and $f \in L^p$.

Proof
($\implies$) If $\|S_N(f) - f\|_p \to 0$, then for each $f \in L^p$ we have

$$\sup_N \|S_n(f)\|_p < \infty.$$

By the uniform boundedness principle, there we have

$$C := \sup_N \|S_N\|_{\text{op}} < \infty.$$

Thus

$$\|S_N(f)\|_p \leq C\|f\|_p$$

for every N and $f \in L^p$ as desired.

($\impliedby$) Suppose now that there is some

$$\|S_N(f)\|_p \leq C\|f\|_p$$

for all N, f.

We have

$$\begin{aligned}
\|S_N(f) - f\|_p &\leq \|S_N(f) - S_N(F_N * f)\|_p + \|S_N(F_N * f) - f\|_p \\
&\leq \|S_N(f - F_N * f)\|_p + \|F_N * f - f\|_p && \deg F_N * f = N \\
&\leq C\|f - F_N * f\|_p + \|F_N * f - f\|_p \\
&\leq (C+1)\|F_N * f - f\|_p \to 0. && N \to \infty
\end{aligned}$$

Here the limit is justified as F_N is a summability kernel.

Corollary 12.3.2.1
If $H : L^p \to L^p$ is a bounded operator on L^p, then $S_N(f) \to f$ in L^p norm for all $f \in L^p$.

12.3.2 p is an Even Integer

Note that as $L^{2k} \subseteq L^2$ for all $k \geq 1$, the action Hf is defined for $f \in L^{2k}$.

Theorem 12.3.3
Let $k \in \mathbb{N}$. There is a constant C_k such that

$$\|Hf\|_{2k} \leq C_k \|f\|_{2k}$$

for all $f \in L^{2k}$.

Proof
By taking $g(x) := f(x) - \hat{f}(0)$ if necessary, we may assume without loss of generality that

$$\hat{f}(0) = 0.$$

Notice that $Hg = Hf$ since $\hat{g}(n) = \hat{f}(n)$ for all $n \neq 0$. Moreover, $\|g\|_p \leq 2\|f\|_p$.

There is also no loss in assuming f is real-valued, since if the result holds for the real-valued functions,

$$\begin{aligned}
\|H(\operatorname{Re} f + i \operatorname{Im} f)\|_p &\leq \|H \operatorname{Re} f\|_p + \|H \operatorname{Im} f\|_p \\
&\leq C_k(\|\operatorname{Re} f\|_p + \|\operatorname{Im} f\|_p) \\
&\leq 2C_k \||f|\|_p \\
&= 2C_k \|f\|_p.
\end{aligned}$$

Recall that $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$. Hence for f real-valued, we have $\hat{f}(n) = \overline{\hat{f}(-n)}$.

From the definition of H, we have

$$\begin{aligned}
Hf &:= \sum_{j=1}^{\infty} i\overline{\hat{f}(j)}e^{ijt} + \sum_{j=1}^{\infty} -i\hat{f}(j)e^{ijt} \\
&= 2 \operatorname{Re} \left(\sum_{j=1}^{\infty} -i\hat{f}(j)e^{ijt} \right)
\end{aligned}$$

so Hf is real.

Recall also that we computed

$$f + iHf = 2\sum_{j=1}^{\infty} \hat{f}(j)e^{ijt}.$$

Since all coefficients in this sum are positive, the product

$$(f + iHf)^{2k}$$

does not contain the term $\hat{f}(0)$. Taking powers of infinite series is justified here through through the convergence of the Fourier series.

It follows that

$$\begin{aligned}\int_{\mathbb{T}} (f + iHf)^{2k} = 0 \\ &= \operatorname{Re} \int_{\mathbb{T}} (f + iHf)^{2k} \\ &= \int_{\mathbb{T}} \operatorname{Re}(f + iHf)^{2k}.\end{aligned}$$

The binomial expansion yields

$$(f + +iHf)^{2k} = \sum_{n=0}^{2k} \binom{2k}{n} f^n (iHf)^{2k-n}.$$

So

$$\begin{aligned}\operatorname{Re}(f + iHf)^{2k} &= \sum_{m=0}^{k} \binom{2k}{2m} f^{2m}(i)^{2k-2m}(Hf)^{2k-2m} \\ &= \sum_{m=0}^{k} \binom{2k}{2m} f^{2m}(-1)^{k-m}(Hf)^{2k-2m}.\end{aligned}$$

Since $\int_{\mathbb{R}} \operatorname{Re}(f + iHf)^{2k} = 0$, we must have

$$\int \sum_{m=1}^{k} \binom{2k}{2m} f^{2m}(-1)^{2k-2m}(Hf)^{2k-2m} = (-1)^{k-1} \int (Hf)^{2k}.$$

But Hf is real so

$$\begin{aligned}\int (Hf)^{2k} &= \int |Hf|^{2k} \\ &\leq \int \sum_{m=1}^{k} \binom{2k}{2m} |f|^{2m} |Hf|^{2k-2m}.\end{aligned}$$

Applying Hölder's inequality with $p = \frac{2k}{2m}$ and dual index $q = \frac{2k}{2k-2m}$ and applying standard computations

$$\|Hf\|_{2k}^{2k} \leq \sum_{m=1}^{k} \binom{2k}{2m} \|f\|_{2k}^{2m} \|Hf\|_{2k}^{2k-2m}.$$

Dividing through by $\|f\|_{2k}^{2k}$ yields

$$\left(\frac{\|Hf\|_{2k}}{\|f\|_{2k}}\right)^{2k} \leq \sum_{m=1}^{k} \binom{2k}{2m} \left(\frac{\|Hf\|_{2k}}{\|f\|_{2k}}\right)^{2k-2m}.$$

Put

$$R_k := \frac{\|Hf\|_{2k}}{\|f\|_{2k}}$$

so that we can view this as

$$R_k^{2k} \leq \sum_{m=1}^{k} \binom{2k}{2m} R_k^{2k-2m}.$$

If $R_k \leq 1$, then we are done. Observe that any constant bounding R_k would be the desired constant.

Otherwise,

$$\begin{aligned} R_k^{2k} &\leq R_k^{2k-2} \sum_{m=1}^{k} \binom{2k}{2m} & \qquad R_k^{2k-2m} \leq R_k^{2k-2} \\ &\leq 2^{2k} R_k^{2k-2}. \end{aligned}$$

Thus

$$R_k^2 \leq 2^{2k}.$$

Observe that this is independent of the choice of f. Taking square roots yields the desired result.

12.3.3 $1 < p < \infty$

Let us now extend the result to all $1 < p < \infty$.

Proposition 12.3.4
For any $p < \infty$, constant C, and $F \in L^p$,

$$m\{|f| > C\} \leq \frac{\|F\|_p^p}{C^p}.$$

Proposition 12.3.5
For all $p < \infty$

$$\|f\|_p^p = p \int_0^\infty t^{p-1} m\{|f| > t\} dt.$$

Proof
Fubini's Theorem.

Theorem 12.3.6
For every $1 < p < \infty$, there is a constant C such that

$$\|Hf\|_p \leq C_p \|f\|_p$$

for all $f \in L^p$.

Proof
We have already done the proof for positive even integers.

Case I: $p > 2$ Pick an integer k such that $p < 2k$. Let $f \in L^p$. For $\alpha > 0$, define

$$f^\alpha(x) := \begin{cases} f(x) & |f(x)| \geq \frac{\alpha}{2} \\ 0, & \text{else} \end{cases}.$$

as well as

$$f_\alpha(x) := f - f^\alpha.$$

We have

$$\begin{aligned} \|f_\alpha\|_{2k}^{2k} &= \int |f_\alpha|^{2k} \\ &= \int_{\{|f| < \alpha/2\}} |f|^{2k-p} |f|^p \\ &\leq \left(\frac{\alpha}{2}\right)^{2k-p} \|f\|_p^p \\ &< \infty. \end{aligned}$$

as well as

$$\begin{aligned}\|f^\alpha\|_2^2 &= \int_{\{|f|\geq\alpha/2\}} |f|^{2-p}|f|^p \\ &\leq \left(\frac{\alpha}{2}\right)^{2-p} \|f\|_p^p && 2-p<0 \\ &< \infty.\end{aligned}$$

We know that $H : L^{2k} \to L^{2k}$ is a bounded operator, hence a previous proposition shows

$$m\{|Hf_\alpha| > \alpha/2\} \leq \frac{\|Hf_\alpha\|_2^2}{(\alpha/2)^{2k}} \leq \frac{C_k}{\alpha^{2k}}\|f_\alpha\|_{2k}^{2k}.$$

Similarly

$$m\{|Hf^\alpha| > \alpha/2\} \leq \frac{\|Hf^\alpha\|_2^2}{(\alpha/2)^2} \leq \frac{C}{\alpha^2}\|f^\alpha\|_2^2.$$

Put

$$\begin{aligned}E_\alpha &:= \{|Hf| > \alpha\} \\ E_{1,\alpha} &:= \{|Hf_\alpha| > \alpha/2\} \\ E_{2,\alpha} &:= \{|Hf^\alpha| > \alpha/2\}.\end{aligned}$$

Then $E_\alpha \subseteq E_{1,\alpha} \cup E_{2,\alpha}$ so that $mE_\alpha \leq mE_{1,\alpha} + mE_{2,\alpha}$.

Another previous proposition shows that

$$\begin{aligned}\|Hf\|_p^p &= p\int_0^\infty \alpha^{p-1} mE_\alpha d\alpha \\ &\leq p\int_0^\infty \alpha^{p-1}\left(\frac{C_k}{\alpha^{2k}}\|f_\alpha\|_{2k}^{2k} + \frac{C}{\alpha^2}\|f^\alpha\|_2^2\right). && mE_{1,\alpha} + E_{2,\alpha}\end{aligned}$$

An application of Fubini's theorem shows

$$\begin{aligned}p\int_0^\infty \alpha^{p-1}\frac{C_k}{\alpha^{2k}}\|f_\alpha\|_{2k}^{2k} d\alpha &= C(p)\int_0^\infty \alpha^{p-1-2k}\left(\int_{\mathbb{T}} |f_\alpha(x)|^{2k} dm\right) d\alpha \\ &= C(p)\int_0^\infty \alpha^{p-1-2k}\left(\int_{\{|f|\leq\alpha/2\}} |f_\alpha(x)|^{2k} dm\right) d\alpha \\ &= C(p)\int_{\mathbb{T}} |f|^{2k}\left(\int_{2|f(x)|}^\infty \alpha^{p-1-2k} d\alpha\right) dm. && (*) \\ &= C(p)\int_{\mathbb{T}} |f|^{2k}\left(\frac{\alpha^{p-2k}}{p-2k}\bigg|_{2|f(x)|}^\infty\right) dm\end{aligned}$$

(∗) Here we fix x and integrate over α then integrate over x.

Since $p < 2k$, evaluating the inner integral yields

$$p\int_0^\infty \alpha^{p-1}\frac{C_k}{\alpha^{2k}}\|f_\alpha\|_{2k}^{2k} \leq C_1(p)\int_{\mathbb{T}}|f|^{2k+p-2k} = C_1(p)\|f\|_p^p.$$

Here $C_1(p)$ is a new constant depending only on p.

A similar argument shows that

$$p\int_0^\infty \alpha^{p-1}\frac{C}{\alpha^2}\|f^\alpha\|_2^2 \leq C_2(p)\|f\|_p^p$$

and therefore

$$\|Hf\|_p^p \leq C(p)\|f\|_p^p.$$

This shows the result for $p > 2$. To see the result for $1 < p < 2$, we rely upon another result from Functional Analysis.

We can define a bounded linear operator $H^* : L^p \to L^p$ by the rule that H^*f is the function with the property that

$$\int H^*f\bar{g} = \int f\overline{Hg}$$

for all $q \in L^q$ where q is the dual index to p.

Moreover, the operator norm of H^* coincides with the operator norm of H when viewed as a map $L^q \to L^q$. The latter is bounded since $1 < p < 2$ implies $2 < q < \infty$.

For all trigonometric polynomials f, g, Parseval's theorem gives

$$\begin{aligned}\int -Hf\bar{g} &= \sum_n \widehat{-Hf}(n)\overline{\hat{g}(n)} \\ &= \sum_n i\cdot\operatorname{sgn}(n)\hat{f}(n)\overline{\hat{g}(n)} \\ &= \sum_n \hat{f}(n)\overline{(-i)\operatorname{sgn}(n)\hat{g}(n)} \\ &= \sum_n \hat{f}(n)\overline{\widehat{Hg}(n)} \\ &= \int f\overline{Hg}.\end{aligned}$$

It follows that

$$H^* = -H$$

and $\|-H\|_{\text{op}} = \|H\|_{\text{op}} < \infty$.

The Hilbert transform is an important operator used in the study of many problems of Fourier analysis.

Corollary 12.3.6.1
For every $1 < p < \infty$,

$$S_N(f) \to f$$

in L^p for all $f \in L^p$.