PMATH351: Real Analysis

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University of Waterloo

*from Professor Kenneth Davidson's Lectures

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1 Preface

1.1 Ken Davidson

MC 5324

Office hours W 10 30 - 11 30

1.2 Website

`www.math.uwaterloo.ca/~krdavids/`

1. post assignments on website
2. course outline
3. solutions, supplementary material

1.3 Crowdmark

for submitting assignments

1.4 Learn

only for posting grades

1.5 No Text

1.6 Reference

1. Garling, a course in mathematical analysis v.2
2. Elementary real analysis by Thompson, Bruckner & Bruckner
3. Carothers, Real analysis
4. D. & Donsig, Real analysis & real applications (with missing material on the website)
5. L. Marcoux, PMATH351 Notes on his website
6. B. Forrest, PMATH 351 Notes

1.7 Course Outline

1.7.1 Metric Spaces

1. topology, open & closed sets, convergence, completeness, compactness
2. more topology, continuity, more completeness, compactness, $C(K)$, connectedness
3. completions of metric spaces, constructions of $\mathbb{R}$
4. polynomial approximations & Stone-Weierstrass Theorem
5. existence and uniqueness for ODEs

1.8 Grades

1.8.1 Assignments

6-8 worth 25%

1.8.2 Midterm

(TBA) late October 25%

1.8.3 Final Exam

50 %

2 The Axiom of Choice, Zorn's Lemma, and Cardinality

2.1 Notation

Definition 2.1.1 ($\mathbb{N}$)
The Natural Numbers NOT including 0

Definition 2.1.2 ($\mathbb{Z}$)
The Integers

Definition 2.1.3 ($\mathbb{Q}$)
The Rationals

Definition 2.1.4 ($\mathbb{R}$)
The Reals

Definition 2.1.5 (subset)
For sets A, B we say A is a subset of B and write $A \subseteq B$ if

$$a \in A \implies a \in b$$

Definition 2.1.6 (strict subset)
We write $A \subset B$ if

$$A \subseteq B \wedge A \neq B$$

We will use the subset notation for most of this class and explicitly stating that $A \neq B$ if we require a strict subset.

Definition 2.1.7 (Power Set)

$$\mathcal{P}(X) = \{A | A \subseteq X\}$$

Definition 2.1.8 (Union)

$$A \cup B = \{x | x \in A \vee x \in B\}$$

In general,

$$\bigcup_{\alpha \in I} A_\alpha = \{x | \exists \alpha, x \in A_\alpha\}$$

Definition 2.1.9 (Intersection)

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

In general,

$$\bigcap_{\alpha \in I} A_\alpha = \{x | \forall \alpha, x \in A_\alpha\}$$

Definition 2.1.10 (Complement)
Let A, B be sets, then

$$A \setminus B = \{a \in A | a \notin B\}$$

Note that if $B \subseteq A$ then

$$A \setminus B = A^c$$

is the complement of A in B.

Theorem 2.1.1 (DeMorgan's Law)

$$\left(\bigcup_{\alpha \in I} A_\alpha\right)^c = \bigcap_{\alpha \in I} (A_\alpha^c)$$
$$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} (A_\alpha^c)$$

Proof
This follows from definition. If a is in the complement of a union of sets A_α, then $a \notin A_\alpha$ for all α.

Similarly, if a is in the complement of the intersection of sets A_α, then $a \in A_\alpha$ for some α.

2.2 Products and the Axiom of Choice

Definition 2.2.1 (Cartesian Product)
For sets $X_1, \dots, X_n$, define their (Cartesian) Product by

$$X_1 \times \dots, X_n = \prod_{i=1}^{n} X_i = \{(x_1, \dots, x_n | x_i \in X_i)\}$$

We write X^n for the n-th product of the same set X.

Definition 2.2.2 (Finite Cardinality)
For a finite set X let

$$|X| = \sum_{x \in X} 1$$

Theorem 2.2.1
Let $\{X_i\}$ be a finite collection of finite sets, then

$$\left|\prod X_i\right| = \prod |X_i|$$

How do we determine the product of arbitrary sets?

Note each $(x_i) \in \prod X_i$ defines a function

$$\begin{aligned} f_{(x_1,\dots,x_n)} : [n] &\to \bigcup X_i \\ i &\mapsto x_i \end{aligned}$$

Now, given $f : [n] \to \bigcup X_i$ and $f(i) \in X_i$, we can define $(x_i) \in \prod X_i$ by

$$x_i = f(i)$$

Here, f is a choice function.

Definition 2.2.3 (Arbitrary Product)
For a collection of sets $\{X_\alpha\}, \alpha \in I$, define

$$\prod_{\alpha \in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha \in I} X_\alpha | f(\alpha) \in X_\alpha \right\}$$

The definition above then begs the question whether for an arbitrary collection of sets $\{X_\alpha\}, \alpha \in I$ is

$$\prod X_\alpha = \varnothing$$

Axiom 2.2.1 (Zermelo's Axiom of Choice)
For an arbitrary collection of sets $\{X_\alpha\}, \alpha \in I$

$$\prod X_\alpha \neq \varnothing$$

The above is equivalent to the Axiom of Choice, stated below.

Axiom 2.2.2 (Axiom of Choice)
For any nonempty set X there is a function

$$f : \mathcal{P}(X) \setminus \varnothing \to X$$

such that for ever $\varnothing \neq A \subseteq X, f(A) \in A$

2.3 Relations & Zorn's Lemma

Definition 2.3.1 (Relation)
Given sets X, Y, A relation is a subset $R \subseteq X \times Y$. We normally write xRy if $(x, y) \in R$.

If $X = Y$, we say R determines a relation on X.

Definition 2.3.2 (Reflexive)
A relation R on X, is reflective if

$$xRx$$

for all $x \in X$.

Definition 2.3.3 (Symmetric)
A relation R on X, is symmetric if

$$xRy \implies yRx$$

for all $x, y \in X$.

Definition 2.3.4 (Anti-Symmetric)
A relation R on X, is reflective if

$$xRy \wedge yRx \implies x = y$$

for all $x, y \in X$.

Definition 2.3.5 (Transitive)
A relation R on X, is reflective if

$$xRy \wedge yRz \implies xRz$$

for all $x, y, z \in X$.

Definition 2.3.6 (Partial Order)
A relation R on a set X is a partial order of X if it is

1. reflexive
2. anti-symmetric
3. transitive

We say (X, R) is a partially ordered set or poset for short.

Definition 2.3.7 (Total Order)
If in addition to the definition above, we have for all $x, y \in X$

$$xRy \vee yRx$$

then R is said to be a total order.

Definition 2.3.8 (Upper Bound)
Let $(X, \preceq)$ be a poset, $A \subseteq X$.
We say $x \in X$ is an upper bound of A, if

$$y \preceq x, \forall y \in A$$

We refer to a toally ordered set as a chain for short.

Definition 2.3.9 (Supremum, Least Upper Bound)
If in addition to the definition above, $x \preceq z$ for all upper bounds z, then x is the least upper bound of A, or the supremum of A.

We denote the supremum x of a set A with $\sup(A)$ and if $x \in A$, then with $\max(A)$.

Definition 2.3.10 (Lower Bound)

Definition 2.3.11 (Infimum, Greatest Lower Bound)

We denote the infimum x of a set A with $\inf(A)$ and if $x \in A$, then with $\min(A)$.

Axiom 2.3.1 (Least Upper Bound)
Let $(\mathbb{R}, \leq)$ be the canonical chain.
If $A \subseteq \mathbb{R}$ has an upper bound, it must have a supremum.

Definition 2.3.12 (Maximal)
Let $(X, \preceq)$ be a poset.
$x \in X$ is maximal if

$$x \leq y \implies x = y$$

Proposition 2.3.1
Any finite non-empty poset has a maximal element.

Axiom 2.3.2 (Zorn's Lemma)
Let $(X, \preceq)$ be a non-empty poset.
If every totally ordered subset of X has an upper bound, then the poset has a maximal element.

Zorn's Lemma is logically equivalent to the Axiom of Choice!

Definition 2.3.13 (Basis)
Let V be a non-zero vector space. Note that $(V, \subseteq)$ is a poset ordered by inclusion. Let

$$L := \{A \subseteq V | A \text{ is linearly independent}\}$$

then a basis is simply a maximal element of L

Theorem 2.3.2
Every nonempty vector space has a basis.

Proof
Let $C = \{A_\alpha | \alpha \in I\}$ be a chain in L

Let

$$A := \bigcup_{\alpha \in I} A_\alpha$$

We claim A is linearly independent.

Indeed, suppose $x_1, \ldots, x_n \in A$ and $\beta_i \in \mathbb{R}$ such that

$$\sum \beta_i x_i = 0$$

Now, for every x_i, there is some $\alpha_i \in I$ such that

$$x_i \in A_{\alpha_i}$$

But since the A_{α_i}'s are part of a coset, there is some j such that

$$\forall i, A_{\alpha_i} \subseteq A_{\alpha_j}$$

But since A_{α_j} is linearly independent, so are the x_i's as they are included inside our maximal element.

Now by Zorn's Lemma, every chain must have an upper bound, so vector by definition has a basis!

Definition 2.3.14 (Well-Ordered)
A poset is well ordered if every non-empty subset has a least element.

Axiom 2.3.3 (Well-Ordering Principle)
Every set X has a partial order $\preceq$ such that $(X, \preceq)$ is well-ordered.

Theorem 2.3.3
TFAE:

1. The Axiom of Choice
2. Zorn's Lemma
3. Well Ordering Principle

Proof (3 $\implies$ 1)
We can define a choice function on the powerset by taking the least element.

2.4 Equivalence Relations & Cardinality

Definition 2.4.1 (Equivalence Relation)
A relation $\sim$ on a set X is an equivalence relation if it is

1. reflexive
2. symmetric
3. transitive

Given some $x \in X$, we define

$$[x] := \{y \in X | y \sim x\}$$

to be its equivalence class, for which x is a representative of the equivalence class.

Proposition 2.4.1

1. no equivalence classes are empty
2. any two non equal equivalence classes are mutually exclusive
3. the union of all equivalence classes is the set itself

Definition 2.4.2 (Partition)
A parition of a set X is a collection

$$P := \{A_\alpha \subseteq X | \alpha \in I\}$$

such that

1. $\forall \alpha, A_\alpha \neq \varnothing$
2. $\alpha \neq \beta \in I \implies A_\alpha \cap A\beta = \varnothing$
3. $X = \bigcup A_\alpha$

Note that this means any equivalence relation induces a partition!

Definition 2.4.3 (Equivalence)
We say two sets X, Y are equivalent if they are bijective and write $X \sim Y$.

Proposition 2.4.2
$[n]$ is not equivalent to any subset of itself.

Corollary 2.4.2.1
Any finite set is then not equivalent to any subset of itself.

Note that the above is simply a restatement of the Pigeonhole Principle!

Definition 2.4.4 (Countable)
A set X is countable if $|X| \leq |\mathbb{N}|$, else it is uncountable.

Proposition 2.4.3
Every infinite set contains a countably infinite subset.

Proof
Axiom of Choice

Corollary 2.4.3.1
A set is infinite if and only if it is equivalent to a proper subset of itself.

Proof
Define a function which maps the countably infinite subset to its neighbour one down, elsewise acting as the identity.

Informally, if there is an injective function $f : X \to Y$, we write

$$|X| \preceq |Y|$$

Theorem 2.4.4 (Cantor-Schroeder-Bernstein)
Let $A_2 \subseteq A_1 \subseteq A_0 = A$.
If $A \sim A_2$, then $A_0 \sim A_1$.

Proof
Assume that $\phi : A_0 \to A_2$ is bijective.

We can recursively define

$$A_{n+2} = \phi(A_n)$$

We then have a sequence of $\{A_n\}$ such that

$$A = (A_0 \setminus A_1) \cup (A_1 \setminus A_2) \cup \cdots \cup A_\infty$$

Where

$$A_\infty = \bigcap_{n=0}^{\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

Note that $A_3 = \phi(A_1) \subseteq A_2$, since $\phi(A_1) \subseteq \phi(A_0) = A_2$.

This continues as $A_4 = \phi(A_2) \subseteq A_3$, since $\phi(A_2) \subseteq \phi(A_1) = A_3$, etc.

Define

$$M := \bigcup_{k=0}^{\infty} A_{2k} \setminus A_{2k+1}$$
$$M_1 := \bigcup_{k=1}^{\infty} A_{2k} \setminus A_{2k+1}$$

Define the odd sequence to be

$$N := \bigcup_{k=0}^{\infty} A_{2k+1} \setminus A_{2k+2}$$

and the odd sequence similarly.

Note

$$A_0 = M \cup N \cup A_\infty$$
$$A_1 = M_1 \cup N \cup A_\infty$$

By construction, we have that

$$(A_k \setminus A_{k+1}) \sim (A_{k+2} \setminus A_{k+3})$$

This is due to ϕ.

We can now define

$$\begin{aligned} f :& A_0 \to A_1 \\ f(x) =& \begin{cases} \phi(x), & x \in M \\ x, & x \in N \\ x, & x \in A_\infty \end{cases} \end{aligned}$$

It remains to show that f is bijective. It is certainly bijective when evaluated with restriction to the odd sequences and A_∞, we essentially shift the even sequences one down, with ϕ and that is a bijection, so we also have a bijection with restriction to the even sequences.

It follows that f is bijective and we are done by definition.

Corollary 2.4.4.1
Let $A' \subseteq A, B' \subseteq B, A \sim B', B \sim A'$, then

$$A \sim B$$

Proof
Let $f : A \to B_1, g : B \to A_1$ be bijective.

Let $A_2 = g \circ f(A) = g(B_1)$.

By Cantor-Schroeder-Bernstein, we have $A_1 \sim A$ as

$$A_2 \subseteq A_1 \subseteq A \wedge A_2 \sim A$$

But $A_1 \sim B$ so the result follows.

Corollary 2.4.4.2
If $|X| \leq |Y|, |Y| \leq |X|$, then $|X| = |Y|$.

Proof
if $|X| \leq |Y|$, $\exists Y_1 \subseteq Y, X \sim Y_1$.

Similary, $\exists X_1 \subseteq X, Y \sim X_1$.

The result follows from the previous corollary.

Corollary 2.4.4.3
An infinite set X is countably infinite if and only if there is an injection

$$f : X \to \mathbb{N}$$

Proof
Assume X is countably infinite, there is a bijection and therefore injection by definition.

Elsewise, suppose there is a desired f. By our work prior, there must be a countably infinite subset $X_1 \subseteq X$.

Define $g(n) = x_n \in X_1$ for $n \in \mathbb{N}$.

Then, note that

$$\begin{aligned} \mathbb{N} &\sim g(\mathbb{N}) = X_1 \subseteq X \\ X &\sim f(X) \subseteq \mathbb{N} \end{aligned}$$

So by CSB, we have $X \sim \mathbb{N}$.

Proposition 2.4.5
Suppose there is a surjection $g : X \to Y$, then there must be an injection

$$f : Y \to X$$

Note that this means $|Y| \preceq |X|$.

Proof
For all $y \in Y$, we must have

$$g^{-1}(\{y\}) \neq \varnothing$$

Then, by the Axiom of Choice, there is a choice function

$$h : \mathcal{P}(X) \to X$$

We can define

$$\begin{aligned} f :& Y \to X \\ & y \mapsto h(g(\{y\})) \end{aligned}$$

Theorem 2.4.6
$[0,1]$ is uncountable.

Proof
Cantor's Diagnalization Argument.

Corollary 2.4.6.1
$\mathbb{R}$ is uncountable.

Proof
There is a bijection between $[0,1]$ and $\mathbb{R}$ given by

$$x \mapsto \tan\left(\pi x - \frac{\pi}{2}\right)$$

Theorem 2.4.7 (Comparability of Cardinals)
Given two sets X, Y, either

$$|X| \preceq |Y| \vee |Y| \preceq |X|$$

Proof
Define

$$S := \{(A, B, f) | A \subseteq X, B \subseteq Y, f : A \to B \text{ is a bijection}\}$$

We can order S by

$$(A_1, B_1, f) \leq (A_2, B_2, g) \iff A_1 \subseteq A_2, B_1 \leq B_2, f = g\Big|_{A_1}$$

Let $C = (A_\alpha, B_\alpha, f_\alpha), \alpha \in I$ be a chain in S.

Let

$$A := \bigcup_\alpha A_\alpha, B := \bigcup_\alpha B_\alpha$$

and define $f : A \to B$ by

$$f(x) = f_\alpha(x)$$

if $x \in A_\alpha$.

Note that this is well defined, since all the f_α are equivalent under restriction.

We now claim that f is bijective.

To see injection, suppose $x \neq y \in A$, we can fine $x \in A_\alpha, y \in A_\beta$, we can further assume $A_\alpha \subseteq A_\beta$, this means

$$f(x) = f_\alpha(x) = f_\beta(x) \neq f_\beta(y) = f(y)$$

To see surjection, note that any $y \in B \implies \exists \alpha, y \in B_\alpha$. But f_α is bijective so we are done.

But then (A, B, f) is an upper bound for C, which by Zorn's Lemma implies there is a maximal element

$$(A_0, B_0, f_0)$$

If either $A_0 = X$ or $B_0 = Y$, then we are done.

Suppose otherwise, we have $x_0 \in X \setminus A_0, y_0 \in Y \setminus B_0$. But then

$$\begin{aligned} f_1 : & A_1 := X \cup \{x_0\} \to B_1 := Y \cup \{y_0\} \\ & x \mapsto \begin{cases} f(x), & x \in X \\ y_0, & x = x_0 \end{cases} \end{aligned}$$

is bijective with

$$(A_0, B_0, f_0) < (A_1, B_1, f_1)$$

contradicting the maximality of our original maximum element, so we are done.

2.5 Cardinal Arithmetic

2.5.1 Sums of Cardinals

Note that for two disjoint sets X, Y, we have

$$|X \cup Y| = |X| + |Y|$$

Definition 2.5.1 (Sum of Cardinals)
Let X, Y be disjoint sets, then

$$|X| + |Y| := |X \cup Y|$$

By the above definition, we will get that

$$\aleph_0 + \aleph_0 = \aleph_0, c + c = c$$

we say such cardinal numbers are idepotent.

Lemma 2.5.1
Every infinite set X can be decomposed as a union of pair-wise disjoint, countably infinite subsets.

Proof
Let $\mathcal{D}$ be the set of all families of pair-wise disjoint, countably infinite sets of X.

Define a partial order by includion.

Now, let $\mathcal{C} := \{C_\alpha | \alpha \in I\}$ be a chain in $\mathcal{D}$.

We claim $\mathscr{C} := \bigcup_\alpha C_\alpha$ is an upper bound for $\mathcal{C}$.

Indeed, it certainly is composed of countably infinite subsets of X, so it suffices to prove pair-wise disjointedness.

Let $c_0, c_1 \in \mathscr{C}$, we must have $c_0 \in C_{\alpha_0}, c_1 \in C_{\alpha_1}$.

Since $\mathcal{C}$ is a chain we assume $C_{\alpha_0} \subseteq C_{\alpha_1}$, but then $c_0, c_1 \in C_{\alpha_1} \in \mathcal{C}$, so they must be pair-wise disjoint.

We may conclude by Zorn's Lemma that there is a maximal element $\mathscr{M}$ in $\mathcal{D}$.

Now, we claim we can modify $\mathscr{M}$ so that its union includes X.

consider $S := X \setminus \bigcup \mathscr{M}$.

Note that S cannot be infinite, or else it contradicts maximality of $\mathscr{M}$.

But any finite (or empty) set can be added to any element of $\mathscr{M}$ and maintain countable infiniteness.

This concludes the proof.

Lemma 2.5.2
Let X be an infinite set.

$$|X| + |X| = |X|$$

Proof
Decompose $X = \bigcup \mathscr{M}$, a family of countably infinite subsets of X.

We can then decompose each of these subsets into countably infinite subsets again by enumerating them and taking the odd / even numbers for example. Indeed, we have

$$\begin{aligned}\bigcup \mathscr{M} &= \bigcup \{C_\alpha | \alpha \in I\} \\ &= \bigcup \{\{c_{\alpha,i} | i \in \mathbb{N}\} | \alpha \in I\} \\ &= \bigcup \mathscr{P}_1 := \bigcup \{\{c_{\alpha,i} | i \in \mathbb{N}, i \text{ is odd}\} | \alpha \in I\} \cup \bigcup \mathscr{P}_2 := \bigcup \{\{c_{\alpha,i} | i \in \mathbb{N}, i \text{ is even}\} | \alpha \in I\}\end{aligned}$$

We claim $|X| = |\bigcup \mathscr{P}_1| = |\bigcup \mathscr{P}_2|$ and it suffices to show the first equality.

Since $\bigcup \mathscr{P}_1 \subseteq X$, one side is trivial.

The other side consists of essentially showing that we can send the naturals to the odds, which is trivial. Define the injection

$$\begin{aligned} h : X &\to \bigcup \mathscr{P}_1 \\ c_{\alpha,i} &\mapsto c_{\alpha,2i+1} \end{aligned}$$

Now we have

$$|X| = \left|\bigcup \mathscr{P}_1\right| + \left|\bigcup \mathscr{P}_2\right| = |X| + |X|$$

Theorem 2.5.3
If X is infinite, then we have

$$|X| + |Y| = \max\{|X|, |Y|\}$$

Proof
We have

$$\max\{|X|, |Y|\} \leq |X| + |Y| \leq \max\{|X|, |Y|\} + \max\{|X|, |Y|\} = \max\{|X|, |Y|\}$$

By Cantor-Schroeder-Bernstein, the result follows.

2.5.2 Product of Cardinals

Definition 2.5.2
Let X, Y be sets

$$|X| \cdot |Y| := |X \times Y|$$

Lemma 2.5.4
If A is an infinite set, then
$$|A| \cdot |A| = |A|$$

Proof
Let $|A| = \alpha$
$$\mathcal{F} := \{(X, f) | X \subseteq A, f : X \to X \text{ is a bijection}\}$$
with a partial order of
$$(X_1, f_1) \leq (X_2, f_2) \iff X_1 \subseteq X_2,\ f_2\Big|_{X_1} = f_1$$

Note that this is similar to the case of Comparability of Cardinals and we can also derive a maximal element (Y, g) using similar ways.

Now, $|Y| = |Y| \cdot |Y|$, so we need only show that $Y = A$.

Suppose otherwise, we must have
$$|A| = |Y| + |A \setminus Y|$$
with $|Y| < |A| \implies |A| = |A \setminus Y|$.

It follows that $\exists Z \subseteq A \setminus Y$ with $|Z| = |Y|$.

So, the Cartesian Products below must be pair-wise disjoint!
$$\begin{aligned} |(Y \times Z) \cup (Z \times Y) \cup (Z \times Z)| &= |Y \times Z| + |Z \times Y| + |Z \times Z| \\ &= |Y| \cdot |Y| + |Y| \cdot |Y| + |Y| \cdot |Y| \\ &= |Y| \\ &= |Z| \end{aligned}$$

This means there must be a bijection $h : Z \to (Y \times Z) \cup (Z \times Y) \cup (Z \times Z)$.

Define
$$m : Y \cup Z \to (Y \cup Z) \times (Y \cup Z)$$
$$x \mapsto \begin{cases} g(x), & x \in Y \\ h(x), & x \in Z \end{cases}$$

But then m is a bijection so
$$(Y, g) \leq (Y \cup Z, m) \in \mathcal{F}$$
which is a contradiction.

This terminates the proof.

Theorem 2.5.5
Let X, Y be infinite sets
$$|X| \cdot |Y| = \max\{|X|, |Y|\}$$

Proof
Define $\alpha := \max\{|X|, |Y|\}$.

We certainly have
$$|X| \cdot |Y| \leq \alpha \cdot \alpha = \alpha$$

Conversely
$$\alpha \leq |X \times Y| = |X| \cdot |Y|$$

so we are done.

2.5.3 Exponentiation of Cardinals

Recall that
$$\prod_{x \in X} Y_x := \left\{ f : X \to \bigcup_{x \in X} Y_x | f(x) \in Y_x \right\}$$

By fixing a certain Y, we can define

Definition 2.5.3

$$Y^X := \prod_{x \in X} Y = \{f : X \to Y\}$$

This naturally leads to the following definition

Definition 2.5.4
Let $X, Y \neq \varnothing$, we have
$$|Y|^{|X|} := \left|Y^X\right|$$

Theorem 2.5.6
Let $X, Y, Z \neq \varnothing$, then

1. $|Y|^{|X|} \cdot |Y|^{|Z|} = |Y|^{|X|+|Z|}$
2. $\left(|Y|^{|X|}\right)^{|Z|} = |Y|^{|X| \cdot |Z|}$

Let $A \subseteq X$ be arbitrary.

Definition 2.5.5 (Characteristic Function)

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Note that the characteristic function is a choice function! Also note the following bijection

$$\Gamma : \mathcal{P}(X) \to \{0,1\}^X$$
$$A \mapsto X_A$$

This tells us that

$$|\mathcal{P}(X)| = 2^{|X|}$$

We now show that the powerset is always greater than the set itself.

Theorem 2.5.7 (Russel's Paradox)
For any set X

$$X \preceq \mathcal{P}(X)$$

Proof
Since we know the cardinals are comparable, it suffices to show that there are not surjections

$$X \to \mathcal{P}(X)$$

Suppose otherwise and let f be such a function. In addition, define

$$A := \{x \in X | x \notin f(x)\}$$

Since A is a subset of X, there must be some $x_0 \in X, f(x_0) = A$.

But x_0 cannot be in A or else it would not be in A by the definition of A, nor can it not be in A, or else it would be in A be the definition of A.

We have arrived at the desired contradiction and conclude the proof.

This shows us that $\aleph_0 \prec c$.

We now ask if there can be a set with cardinality in between the natural numbers and the reals. This is not derivable from standard set theory so we instead include the following axioms which are independent of but consistent with the previously introduced axioms.

Axiom 2.5.1 (Continuum Hypothesis)
If $\aleph_0 \preceq |X| \preceq c$, then

$$|X| = \aleph_0 \vee |X| = c$$

Axiom 2.5.2 (Generalized Continuum Hypothesis)
If $|X| \preceq |Y| \preceq 2^{|X|}$ then

$$|Y| = |X| \vee |Y| = 2^{|X|}$$

3 Normed Vector Spaces & Metric Spaces

3.1 Definitions & Basic Results

Definition 3.1.1 (Norm)
Let V be a vector space over $\{\mathbb{R}, \mathbb{C}\}$. A norm on V is a function

$$\|\cdot\| : V \to \mathbb{R}$$

which satisfies the following properties:

1. $\forall v \in V, \|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$ (positive definite)
2. $\forall v \in V, \alpha \in \mathbb{R}, \|\alpha v\| = |\alpha|\|v\|$ (positive homogeneity)
3. $\forall v, w \in V, \|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)

Note if the first condition fails but the rest hold, it is called a seminorm.

The pair $(V, \|\cdot\|)$ is a normed vector (linear) space.

Example 3.1.1
The Euclidean Norm on $\mathbb{R}^n, \mathbb{C}^n$

Example 3.1.2
Inner Product Spaces V with Inner Product $\langle \cdot, \cdot \rangle$ induces a norm

$$\|v\|_2 := \sqrt{\langle v, v \rangle}$$

Proof
positive definite, homogeneity are trivial.

To see the triangle inequality note that by the Cauchy Schwarz Inequality ($|\langle u, v \rangle| \leq \|u\|\|v\|$)

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v \rangle \\
&= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\
&= \langle u, u \rangle + 2\,\mathrm{Re}(\langle u, v \rangle) + \langle v, v \rangle \\
&\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

Example 3.1.3
If $X \subseteq \mathbb{R}^n$ is closed and bounded, $C(X), C_{\mathbb{R}}(X)$ are the spaces of continuous $\mathbb{C}, \mathbb{R}$-valued

functions with norm

$$\|f\|_\infty = \sup_{x\in X}|f(x)| < \infty$$

by the Extreme Value Theorem.

Example 3.1.4
If $X \in \mathbb{R}^n, C_b(X)$ is the space of bounded continuous functions, $\|\cdot\|_\infty$ is a norm

Example 3.1.5
$\|x\|_1 := \sum_{i=1}^n |x_i|$ on $\mathbb{R}^n, \mathbb{C}^n$ is a norm

Theorem 3.1.6 (Minkowski's Inequality)
Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $f, g \in C([a,b])$, we have

$$\left(\int_a^b |(f+g)(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}}$$

with equality if and only if f, g lie on a ray.

Lemma 3.1.7
The polynomial $\varphi(x) = x^p$ is convex for $p \geq 1, x > 0$ and strictly convex for $p \geq 2$

Proof

$$\varphi'(x) = px^{p-1}$$
$$\varphi^{(2)}(x) = p(p-1)x^{p-2} \underbrace{\geq}_{> \text{if } p \geq 2} 0$$

Proof (Minkowski's Inequality II, inequality)
Let $f, g \in L^p(-\infty, \infty)$.

If $f = 0$ or $g = 0$, then the result is trivial.

Elsewise, let

$$A := \|f\|_p > 0$$
$$B := \|g\|_p > 0$$

Define $f_0 = f/A, g_0 = g/B$.

Note that $\|f_0\|_p = \|g_0\|_p$ and

$$\frac{f+g}{A+B} = \frac{Af_0}{A+B} + \frac{Bg_0}{A+B}$$

Now by convexity

$$\begin{aligned}
\left(\frac{\|f+g\|}{A+B}\right)^p &= \frac{1}{(A+B)^p}\int |f+g|^p \\
&= \int \left| \underbrace{\frac{A}{A+B}}_{t} f_0 + \underbrace{\frac{B}{A+B}}_{1-t} g_0 \right|^p \\
&\leq \int \frac{A}{A+B}|f_0|^p + \frac{B}{A+B}|g_0|^p \\
&\leq \frac{A}{A+B}\|f_0\|_p^p + \frac{B}{A+B}\|g_0\|_p^p \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\frac{\|f+g\|}{A+B} &\leq 1 \\
&= \frac{A\|f_0\|_p}{A+B} + \frac{B\|g_0\|_p}{A+B} \\
&= \frac{\|f\|_p}{A+B} + \frac{\|g\|_p}{A+B}
\end{aligned}$$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof (Minkowski's Inequality, Equality)
Consider the following line from the proof above

$$\int \left| \underbrace{\frac{A}{A+B}}_{t} f_0 + \underbrace{\frac{B}{A+B}}_{1-t} g_0 \right|^p \leq \int \frac{A}{A+B}|f_0|^p + \frac{B}{A+B}|g_0|^p$$

By strict convexity, if $f_0 ! = g_0$, then the inequality would also become strict.

This means that for equality to happen we must have

$$|f_0(x)| = |g_0(x)| \wedge \operatorname{sgn} f_0(x) = \operatorname{sgn} g_0(x) \implies f_0 = g_0$$

But $f_0 = g_0 \implies f = \frac{A}{B} g \in \mathbb{R}_+ f$.

So we have equality if and only if one of f, g is the zero function, else if $g \in \mathbb{R}_+ f$.

Corollary 3.1.7.1
Minkowski's Inequality $\implies$ Minkowski's Inequality for sequences

Proof
We will show that there is a linear map (isometry) $T : l_p \to L^p(\mathbb{R})$ such that

$$\|T\{x_n\}\|_p = \|\{x_n\}\|_p$$

Note that the two norms are different and respectively for their normed vector spaces.

First, define $\varphi : [0,1] \to \mathbb{R}$ continuous such that $\|\varphi\|_p = 1$. Then, define $\varphi(0) = \varphi(1) = 0$. Next, define φ to be linear with positive slope from $[0, \frac{1}{2}]$ and linear with negative slope from $(\frac{1}{2}, 1]$.

Define $T : l_p \to L^p(\mathbb{R})$

$$(Tx)(t) = \sum_{n=1}^{\infty} x_n \varphi(t-n)$$

and note that $(Tx)(t)$ is zero if $t \in \mathbb{N}$ and elsewise nonzero only on $(\lfloor t \rfloor, \lceil t \rceil)$. We are essentially scaling φ up to the size of each element in the sequence x_n over the inverval $(n, n+1)$.

Then we have

$$\begin{aligned}(\|Tx\|_p)^p &= \int_{-\infty}^{\infty} \left|\sum x_n \varphi(t-n)\right|^p dx \\ &= \sum_{n=0}^{\infty} |x_n|^p \|\varphi(t-n)\|_p^p dx \\ &= (\|x\|_p)^p\end{aligned}$$

Definition 3.1.2
Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed linear spaces.
Let $T : X \to Y$ be linear.
Define

$$\|T\| := \sup\{\|T(x)\|_Y | x \in X, \|x\|_X \leq 1\}$$

We say that T is bounded if $\|T\| < \infty$

Definition 3.1.3 (Metric)
Let X be a set. A metric on X is a function

$$d : X \times X \to \mathbb{R}$$

which fulfills the following properties:

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry)
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$ (triangle inequality)

We say that the pair (X, d) is a metric space.

Example 3.1.8
Any normed vector space induces a metric space through the metric

$$d_{\|\cdot\|}(x, y) = \|x - y\|$$

Some of the most important metric spaces are vector spaces with abstract distance functions.

Example 3.1.9 (Discrete Metric)
Let X be any set and define

$$d(x, y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

Example 3.1.10 (Hamming Metric)
Let $X = \mathcal{P}([n])$ for two subsets, define

$$\begin{aligned} d(A, B) &= |A \triangle B| \\ &= |(A \cup B) \setminus (A \cap B)| \end{aligned}$$

Proof
properties 1, 2 are trivial. We only need to show the triangle inequality.

To do this we claim

$$A \triangle C \subseteq (A \triangle B) \cup (B \triangle C)$$

Note that we can decompose A, B, C into disjoint sets

$$A \setminus (B \cup C) \tag{1}$$
$$B \setminus (A \cup C) \tag{2}$$
$$C \setminus (A \cup B) \tag{3}$$
$$(A \cap B) \setminus C \tag{4}$$
$$(B \cap C) \setminus A \tag{5}$$
$$(A \cap C) \setminus B \tag{6}$$
$$A \cap B \cap C \tag{7}$$

Note that

$$A \triangle C = \bigcup \{(1), (3), (4), (5)\}$$

which is a subset of

$$(A \triangle B) \cup (B \triangle C) = \bigcup \{(1), (2), (3), (4), (5)\}$$

note that we actually count (3) twice in the actual computation of the Hamming Metric but it suffices to show that even adding it once would fufill the example.

Example 3.1.11 (Geodesic Distance)
Consider $X = S^2$ or the surface of a ball in $\mathbb{R}^3$

then $d(x, y) =$ shortest path from $x \to y$ is a metric.

Example 3.1.12 (Hadamard Distance)

Example 3.1.13 (Hausdorf Metric)
Fix a closed subset $Y \subseteq \mathbb{R}^n$

Let $\mathscr{H}(Y)$ be the set of closed and bounded (compact) subsets of Y.

For $a \in A \in \mathscr{H}(Y), B \in \mathscr{H}(Y)$ define

$$d(a, B) := \inf_{b \in B} \|a - b\| = \|a - b_0\|$$

for some $b_0 \in B$ due to compactness.

then $d_H : \mathscr{H}(Y) \times \mathscr{H}(Y) \to \mathbb{R}$

$$d_H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

is a metric.

Proof (1)
Suppose $d_H(A, B) = 0$, if $a \in A, \exists\{b_n\} \subseteq B, b_n \to a$. This means that $a \in B$ by closedness, and similarly for $b \in B$.

This shows that $A = B$

Proof (2)
trivial

Proof (3)
Let $A, B, C \in \mathscr{H}(Y)$ take $a \in A, b \in B$ then

$$\begin{aligned}
d(a, C) &= \inf_{c \in C} \|a - c\| \\
&\leq \inf_{c \in C} (\|a - b\| + \|b - c\|) \\
&= \|a - b\| + d(b, C) \\
&\leq \|a - b\| + d(B, C) \\
&\therefore \\
d(a, C) &\leq \inf_{b \in B} \|a - b\| + d(B, C) \\
&= d(a, B)_d(B, C) \\
&\leq d(A, B) + d(B, C) \\
&\therefore \\
\sup_{a \in A} d(a, C) &\leq d(A, B) + d(B, C) \\
\sup_{c \in C} d(c, A) &\leq d(A, B) + d(B, C) && \text{vice versa} \\
&\therefore \\
d(A, C) &= d(A, B) + d(B, C)
\end{aligned}$$

Example 3.1.14 (*p*-adic distance)
Let $p \in \mathbb{Z}$ be prime, we will "put" a norm $\|\cdot\|_p$ on $\mathbb{Q}$ as follows:

Define $\|0\|_p = 0$, else $\|x\|_p = p^{-a}$ where

$$x = p^a \frac{r}{s}, r, s, a \in \mathbb{Z}, \gcd(r, p) = 1, \gcd(s, p) = 1$$

the metric is then

$$d_p(x, y) = \|x - y\|_p$$

Proof
The first two conditions are trivial, we need only prove the triangle inequality. We actually claim that

$$d_p(x,y) \leq \max\{d_p(x,z), d_p(z,y)\}$$

Suppose $x, y \in \mathbb{Q}$ with both non-zero, otherwise the proof is trivial.

$$x = p^a \frac{r}{s}, y = p^b \frac{u}{v}, r, s, u, v, a, b \in \mathbb{Z}, \gcd(\alpha, p) = 1, \alpha \in \{r, s, u, v\}$$

Suppose without loss of generality that $a \leq b$ so that

$$x - y = p^a \left(\frac{r}{s} - p^{b-a}\frac{u}{v}\right) = p^a \left(\frac{rv - p^{b-a}su}{sv}\right)$$

If $a < b$ then consider $rv - p^{b-a}su \equiv rv \mod p$. It follows that

$$d_p(x,y) = p^{-a} = \|x\|_p = d(x,0)$$

Suppose elsewise that $a = b$, then there is some $c \geq 0$ such that

$$rv - su = p^c w, \gcd(w,p) = 1$$

Then

$$d_p(x,y) = p^{-a-c} \leq p^{-a} = \|x\|_p = d(x,0)$$

If $z \in \mathbb{Q}$ then

$$\begin{aligned}
d_p(x,y) &= \|x - y\|_p \\
&= \|x - z - (y - z)\|_p \\
&= d_p(x - z, y - z) \\
&\leq \max\{\|x - z\|_p, \|y - z\|_p\} \\
&= \max\{d_p(x,z), d_p(y,z)\}
\end{aligned}$$

4 Topology

4.1 Topology of Metric Spaces

Definition 4.1.1 (Open Ball)
Let (X, d) be a metric space.
Let $x_0 \in X, \epsilon > 0$.
The open ball of radius ϵ centered around x_0 is the set

$$B(x_0, \epsilon) := \{x \in X | d(x_0, x) < \epsilon\}$$

We can also write $b_r(x)$ for open ball.

Definition 4.1.2 (Closed Ball)
Let (X, d) be a metric space.
Let $x_0 \in X, \epsilon > 0$.
The closed ball of radius ϵ centered around x_0 is the set

$$B[x_0, \epsilon] := \{x \in X | d(x_0, x) \leq \epsilon\}$$

We can also write $b_{\leq r}(x)$ for closed ball.

Definition 4.1.3 (Open)
We say $U \subseteq X$ is open if every $x \in X$ has a ball $B(x, \epsilon) \in X$ for some $\epsilon > 0$

Definition 4.1.4 (Closed)
We say $C \subseteq X$ is closed if C^c is open.

Definition 4.1.5 (Neighbourhood)
Let (X, d) be a metric space and $x \in X$.
$N \subseteq X$ is a a neighbourhood of x if $x \in \text{int}(N)$

Note that a neighbourhood is not necessarily open but contains an open ball around single point.

Proposition 4.1.1
$b_r(x) \subseteq (X, d)$ is open.

Proof
For any point $y \in b_r(x)$
$$b_{r-d(x,y)}(y) \subseteq b_r(x)$$
(triangle inequality)

Proposition 4.1.2
$b_{\leq r}(x) \subseteq (X, d)$ is closed.

Proof
We show the complement is open.

Note $(b_{\leq r}(x))^c := \{y \in X | d(x, y) > r\}$.

For any y in the complement
$$b_{d(x,y)-r} \subseteq (b_{\leq r}(x))^c$$

Example 4.1.3
$\{(x, y) \in R^2 | xy > 1\}$ is open.

$\{(x, y) \in R^2 | xy \geq 1\}$ is closed.

Example 4.1.4
Consider $(\mathbb{N}, d)$, a metric space with the 2-adic metric.

$$d_2(n, m) = \begin{cases} 0, & n = m \\ 2^{-k}, & n - m = 2^k l, 2 \nmid l \end{cases}$$

$$\begin{aligned} b_{2^{-k}}(n) &= \{m : d(n, m) < 2^{-k}\} \\ &= \{m : d(n, m) \leq 2^{-k}\} \\ &= \{m : 2^{k+1} | m - n\} \\ &= n + 2^{k+1}\mathbb{Z} \cap \mathbb{N} \end{aligned}$$

Proposition 4.1.5
Let (X, d) be a metric space

1. $X, \varnothing$ are open
2. If $\{U_\alpha | \alpha \in I\}$ is an arbitrary collection of subsets of X, $\bigcup_\alpha U_\alpha$ is open
3. If $\{U_i | i \in [n]\}$ is a finite collection of subsets of X, $\bigcap_i U_i$ is open

Proposition 4.1.6
Let (X, d) be a metric space

1. $X, \varnothing$ are closed
2. If $\{U_\alpha | \alpha \in I\}$ is an arbitrary collection of subsets of X, $\bigcap_\alpha U_\alpha$ is closed
3. If $\{U_i | i \in [n]\}$ is a finite collection of subsets of X, $\bigcup_i U_i$ is closed

Definition 4.1.6 (Topology)
A topology on a set X is a collection of sets $\tau \subseteq X$ such that

1. $X, \varnothing \in \tau$
2. $\{U_\alpha \in \tau | \alpha \in I\} \implies \bigcup_{\alpha \in I} U_\alpha \in \tau$
3. $\{U_i \in \tau | i \in [n]\} \implies \bigcap_{i \in [n]} U_i \in \tau$

We say the elements of τ are τ-open sets or just open sets.

We call the pair (X, τ) a topological space.

If (X, τ) is a topological space, let τ_d denote the topology consisting of subsets of X which are open with respect to the metric d.

Proposition 4.1.7
Let (X, d) be a metric space

1. for any $x \in X, \epsilon > 0$, $B(x, \epsilon)$ is open
2. A subset is open if and only if it is the union of open balls
3. for any $x \in X, \epsilon > 0$, $B[x, \epsilon]$ is closed
4. Every finite subset is closed

Proof (2)
A union of open balls is clearly open.

To decompose a subset into a union of open balls, simply take the union of all open balls around every member of the subset.

Proof (3)
We show the complement is open.

Let $z \in B[x, \epsilon]^c$, and $r = d(z, x) - \epsilon$.

We claim that $B(z, r) \subseteq B[x, \epsilon]^c$.

Suppose otherwise. Let $w \in B(z,r), B[x,\epsilon]^c$. It must be true that

$$\begin{aligned} d(z,x) &\leq \underbrace{d(z,w)}_{<r} + \underbrace{d(w,x_0)}_{\leq \epsilon} \\ &< d(z,x) \end{aligned}$$

which is clearly a contradiction.

Theorem 4.1.8
Let $I \subseteq R$ be open, then it is the countable union of disjoint open intervals.

Proof (sketch)
Partition I by having two members being equivalent if the closed interval between then is a subset of I.

Note that every equivalence class is necessarily non-empty and in particular not a singleton as there is an open ball around every member of I.

We can label all equivalent classes by choosing a rational inside the interval defined by the equivalence class.

Since the rationals are countable, so are the equivalence classes.

So we can decompose every open set as a union of countable intervals, but how about closed sets?

Example 4.1.9 (Cantor Set)
Let $P_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

Recusively define P_n for $n > 1$ by removing the middle third from each of the 2^{n-1} closed intervals from P_{n-1}.

1. P_n is closed
2. P_n contains no interval of length greater than $\frac{1}{3^n}$

Let $P := \bigcap_{i=1}^{\infty} P_i$, this is referred to as the Cantor (Ternary) Set.

1. $x \in P \iff x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with $a_n = 0, 2$
2. P is uncountable as it is essentially all ternary strings with no digits being 1
3. P contains no intervals of positive length

Example 4.1.10 (Discrete Metric)
Let (X, d) be a metric space and d be the discrete metric.

Note that each singleton is open since

$$\{x\} = B(x, 1)$$

But then every set is open as

$$X \supseteq U := \bigcup_{x \in U} \{x\}$$

It follows that every set is also closed

We know that $\mathbb{R}, \varnothing$ are both open and closed in $\mathbb{R}$, are there any other such sets?

4.2 Convergence of Sequences & Topology in a Metric Space

Definition 4.2.1 (Sequential Limit)
Let (X, d) be a metric space and $\{x_n\} \subseteq X$ be a sequence in X.
We say the sequence converges to a point $x_0 \in X$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d(x_0, x_n) < \epsilon$$

and write

$$\lim_{n \to \infty} x_n = x_0$$

Proposition 4.2.1 (Uniquess of the Limit)
Let (X, d) be a metric space and $\{x_n\} \subseteq X$ be a sequence in X.
If $y_0 = \lim_{n \to \infty} x_n = x_0$ then $x_0 = y_0$

Proof
Suppose $d(x_0, y_0) \neq 0$.

Let $0 < \epsilon = d(y_0, x_0)$

There is some $N \in \mathbb{N}$ such that

$$d(x_0, x_N), d(y_0, x_n) < \frac{\epsilon}{2}$$

But then by the Triangle Inequality:

$$d(x_0, y_0) \leq d(x_0, x_N) + d(x_N, y_0) < \epsilon$$

which is impossible.

Definition 4.2.2 (Limit Point)
Let $A \subseteq X$. We say $x_0 \in X$ is a limit point of A if there is some $\{x_n\} \subseteq A$ with $x_n \to x_0$.

We denote the set of all limit points of A by $\text{Lim}(A)$.

Limit points are also referred to as cluster points.

Definition 4.2.3 (Accumulation Point)
$x_0 \in X$ is an accumulation point of $A \subseteq X$ if there is some seqeuence $\{a_n\} \subseteq A$ of distinct points such that

$$a_n \to x_0$$

Definition 4.2.4 (Isolated Point)
$x_0 \in X$ is an isolation point of $A \subseteq X$ if there is some $r > 0$ such that

$$b_r(x_0) \cap A = \{x_0\}$$

We now state a very important result which shows that the topology of a metric space is determined by the converging sequences.

Proposition 4.2.2
$A \subseteq X$ is closed if and only if $\text{Lim}(A) \subseteq A$.

Proof
Suppose A is closed and $\{x_n\} \to x_0$ be a sequence in A converging to some point.

If $x_0 \in A^c$, then there is a ball of nonzero radius $B(x_0, \epsilon) \subseteq A^c$ which contradicts the definition of a converging sequence.

Conversely, suppose A is not closed, so A^c is not open.

There is some point $x_0 \in A^c$ such that for every k

$$b_{\frac{1}{k}}(x_0) \cap A \neq \varnothing$$

Let $x_k \in b_{\frac{1}{k}}(x_0) \cap A$ and notice this defines a sequence in A converging to $x_0 \notin A$. So

$$\text{Lim}(A) \not\subseteq A$$

as desired.

Definition 4.2.5 (Boudary Point)
Let (X, d) be a metric space, $x \in X$, and $A \subseteq X$.
x is a boundary point of A if for every neighbourhood N of x

$$N \cap A \neq \varnothing \wedge N \cap A^c \neq \varnothing$$

We denote the set of all boundary points with $\text{bdy}(A)$.

Proposition 4.2.3
Let (X, d) be a metric space, $x \in X$, and $A \subseteq X$.

$$x \in \text{bdy}(A)$$
$$\iff$$
$$\forall \epsilon > 0, B(x, \epsilon) \cap A \neq \varnothing, B(x, \epsilon) \cap A^c \neq \varnothing$$

Proposition 4.2.4
Let $A \subseteq (X, d)$
A is closed if and only if $\text{bdy}(A) \subseteq A$.

Proof
First, suppose A is closed. Consider $x \in A^c$, we will show that $x \notin \text{bdy}(A)$.

By openness, there is a ball $B(x, \epsilon) \subseteq A^c \implies B(x, \epsilon) \cap A = \varnothing$ so $x \notin \text{bdy}(A)$. So no point in the boundary can be outside of A.

Conversely, suppose that $\text{bdy}(A) \subseteq A$, we will show that A^c is open.

Let $x \in A^c$, then $x \notin \text{bdy}(A)$. This means that there is a ball $B(x, \epsilon) \cap A = \varnothing$ meaning that the ball sits inside the complement. This satsfies the definition of openness.

Definition 4.2.6 (Sequential Limit Point)
A point x_0 is a limit point of the sequence $\{x_n\}$ if there is a subsequence

$$x_{n_k} \to x_0$$

We denote the set of all limit points of a sequence $\lim *\{x_n\}$ and note that in general

$$\text{Lim}\{x_n\} \subseteq \lim *\{x_n\}$$

Example 4.2.5
The constant sequence $\{1, 1, 1, \ldots\}$

We close this section with a remark about the discrete metric.

Let (X, d) be a metric space with the discrete metric.

Assume $x_n \to x_0$, there is some $N \in \mathbb{N}$ with $n \geq N \implies d(x_n, x_0) < \frac{1}{2}$.

But the only points which satisfy this definition is $\{x_0\}$.

It follows that the only converging sequences are the ones which constant after a finite amount of terms.

4.3 Boundaries, Interiors, & Closures of a Set

Definition 4.3.1 (Closure)
Let (X, d) be a metric space, the closure of a subset $A \subseteq X$ is

$$\bar{A} := \bigcap \{F \subseteq X | A \subseteq F, F \text{ is closed}\}$$

In other words, the closure is the "smallest" set containing A which is closed.

Proposition 4.3.1
Let $A \subseteq (X, d)$, then $\bar{A}$ is equivalent to the following

(1) $A \cup \{x \in X : x \text{ is an accumulation point}\}$

(2) $\{x \in X : x \text{ is a limit point of } A\}$

(3) $\{x \in A : x \text{ is an isolated point}\} \cup \{x \in X : a \text{ is an accumulation point}\}$

The proof essentially requires us to show that the set of limit points of $A, \bar{A}$ are the same.

Proof
We first note that all three sets are actually equivalent so it suffices to show the closure is the set of limit points of A.

Since $\bar{A}$ is a closed set containing A, it must contain A and by closedness, contain all the limit points of A. This shows that

$$\text{Lim}(A) \subseteq \bar{A}$$

Conversely, let $\{a_n\}_{n \geq 1} \to x$ be a sequence of limit points of A (ie a sequence in $\bar{A}$).

This means for each n, there is a sequence

$$a_k^{(n)} \in A, \lim a_k^{(n)} = a_n$$

For each $n \in \mathbb{N}$ choose k_n such that

$$d(a_{k_n}^{(n)}, a_n) < \frac{1}{n}$$

This defines a sequence of A.

Let $\epsilon > 0$, choose N_1 such that

$$\frac{2}{N_1} \leq \epsilon$$

No, choose N_2 such that for $n \geq N_2$

$$d(a_n, x) < \frac{1}{N_1}$$

Then, for $n \geq N := \max(N_1, N_2)$ we have

$$d(a_{k_n}^{(n)}, x) \leq d(a_{k_n}^{(n)}, a_n) + d(a_n, x) < \frac{2}{N} \leq \epsilon$$

This shows that x is indeed a limit point of A and that

$$\bar{A} \subseteq \text{Lim}(A)$$

All in all, the closure is the set of limit points of A.

Proposition 4.3.2
$\bar{A} = A \cup \text{bdy}(A)$.

Proof
We first show that $F := A \cup \text{bdy}(A) \subseteq \bar{A}$, from our work prior, it suffices to show that $\text{bdy}(A) \subseteq \bar{A}$.

But since the closure is closed it must contain the boundary by part 1 so we are done.

Now it remains to show that F is closed which means $\bar{A} \subseteq F$.

Let $x \in F^c$, so x is neither in the boundary or A. This means that there is a ball $B(x, \epsilon) \cap A = \varnothing$. Note that no point $z \in B(x, \epsilon)$ can be inside the boundary as the ball is a neighbourhood of z. Elsewise, $B(x, \epsilon) \cap A \neq \varnothing$, which is a contradiction. It follows that $B(x, \epsilon) \subseteq F^c$, demonstrating openness of the complement and therefore the closedness of F.

Definition 4.3.2 (Interior)
Let (X, d) be a metric space, the interior of a subset $A \subseteq X$ is

$$\operatorname{int}(A) := \bigcup \{F \subseteq X | F \subseteq A, F \text{ is open}\}$$

So the interior is the "biggest" open subset of A.

Proposition 4.3.3
$\operatorname{int} A = A^{c-c}$ where $A^{-} = \bar{A}$.

Proof
$A^{c-} \supseteq A^c$ is closed, so A^{c-c} is open and disjoint from A^c, which means it is contained in $\operatorname{int} A$.

Conversely, if $x \in \operatorname{int} A$, there is a ball of radius r such that $b_r(x) \subseteq A$.

This means that $\{y : d(x, y) \geq r\} \supseteq A^c$ and is closed, meaning it must contain A^{c-}, so $A^{c-c} \supseteq b_r(x)$. As a result, we have $A^{c-c} \supseteq \operatorname{int} A$.

All in all, $A^{c-c} = \operatorname{int} A$.

Proposition 4.3.4
Let $A \subseteq B \subseteq X$

1. $\bar{A} \subseteq \bar{B}$
2. $\operatorname{int}(A) \subseteq \operatorname{int}(B)$
3. $\bar{A}^c = \operatorname{int}(A^c)$
4. $\operatorname{int}(A) = A \setminus \operatorname{bdy}(A)$

Proposition 4.3.5
1. $\overline{A \cup B} = \bar{A} \cup \bar{B}$
2. $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$

Proof (1)
Since $\bar{A} \cup \bar{B}$ is closed and contains $A \cup B$, we have

$$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$$

Next, Since $\overline{A \cup B}$ is closed and contains A,

$$\bar{A} \subseteq \overline{A \cup B}$$

and vice versa for B so we are done.

Proof (2)
Since $\text{int}(A) \cap \text{int}(B)$ is open and contained in $A \cap B$ we have

$$\text{int}(A) \cap \text{int}(B) \subseteq \text{int}(A \cap B)$$

Likewise, since the open set $\text{int}(A \cap B) \subseteq A \cap B$ is contained in A, we have

$$\text{int}(A \cap B) \subseteq \text{int}(A)$$

and likewise for B so we are done

Let (X, d) be a metric space and $x \in X$. Now, the question falls upon whether

$$B[x, \epsilon] = \overline{B(x, \epsilon)}$$

Definition 4.3.3 (Dense)
We say a proper subset $A \subset X$ is dense in X if

$$\bar{A} = X$$

Note that this essentially means that no smaller subset of X is closed and contains A. To show that this subset does not exist, we need only prove that the complement of A has no open balls.

Definition 4.3.4 (Seperable)
A metric space (X, d) is seperable if it contains a countably dense subset, else it is non-seperable.

Example 4.3.6
$\mathbb{R}^n$ is seperable due to $\mathbb{Q}^n$

Proposition 4.3.7
l_1 is seperable

Proof (sketch)
Take the set of all rational sequences which only the first m entries can be non-zero for all $m \in \mathbb{N}$.

Since all series converge in our space, the tails of any series are arbitrarily small.

This means we can approximate any series with an element in our subset by appropriately approximating the first something numbers.

Proposition 4.3.8
l_∞ is not seperable.

Proof
Consider the collection of disjoint balls of size $\frac{1}{2}$ centered around points whose elements are defined by the characteristic function of the subsets of $\mathbb{N}$.

Any dense subset of the metric space must have at least one element in each subset of the collection.

There are uncountable disjoint balls as it is essentially the powerset of the natural numbers.

Proposition 4.3.9
$(C(\mathbb{R}), \|\cdot\|_\infty)$ is not seperable (due to ℓ_p)

5 More Topology

5.1 Continuity

Definition 5.1.1 (Pointwise Continuity)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$.
f is continuous at $x_0 \in X$ if

$$\forall \epsilon > 0, \exists \delta > 0, d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon$$

elsewise it it discontinuous at that point.

Definition 5.1.2 (Function Continuity)
We say f is continuous it if continuous at every point of its domain.

Definition 5.1.3
Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$.
f is sequentially continuous at $x_0 \in X$ if whenever $\{x_n\} \subseteq X \to x_0$, we have

$$f(x_n) \to f(x_0)$$

Theorem 5.1.1 (Sequential Characterization of Continuity)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$.
f is continuous at $x_0 \in X$ if and only if it is sequentially continuous.

Proof ($\implies$)
First assume f is continuous at x_0.

Let $\epsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon$$

Let $\{x_n\} \to x_0$ be some converging sequence.

By definition there is some $N \in \mathbb{N}$ such that $n \geq N \implies d_X(x_n, x_0) < \delta$.

This means
$$\forall n \geq N, d_Y(f(x), f(x_0)) < \epsilon$$
But then $f(x_n) \to f(x_0)$ by definition.

Proof ($\Longleftarrow$)
Now suppose that f is not continuous at x_0.

There is some $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there is a point x_n with

$$d_X(x_n, x_0) < \frac{1}{n}, d_Y(f(x_n), f(x_0)) \geq \epsilon_0$$

This defines a sequence $\{x_n\}$ such that

$$x_n \to x_0, f(x_n) \not\to f(x_0)$$

concluding our proof.

Definition 5.1.4 (Pullback)
Every function $f : X \to Y$ induces a function

$$\begin{aligned} f^{-1} : \mathcal{P}(Y) &\to \mathcal{P}(X) \\ B &\mapsto \{x \in X | f(x) \in B\} \end{aligned}$$

The map f^{-1} is referred to as the pullback of f.

We now establish continuity in terms of pullbacks and continuous sequences.

Theorem 5.1.2
Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$.
f is continuous at $x_0 \in X$ if and only if:
If W is a neighbourhood of $y_0 = f(x_0)$ then $V := f^{-1}(W)$ is a neighbourhood of x_0.

Proof
First, suppose f is continuous at x_0. Let W be as in the statement of the theorem.

By definition of a neighbourhood, there is some $\epsilon_0 > 0$ such that $B(y_0, \epsilon_0) \subseteq W$.

By the definition of continuity there is some $\delta_0 > 0$ such that

$$x \in B(x_0, \delta_0) \implies f(x) \in B(y_0, \epsilon_0)$$

This means that $B(x_0, \delta_0) \subseteq V$ which in turn means that x_0 is in the interior of V and that V is indeed a neighbourhood of x by definition.

Conversely, let $\epsilon > 0$ be arbitrary. Then $W := B(y_0, \epsilon)$ is a neighbourhood of y_0. Let $V := f^{-1}(W)$, which is by assumption a neighbourhood of x_0.

In particular there is some $\delta_0 > 0$ such that $B(x_0, \delta_0) \subseteq V$. So

$$x \in B(x_0, \delta_0) \implies d_Y(f(x_0), f(x)) < \epsilon$$

which satisfies the definition of continuity.

Theorem 5.1.3
Let $(X, d), (Y, \rho)$ be metric spaces and $f : X \to Y$. TFAE:

(1) f is continuous

(2) If $\{x_n\}$ is a sequence in X with $x_n \to x_0$, then $f(x_n) \to f(x_0)$ in Y

(3) If W is an open set in Y, then $V := f^{-1}(W)$ is an open set in X

Note that the third is a topological condition which only refers to the open sets of X, Y.

Proof (1 $\iff$ 2)
This is precisely the sequential characterization of continuity.

Proof (1 $\implies$ 3)
Let $V \subseteq Y$ be open and $X \in f^{-1}(V)$. As V is open, there must be some ϵ such that

$$b_\epsilon(f(x)) \subseteq V$$

By continuity, there is some $\delta > 0$ such that

$$f(b_\delta(x)) \subseteq b_\epsilon(f(x)) \subseteq V$$

Proof (3 $\implies$ 1, 2)
Let $\{x_n\} \subseteq X \to x \in X$.

Let $\epsilon > 0$ and take

$$V = b_\epsilon(f(x))$$

By (3), $f^{-1}(V)$ is open, so for any $x \in f^{-1}(V)$, there is some $\delta > 0$ such that

$$b_\delta(x) \subseteq V$$

this shows (1).

To see (2), since $x_n \to x$, there is some $N \in \mathbb{N}$ such that $n \geq N$ means

$$d(x_n, x) < \delta \implies \rho(f(x_n), f(x)) < \epsilon$$

this is precisely (2).

We conclude the proof.

Proof ($\neg 1 \implies \neg 2$)
If f is not continuous at $x \in X$, then there is some ϵ_0 such that

$$f(b_\delta(x)) \not\subseteq b_{\epsilon_0}(f(x))$$

for any $\delta > 0$.

For each $\delta = \frac{1}{n}$, pick $x_n \in b_\delta(x)$ such that

$$\rho(f(x), f(x_n)) \geq \epsilon_0$$

so sequential continuity does not hold.

Example 5.1.4
Consider a metric space (X, d) with the discrete metric.

Since any subset of X is open, any function on $f : (X, d) \to (Y, d_Y)$ is continuous.

Definition 5.1.5 (isometry)
Let $(X, d), (Y, \rho)$ be metric spaces.
We say X is isometric to Y if there is an isometry

$$f : X \to Y$$

which is a function with

$$\rho(f(x), f(x')) = d(x, x')$$

for all $x, x' \in X$.

Definition 5.1.6 (Lipschitz)
f is Lipschitz if there is some $c \in \mathbb{R}$ such that

$$\rho(f(x), f(x')) \leq c d(x, x')$$

for all $x, x' \in X$.

Definition 5.1.7 (Bi-Lipschitz)
f is Bi-Lipschitz if there is some $c, C \in \mathbb{R}_+$ such that

$$c d(x, x') \leq \rho(f(x), f(x')) \leq C d(x, x')$$

for all $x, x' \in X$.

Definition 5.1.8 (Uniform Continuity)
f is uniformly continuous if for all $\epsilon > 0$ there is some δ such that

$$d(x, x') < \delta \implies \rho(f(x), f(x')) < \epsilon$$

Definition 5.1.9 (Homeomorphism)
f is said to be a homeomorphism if it is bijective and ϕ, ϕ^{-1} are both continuous.
We say X, Y are homeomorhic if there is such an f.

Proposition 5.1.5
Let $(X, d), (Y, \rho)$ be metric spaces.
If $f : X \to Y$ is Bi-Lipschitz and surjective, then f is a homeomorphism.

Lemma 5.1.6
Lipschitz functions are uniformly continuous.

Proof
We show that f is injective.

Suppose $x \neq x' \in X$, then

$$0 < cd(x, x') \leq \rho(f(x), f(x'))$$

so $f(x) \neq f(x')$.

Next, we claim f^{-1} is Lipschitz.

Let $y, y' \in Y, y = f(x), y' = f(x')$ so that

$$cd(x, x') \leq \rho(f(x), f(x'))$$
$$d(f^{-1}(y), f^{-1}(y')) \leq \frac{1}{c}\rho(y, y')$$

Note that if we define

$$d_2(x, y) = \rho(f(x), f(y))$$

then d_2 is an equivalent metric of X by construction.

Definition 5.1.10 (Continuity on a Set)
Given a function $f : X \to Y$ and a subset $A \subseteq X$. The restriction of $f(x)$ to A is the function

$$\begin{aligned} f\Big|_A &: A \to Y \\ x &\mapsto f(x) \end{aligned}$$

We say f is continuous on A if $f\Big|_A$ is continuous on the metric space (A, d_A).

Note that the sequential characterization of continuity applies with sequences in the subset.

5.2 Complete Metric Spaces: Cauchy Sequences

If we wish to test for convergence of a sequence by definition, we must have a limit in mind. This leads to the question whether there is some simpler test for the convergence of sequences.

Definition 5.2.1 (Cauchy Sequence)
Let (X, d_X) be a metric spaces.
A sequence $\{x_n\}$ in X is said to be Cauchy if for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$n, m \geq N \implies d(x_n, x_m) < \epsilon$$

Definition 5.2.2 (Complete)
We a metric space is complete if every cauchy sequence converges within it.

Is every cauchy sequence convergent?

The answer is no. Consider $X = (0, 1)$ with the Euclidean Metric. Then $\{x_n\}$ does not converge.

Proposition 5.2.1
If sequence $\{x_n\}$ in a metric space (X, d) converges, then it is cauchy

Proof
Suppose $x_n \to x_0$

Let $\epsilon > 0$ be arbitrary and $N \in \mathbb{N}$ be such that

$$n, m \geq N \implies d(x_n, x_0), d(x_m, x_0) < \frac{\epsilon}{2}$$

So then

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_m, x_0) < \epsilon$$

and $\{x_n\}$ is cauchy by definition.

Definition 5.2.3 (Bounded)
Let (X, d) be a metric spaces.
A subset $A \subseteq X$ is bounded if there is some $x_0 \in X, M \in \mathbb{R}$ such that

$$A \subseteq B[x_0, M]$$

Proposition 5.2.2
Assume a cauchy sequence has a subsequence $\{x_{n_k}\} \to x_0 \in X$, then $\{x_n\} \to x_0$

Proof
Let $\epsilon > 0$ there is some $N_1 \in \mathbb{N}$ such that

$$n, m \geq N_1 \implies d(x_n, x_m) < \frac{\epsilon}{2}$$

By the convergence of the subsequence, there is some $K \in \mathbb{N}$ such that

$$k \geq K \implies d(x_{n_k}, x_0) < \frac{\epsilon}{2}$$

Now let $N := \max(n_K, N_1)$ we have

$$n \geq N \implies d(x_n, x_0) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x_0) < \epsilon$$

so $x_n \to x_0$ by definition.

Example 5.2.3
Let (X, d) be a metric spaces with the discrete metric.

Every cauchy sequence is eventually constant so (X, d) is complete.

Example 5.2.4
$\mathbb{R}, \mathbb{R}^n$ is complete.

We first bound x_n, then subdivide to find sequences of monotonically decreasing and increasing endpoints which contain infinite elements of the sequence.

Then, by the Squeeze Theorem, $x_n \to L$ which is the limit of both endpoint sequences.

Proposition 5.2.5
Every cachy sequence is bounded.

Proof
Let $\{x_n\}$ be cauchy, there is some $N \in \mathbb{N}$ such that

$$n, m \geq N \implies d(x_n, x_m) < 1$$

In particular, for all $n \geq N, d(x_N, x_n) < 1$.

Next, take

$$M := \max\{d(x_N, x_1), \ldots, d(x_N, x_{N-1}), 1\}$$

We certainly have

$$\{x_n\} \subseteq B[x_0, M]$$

Theorem 5.2.6 (Least Upper Bound / Greatest Lower Bound)
Take $(\mathbb{R}, |\cdot|)$ with its normal metric.
Let $S \subseteq \mathbb{R}$ be a bounded subset, then S has a LUB / GLB.

Proof
There is a greatest integer lower bound K and a witness $K + 1 > s_0 \in S$.

In general, there is $k_n \in \{1, 2, \ldots, 9\}$ such that

$$K + \sum_{i=1}^{n} \frac{x_i}{10^{-i}}$$

is a lower bound but not

$$s_n := K + \sum_{i=1}^{n-1} \frac{x_i}{10^{-i}} + \frac{x_n + 1}{10^{-n}} \in S$$

Let

$$M := \lim_{n \to \infty} K + \sum_{i=1}^{n} \frac{x_i}{10^{-i}}$$

and note that $M \leq S$.

But $s_n - M < 10^{-n}$ so

$$\lim_{n\to\infty} s_n = M$$

this shows that

$$M = \inf S$$

Corollary 5.2.6.1
Let $x_n \in \mathbb{R}$ with $x_i \leq x_{i+1} \leq M$ for some $M \in \mathbb{R}$.
Then the limit exists.

Proof
Let $L = \sup\{x_n\}, \epsilon > 0$ and note that

$$L - \epsilon$$

is not an upper bound.

So there is some $x_N > L - \epsilon$, for all $n \geq N$

$$L - \epsilon < x_N \leq x_n \leq L$$

so $0 \leq L - x_n \leq \epsilon$.

We conclude $x_n \to L$.

5.3 Completeness of $\mathbb{R}, \mathbb{R}^n, l_p$

Theorem 5.3.1 (Bolzano-Weierstrass)
Every bounded sequence in $\mathbb{R}$ has a converging subsequence.

Proof
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Theorem 5.3.2 (Completeness of $\mathbb{R}$)
By our work prior, every cauchy sequence is bounded.
By Bolzano-Weierstrass, every such sequence has a converging subsequence.
By a previous proposition, this means the original sequence converges.

Lemma 5.3.3
$\{\vec{x}_n\} \subseteq \mathbb{R}^n$ is cauchy if and only if each component-wise sequence is cauchy.

Proof
Suppose $\{\vec{x}_n\}$ is cauchy, but

$$|\vec{x}_{k,i} - \vec{x}_{l,i}| \leq \|\vec{x}_k - \vec{x}_l\| < \epsilon$$

Conversely, we can bound the overall norm by the max of the component-wise norms.

Corollary 5.3.3.1 (Completeness of $\mathbb{R}^n$)

Proof
If $\vec{x}_k$ is cauchy, $\lim_{k\to\infty} \vec{x}_k = \vec{x}$ where $x_{k,i} \to x_i$.

Proposition 5.3.4
If (X, d) is complete then every (Y, d) where $Y \subseteq X$ is complete if and only if Y is closed.

Proof
If (Y, d) is complete, then any converging sequence is cauchy and converges in Y by definition to Y is certainly closed.

Else if Y is closed any cauchy sequence in Y is also in X so converges by completeness of X. But by closedness of Y is also converges in Y so (Y, d) is complete by definition.

Definition 5.3.1 (Banach Space)
A complete normed linear space.

Lemma 5.3.5
$\{\vec{x}_n\} \subseteq l^p$ is cauchy if and only if each component-wise sequence is cauchy.

Proposition 5.3.6 (Completeness of $(l_p, \|\cdot\|_p)$)
Let $1 \leq p \leq \infty$, then every cauchy sequence of sequences in $(l_p, \|\cdot\|_p)$ converges.

Proof ($p = \infty$)
Let $\{x_n\}$ be a cauchy sequence in the proposed metric space.

Since each component sequence is cauchy, they converge so we can define

$$x_{0,i} := \lim_{n\to\infty} x_{n,i}$$

We claim $x_n \to x_0 \in l_1$

To see this, let $\epsilon > 0$.

Since the original sequence is cauchy there is come $N_0 \in \mathbb{N}$ such that for all $k, m \geq N_0$

$$\|x_k - x_m\|_\infty < \frac{\epsilon}{2}$$

Fix a $k \geq N_0$ then for every component i and every $m \geq N_0$

$$|x_{k,i} - x_{m,i}| \leq \|x_k, x_m\|_\infty < \frac{\epsilon}{2}$$

So then

$$|x_{k,i} - x_{0,i}| = \lim_{m\to\infty} |x_{k,i} - x_{m,i}| \leq \frac{\epsilon}{2} < \epsilon$$

It follows that $\{x_{k,i} - x_{0,i}\}_{i=1}^\infty \in l_p$ and since we are in a normed linear space, then $x_0 \in l_p$.

Convergence follows trivially.

Proof ($1 \leq p < \infty$)
Let $\{x_k\}$ be a sequence in the proposed sequence space.

For similar reasons, we can define

$$x_{0,i} := \lim_{k\to\infty} x_{k,i}$$

We now claim $x_k \to x_0 \in l_p$.

Indeed, let $\epsilon > 0$, there is some $N_0 \in \mathbb{N}$ such that $k, m \geq N_0$ means

$$\|x_k - x_m\|_p < \frac{\epsilon}{2}$$

Fix a $k \geq N_0$. For all $j \in \mathbb{N}$, all $m \geq N_0$ we have

$$\left(\sum_{i=1}^{j} |x_{k,i} - x_{m,i}|^p\right)^{\frac{1}{p}} \leq \|x_k - x_m\|_p < \frac{\epsilon}{2}$$

It follows that for all partial sums up to j

$$\left(\sum_{i=1}^{j} |x_{k,i} - x_{0,i}|^p\right)^{\frac{1}{p}} = \lim_{m\to\infty} \left(\sum_{i=1}^{j} |x_{k,i} - x_{m,i}|^p\right)^{\frac{1}{p}} \leq \frac{\epsilon}{2}$$

By the arbitrary choice of j the series must also be bounded

$$\left(\sum_{i=1}^{\infty}|x_{k,i} - x_{0,i}|^p\right)^{\frac{1}{p}} \leq \frac{\epsilon}{2} < \epsilon$$

As before, this suffices to show $x_k \to x_0 \in l_p$.

Example 5.3.7
$C[a, b]$ with the L^p norm with $1 \leq p < \infty$ is not complete.

Consider

$$g_n(x) := \begin{cases} 0, & x \in [a, \frac{a+b}{2}] \\ \text{linear}, & x \in (\frac{a+b}{2}, \frac{a+b}{2} + \frac{1}{n}] \\ 1, & x \in (\frac{a+b}{2} + \frac{1}{n}, b] \end{cases}$$

$g_n \to f$ where

$$f(x) = \begin{cases} 0, & x \in [a, \frac{a+b}{2}] \\ 1, & x \in (\frac{a+b}{2}, b] \end{cases}$$

but $f \notin C[a, b]$.

5.4 Completeness of the Dual Space

Definition 5.4.1 (Dual Space)
Let V be a normed vector space

$$V^* := \{\varphi \in \mathcal{L}(V, \mathbb{F}) : \varphi \text{ is continuous}\}$$

Proposition 5.4.1
Let $\varphi \in \mathcal{L}(V, \mathbb{F})$. The following are equivalent

(1) φ is continuous

(2) φ is continuous at 0

(3) φ is Lipschitz with Lipschitz Constant

$$\|\varphi\|_* := \sup_{\|v\| \leq 1} |\varphi(v)| < \infty$$

Proof (1 $\implies$ 2)
Continuity is by definition point-wise continuity over the entire domain.

Proof ($\neg 3 \implies \neg 2$)
We first claim φ is Lipschitz if and only if $\|\varphi\|_*$ is a Lipschitz constant.

Suppose φ is Lipschitz, so there is some c such that

$$|\varphi(v)| \leq c\|v\|$$

So

$$\|\varphi\|_* = \sup_{\|v\|\leq 1} |\varphi(v)| \leq \sup C \cdot 1 = C$$

and $\|\varphi\|_*$ is the optimal Lipschitz constant.

Conversely, suppose $\|\varphi\|_*$ is not a Lipschitz constant. Note that

$$v = \|v\| \cdot \frac{v}{\|v\|}$$

so if $\|\varphi\|_*$ is finite, then

$$|\varphi(v)| = \|v\| \left| \varphi\left(\frac{v}{\|v\|}\right) \right| \leq \|\varphi\|_* \|v\|$$

which contradicts our assumptions.

So we must have $\|\varphi\|_* = +\infty$. We can pick

$$v_n \in V, \|v_n\| \leq 1, |\varphi(v_n)| > n^2$$

But

$$\left\| \frac{1}{n} v_n \right\| \leq \frac{1}{n}$$

so $\frac{1}{n} v_n \to 0$. But clearly

$$\varphi\left(\frac{1}{n} v_n\right) \not\to \varphi(0) = 0$$

so φ is discontinuous at $v = 0$ as desired.

Proof ($3 \implies 1$)
Lipschitz continuity implies uniform continuity implies continuity.

Theorem 5.4.2
Let V be a normed vector space over $\mathbb{F}$.
$(V^*, \|\cdot\|_*)$ is a banach space.

Proof
We first show that $\|\cdot\|_*$ is a norm.

(1) $\|\varphi\|_* = 0 \iff \varphi = 0$

(2) $\|\lambda\varphi\|_* = \sup_{\|v\|\leq 1}|\lambda\varphi(v)| = |\lambda|\|\varphi\|_*$

(3) $\varphi, \psi \in V^*$

$$\begin{aligned}\|\varphi + \psi\| &= \sup_{\|v\|\leq 1} |(\varphi+\psi)(v)| \\ &\leq \sup_{\|v\|\leq 1} |\varphi(v)| + |\psi(v)| \\ &\leq \sup_{\|v\|\leq 1} |\varphi(v)| + \sup_{\|v\|\leq 1} |\varphi(v)| \\ &= \|\varphi\|_* + \|\psi\|_*\end{aligned}$$

Then let $\{\varphi_n\}_{n\geq 1}$ be cauchy, for all $\epsilon > 0$, there is some N such that $m, n \geq N$ means that

$$\|\varphi_n - \varphi_m\|_* < \epsilon$$

Fix $v \in V, \|v\| \leq 1$, we have

$$|\varphi_n(v) - \varphi_m(v)| \leq \|\varphi_n - \varphi_m\|_* \|v\| < \epsilon\|v\|$$

hence $\varphi_n(v)$ is also cauchy and converges as $\mathbb{F}$ is complete.

Let

$$\varphi(v) := \lim_{m\to\infty} \varphi_m(v)$$

under $|\cdot|$.

Let $m \to \infty$ we have

$$\begin{aligned}|\varphi_n(v) - \varphi(v)| \leq \epsilon\|v\| \\ \|\varphi_n - \varphi\|_* \leq \epsilon\end{aligned}$$

Now, it only remains to show that $\varphi \in V*$.

If $n \geq N$

$$\|\varphi\|_* \leq \|\varphi_n\|_* + \epsilon$$

so φ is Lipschitz (ie continuous) and so resides in V^*. Linearity simply follows from the fact that each φ_n is linear and the limit is also linear.

Since $\epsilon > 0$ was arbitrary

$$\lim_{n\to\infty} \varphi_n = \varphi \in V^*$$

on $\|\cdot\|_*$.

By definition, V^* is complete.

Definition 5.4.2 (Equivalent)
$(X, d), (X, \rho)$ have equivalent metrics if there is $0 < C \le D < \infty$

$$Cd(x, y) \le \rho(x, y) \le D(x, y)$$

for all $x, y \in X$.

Definition 5.4.3 (Equivalent)
$(X, \|\cdot\|), (X, |||\cdot|||)$ have equivalent norms if there is $0 < C \le D < \infty$

$$C\|x\| \le |||x||| \le D\|x\|$$

for all $x \in X$.

Proposition 5.4.3
$(X, d), (X, \rho)$ equivalent metrics or $(X, \|\cdot\|), (X, |||\cdot|||)$ equivalent norms.
Then the metric spaces are complete if and only if the other is complete as well.
Likewise for the normed vector spaces.

Proof
Trivial.

Theorem 5.4.4
$V = \mathbb{F}^n$ then all norms on V are equivalent.

Proof
We need only show any arbitrary norm is equivalent to the Euclidean Norm.

Let $\|\cdot\|_2, |||\cdot|||$ be the Euclidean and another arbitrary norm.

$$
\begin{aligned}
|||x||| &= \left|\left|\left|\sum_{i=1}^n x_i e_i\right|\right|\right| \\
&\leq \sum_{i=1}^n |||x_i e_i||| && \text{triangle inequality} \\
&= \sum_{i=1}^n |x_i| \cdot |||e_i||| \\
&\leq \left(\sum_{i=1}^n x_i\right)^{\frac{1}{2}} \underbrace{\left(\sum_{i=1}^n |||e_i|||^2\right)^{\frac{1}{2}}}_{=:D} && \text{Cauchy-Schwartz} \\
&= D\|x\|_2
\end{aligned}
$$

This in fact shows that

$$f(x) := |||x|||$$

is continuous with respect to $\|\cdot\|_2$.

Let $S := \{x \in V : \|x\|_2 = 1\}$ be the unit sphere which is closed, bounded, and therefore compact.

By the Extreme Value Theorem, f attains its minimum on S.

There is some $x_0 \in S$ such that

$$\|x_0\|_2 = 1, |||x_0||| = \inf_{\|x\|=1} |||x|||$$

but $x_0 \neq 0$ so $|||x_0||| \neq 0$.

If $v \in V$ then $\frac{v}{\|v\|_2} \in S$.

$$
\begin{aligned}
\left|\left|\left|\frac{v}{\|v\|_2}\right|\right|\right| &\geq |||x_0||| \\
|||v||| &\geq \underbrace{|||x_0|||}_{=:C} \cdot \|v\|_2
\end{aligned}
$$

All in all

$$C\|v\|_2 \leq |||v||| \leq D\|x\|_2$$

so all norms on $\mathbb{F}^n$ are equivalent, so $(\mathbb{F}^n|||\cdot|||)$ is complete and has the same topology as $\mathbb{F}^n$ with the regular Euclidean Norm.

5.5 Compactness

Definition 5.5.1 (Open Cover)
Let $A \subseteq (X, d)$, an open cover of A is a collection

$$\{U_\lambda\}_{\lambda \in \Lambda}$$

of open sets such that A is in the union of the collection.

Definition 5.5.2 (Subcover)
$\{U_\lambda\}_{\lambda \in \Lambda'}$ where $\Lambda' \subseteq \Lambda$.
A subcover is finite if Λ' is finite.

Definition 5.5.3 (Compact)
$A \subseteq (X, d)$ is compact if all open covers of A have a finite subcover.

Definition 5.5.4 (Sequentially Compact)
All sequences $\{a_n\} \subseteq A \subseteq (X, d)$ have a converging subsequence.

Definition 5.5.5 (Finite Intersection Property)
A collection of closed sets $\{F_\lambda\}_{\lambda \in \Lambda}$ has the FIP if

$$\lambda_1, \ldots, \lambda_n \in \Lambda \implies \bigcap_{1 \le i \le n} F_{\lambda_i} \neq \varnothing$$

Definition 5.5.6 (Totally Bounded)
$A \subseteq (X, d)$ is totally bounded if for every $\epsilon > 0$ there are

$$x_1, \ldots, x_n, A \subseteq \bigcup_{i=1}^{n} b_\epsilon(x_i)$$

Example 5.5.1
Finite sets are compact.

Example 5.5.2
$\mathbb{N}, \mathbb{R}$ are not compact.

Proposition 5.5.3
If $A \subseteq (X, d)$ is nonempty and compact, it is bounded.

Proof
Take the open cover

$$A \subseteq \bigcup_{n \in \mathbb{N}} b_n(x_0)$$

around any point $x_0 \in A$.

Any finite subcover necessarily bounds A.

Proposition 5.5.4
If $A \subseteq (X, d)$ is sequentially compact, then A is closed.

Proof
For all $x \in X \setminus A, a \in A, d(a, x) > 0$.

Take the open cover

$$A \subseteq \bigcup_{n \in \mathbb{N}} \left\{ y : d(x, y) > \frac{1}{n} \right\}$$

By compactness

$$A \subseteq \bigcup_{i=1}^{N} \left\{ y : d(x, y) > \frac{1}{n_i} \right\} = \left\{ y : d(x, y) > \frac{1}{r} \right\}$$

where $r := \max\{n_i\}$.

So $b_r(x) \cap A = \varnothing$ so $X \setminus A$ is open, which means A is closed.

Proposition 5.5.5
If $A \subseteq (X, d)$ is sequentially compact, it must be closed.

Proof
Else, there is a sequence converging outside of A which means any subsequence necessarily converges outside of A.

Remark that closed and boundedness is insufficient to guarantee compactness.

Example 5.5.6
Let X be an infinite set with the discrete metric d.

Then

$$\text{diam}(X) = \sup_{x,y} d(x, y) = 1$$

while

$$X = \bigcup_{x \in X} b_1(x) = \bigcup \{x\}$$

However, any finite subcover makes for a finite set which cannot possibly cover X.

Theorem 5.5.7 (Borel-Lebesgue)
Let (X, d) be a metric space, then TFAE

(1) X is compact

(2) If $\{F_\lambda\}_{\lambda \in \Lambda}$ has the FIP, then $\bigcap F_\lambda \neq \varnothing$

(3) X is sequentially compact

(4) X is complete and totally bounded

Proof (1 $\implies$ 2)
Suppose (X, d) is compact.

Let $\{F_\lambda\}_{\lambda \in \Lambda}$ be a collection of closed sets with the FIP.

If

$$\bigcap F_\lambda = \varnothing$$

then

$$\bigcup F_\lambda^c = X$$

is an open cover of X.

However any

$$\bigcap_{i=1}^{N} F_{\lambda_i} \neq \varnothing$$

so any finite subcover

$$\bigcup_{i=1}^{N} F_{\lambda_i}^c \neq X$$

and the space cannot be compact.

Proof (2 $\implies$ 3)
Let $\{x_n\}$ be a sequence in (X, d) and define

$$F_n := \{x_k : k \geq n\}$$

$$x_N \in \cap_{i=1}^M F_{n_i}$$

where $N := \max\{n_i\}$.

So $\{\overline{F_n}\}$ is a collection of closed sets with FIP.

By assumption, $\exists x_0 \in \bigcap \overline{F_n}$.

We have $x_0 \in F_1$ so there is some m_1 such that

$$d(x_{m_1}, x_0) < 1$$

We can do this since if $x_0 \notin F_1$ then it is a limit point, else we can literally just take $x_{m_1} = x$ itself.

For each $k \geq 2$ we have $x_0 \in \overline{F_{m_{k-1}+1}}$ so there is some $m_k > m_{k-1}$ such that

$$d(x_{m_k}, x_0) < \frac{1}{k}$$

Thus $x_{m_k} \to x_0 \in X$ and we are done.

Proof (3 $\implies$ 4)
Suppose (X, d) is sequentially compact and let $\{x_n\} \subseteq X$ be cauchy.

By sequential compactness, it has a converging subsequence

$$\{x_{n_i}\} \to x \in X$$

as x_n is cauchy, we must also have $x_n \to x$ so (X, d) is certainly complete.

Now, suppose for a contradiction that X is NOT totally bounded. There is some $\epsilon > 0$ such that X is not a finite union of $b_\epsilon(x_i)$ for each $x_i \in X$.

This, if $x_1, \ldots, x_n \in X$ with $d(x_i, x_j) \geq \epsilon$ for $i \neq j$ then

$$X \neq \bigcup b_\epsilon(x_i)$$

so we may choose x_{n+1} outside of the finite union.

Note that $d(x_{n+1}, x_i) \geq \epsilon$ for each $1 \leq i \leq n$.

Repeate the procedure and get a sequence which every subseqeuence is not cauchy and therefore cannot converge.

Definition 5.5.7 (ϵ-net)
Let $\{x_i\} \subseteq (X, d)$ be such that

$$X = \bigcup b_\epsilon(x_i)$$

then $\{x_i\}$ is an ϵ-net of X.

Proof (4 $\implies$ 1)
Suppose (X, d) is complete and totally bounded.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be some open cover of X.

For each k, let $x_{k,1}, \dots, x_{k,p_k}$ be a 2^{-k}-net of X.

If each closed ball $\overline{b_{2^{-1}}(x_{1,i})}$ has a finite subcover, then X has a finite subcover.

Indeed, suppose that

$$\overline{b_{2^{-1}}(x_{1,i})} \subseteq \bigcup_{j=1}^{n_i} U_{\lambda_{ij}}$$

then

$$X \subseteq \bigcup_{i=1}^{p_1} \overline{b_{2^{-1}}(x_{1,i})} \subseteq \underbrace{\bigcup_{i=1}^{p_1} \bigcup_{j=1}^{n_i}}_{\text{finite}} U_{\lambda_{ij}}$$

This means that there is some i_1 such that $\overline{b_{2^{-1}}(x_{1,i_1})}$ has no subcover.

But note that

$$\begin{aligned}
\overline{b_{2^{-1}}(x_{1,i_1})} &= \overline{b_{2^{-1}}(x_{1,i_1})} \cap \bigcup_{i=1}^{p_2} \overline{b_{2^{-2}}(x_{2,i})} \\
&= \bigcup_{i=1}^{p_2} \left(\overline{b_{2^{-1}}(x_{1,i_1})} \cap \overline{b_{2^{-2}}(x_{2,i})} \right)
\end{aligned}$$

But if there is a finite subcover of each of the intersections, then there is a finite subcover of $\overline{b_{2^{-1}}(x_{1,i_1})}$.

So there must be some i_2 such that

$$A_2 := \overline{b_{2^{-1}}(x_{1,i_1})} \cap \overline{b_{2^{-2}}(x_{2,i_2})}$$

has no subcover.

Continuing this argument, we can always get $i_1, \dots, i_n$ such that

$$A_n := \bigcap_{k=1}^{n} \overline{b_{2^{-k}}(x_{k,i_k})}$$

has no finite subcover.

So

$$A_n = \bigcup_{i=1}^{p_{n+1}} \left(\overline{b_{2^{-n}}(x_{n,i_n})} \cap \overline{b_{2^{-n-1}}(x_{2,i_{n+1}})} \right)$$

has no subcover with

$$\text{diam}\left(\overline{b_{2^{-n-1}}(x)} \right) \leq 2\frac{1}{2^{n+1}} = 2^{-n} \to 0$$

We claim $(x_{n,i_n})_{i=1}^{\infty}$ is cauchy.

Indeed, for $\epsilon > 0$, there is some N such that $2^{1-N} < \epsilon$. If $n > m \geq N$, then

$$\overline{b_{2^{-n}}(x_{n,i_n})} \cap \overline{b_{2^{-m}}(x_{m,i_m})} \supseteq A_n \neq \varnothing$$

This means there is some $y \in A_n$ and

$$d(x_{n,i_n}, x_{m,i_m,}) \leq d(x_{n,i_n}, y) + d(y, x_{m,i_m}) \leq 2\frac{1}{2^N} < \epsilon$$

By the completeness of (X, d)

$$x_{n,i_n} \to x_0 \in X$$

Choose some λ_0 such that $x_0 \in U_{\lambda_0}$. There is some $r > 0$ such that

$$b_r(x_0) \subseteq U_{\lambda_0}$$

Find N such that $2\frac{1}{2^N} < r$ and that for $n \geq N$

$$d(x_{n,i_n}, x_0) < \frac{r}{2}$$

This means that

$$2^{-n} \leq 2^{-N} < \frac{r}{2}$$

If $x \in \overline{b_{2^{-n}}(x_{n,i_n})}$

$$d(x, x_0) \leq d(x, x_{n,i_n}) + d(x_{n,i_n}, x_0) < \frac{1}{2^{-n}} + \frac{r}{2} < 2\frac{r}{2} = r$$

So

$$\overline{b_{2^{-n}}(x_{n,i_n})} \subseteq b_r(x_0) \subseteq U_{\lambda_0}$$

and in particular,

$$A_N \subseteq b_{2^{-N}}(x_{N,i_N}) \subseteq U_{\lambda_0}$$

So there is a finite subcover of some finite 2^{-n}-net. By definition, (X, d) is compact.

Remark that the first two conditions for compactness clearly only depend on the topology of X.

If we phrase the convergence of a sequence as the follows, then the third condition also only depends on the topology of the X.

Definition 5.5.8 (Topological Sequential Convergence)
If for all open sets U with $x_0 \in U$, there is some $N \in \mathbb{N}$ such that for $n \geq N$

$$x_n \in U$$

Then $x_n \to x_0$

On the otherhand, the fourth condition is tied to the metric.

Consider the following example

Example 5.5.8
$\mathbb{R}$ is homeomorphic to $(0, 1)$ so in a sense, they have the same open sets.

Consider the bijection

$$f(x) = \frac{\arctan(x) + \frac{\pi}{2}}{\pi}$$

However, $\mathbb{R}$ is complete but not totally bounded while $(0, 1)$ is not complete but totally bounded.

So completeness and total boundedness does not correlate with the topology of a set.

Proposition 5.5.9
If $A \subseteq (X, d)$ then A is a compact subset of X if and only if (A, d) is compact.

Proof ($\implies$)
If A is compact subset of X and $\{U_\alpha\}$ is a collection of open sets open in A which cover A.

We wish to find $\{V_\alpha\}$ open in X such that

$$V_\alpha \cap A = U_\alpha$$

If $x \in U_\alpha$, there is some $r_x > 0$ with

$$b^A_{r_x} \subseteq U_\alpha$$

Let

$$V_\alpha := \bigcup_{x \in U_\alpha} b_{r_x}^X(x)$$

and note that it is open in X.

But then

$$V_\alpha \cap A = \bigcup \left(b_{r_x}^X(x) \cap A\right) = \bigcup b{r_x}^A(x) = U_\alpha$$

So then by the compactness of A in X, we know get extract a finite subcover of $\{V_\alpha\}$, which induces a finite subcover of $\{U_\alpha\}$.

By definition (A, d) is compact as desired.

Proof ($\Longleftarrow$)
Suppose (A, d) is compact.

Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of A.

Define

$$U_\alpha := A \cap V_\alpha$$

and note that U_α is open in (A, d) with

$$A \subseteq \bigcup U_\alpha$$

But then the existence of a finite subcover is evident.

The proposition above shows that compactness is an intrinsic property of a set and NOT the larger ambient universe.

Theorem 5.5.10 (Heine-Borel)
$A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof
The forward direction is trivial.

We know that $\mathbb{R}^n$ is complete and A being closed implies A is complete in $\mathbb{R}^n$.

Define

$$R := \sup_{x \in A} \|x\|_2 < \infty$$

which exists by boundedness.

It suffices to show that $[-R, R]^n$ is totally bounded.

Let $\epsilon > 0, \delta := \frac{\epsilon}{n}$.

Consider

$$\{k_1\delta, k_2\delta, \dots, k_n\delta\} \in \mathbb{R}^n : k_i \in \mathbb{Z}, |k_i| \leq \frac{R+1}{\delta}$$

Note that for $x \in A$

$$\forall i, \exists k_i, |k_i\delta - x_i| < \delta$$

Then

$$\|x - (k_1\delta, \dots, k_n\delta)\|_1 = \sum |x_i - k_i\delta| < n\delta = \epsilon$$

Although we used 1-norm, all norms in $\mathbb{R}^n$ are equivalent so we are done.

Proof (alternative)
Apply Bolzano-Weierstrass theorem repeatedly to get sub^{n}sequence which converges by component-wise convergence.

Proposition 5.5.11
If (X, d) is compact with $A \subseteq X$ closed, then A is compact.

Proof
If $\{U_\lambda\}$ is an open cover of A, then

$$\{U\lambda\} \cup A^c$$

is an open cover of X.

The rest is trivial.

5.6 More on Compactness

Definition 5.6.1 (Lebesgue Number)
Let (A, d) be compact and $\{U_\lambda\}$ be an open cover.
Define the lebesgue number as

$$\delta(\{U_\lambda\}) = \inf_{x \in A} \sup\{r > 0 : b_r(x) \subseteq U_\lambda\}$$

Theorem 5.6.1
Let (A, d) be compact and $\{U_\lambda\}$ be an open cover.
Then $\delta(\{\lambda\}) > 0$.

Proof
Let $x \in A$, there is some $\lambda(x)$ such that $x \in U_{\lambda(x)}$.

By definition there is some $r(x) > 0$ such that

$$b_{r(x)}(x) \subseteq U_{\lambda(x)}$$

Now

$$\{b_{\frac{r(x)}{2}}(x) : x \in A\}$$

is an open cover of A.

there is a subcover

$$b_{\frac{r(x_1)}{2}}(x_1), \dots, b_{\frac{r(x_n)}{2}}(x_n)$$

Let $r := \min\left\{\frac{r(x_i)}{2}\right\} > 0$.

For all $x \in A$ there is some x_i

$$x \in b_{\frac{r(x_i)}{2}}(x_i)$$

If $y \in b_r(x)$ then

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < r + \frac{r(x)}{2} \leq r(x_i)$$

This shows that

$$b_r(x) \subseteq b_{r(x_i)}(x_i) \subseteq U_{\lambda(x_i)}$$

By the arbitrary choice of x

$$\delta(\{U_\lambda\}) > 0$$

as desired.

Definition 5.6.2 (Product Space)
The product space of $(X, d), (Y, \rho)$ is

$$(X \times Y, D)$$

where D is defined as

$$D((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), \rho(y_1, y_2)\}$$

Note that

$$b_r(x, y) = b_r(x) \times b_r(y)$$

Theorem 5.6.2
Let $(X, d), (Y, \rho)$ be compact metric spaces. Then

$$(X \times Y, D)$$

is compact.

Proof
Let $\{W_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $X \times Y$.

Fix a $y_0 \in Y$.

For each $x \in X$ there is an indice $\lambda(x)$ such that

$$(x, y_0) \in W_{\lambda(x)}$$

By definition, there is some $r(x) > 0$ such that

$$b_{r(x)}(x, y_0) = b_{r(x)}(x) \times b_{r(x)}(y_0) \subseteq W_{\lambda(x)}$$

By construction $\{b_{r(x)} : x \in X\}$ is an open cover of X. By compactness, we can extract a finite subcover

$$\{b_{r(x_i)}(x_i)\}$$

for $1 \leq i \leq n(y_0)$.

$$X \subseteq \bigcap_{i=1}^{n(y_0)} b_{r(x_{i,y_0})}(x_{i,y_0})$$

Define

$$r(y_0) := \min\{r(x_{i,y_0}) : 1 \leq i \leq n(y_0)\}$$

By construction

$$\begin{aligned} b_{r(y_0)}(x_{i,y_0}, y_0) &= b_{r(y_0)}(x_{i,y_0}) \times b_{r(y_0)}(y_0) \\ &\subseteq b_{r(x_{i,y_0})}(x_{i,y_0}, y_0) \\ &\subseteq W_{\lambda(x)} \end{aligned}$$

For each y_0 get $r(y_0)$ and $X \times b_{r(y_0)}(y_0)$ is covered by a finite set

$$W_{\lambda(x_{1,y_0})}, \dots, W_{\lambda(x_{n(y_0),y_0})}$$

So

$$\{b_{r(y_0)}(y_0) : y_0 \in Y\}$$

is an open cover of Y.

Again by compactness, we can extract a finite subcover

$$b_{r(y_1)}(y_1)\dots, b_{r(y_m)}(y_m)$$

So $X \times b_{r(y_j)}(y_j)$ is covered by

$$W_{\lambda(x_{i,y_j})}, \dots, W_{\lambda(x_{n(y_j),y_j})}$$

Overall

$$X \times Y \subseteq \bigcup \{W_{\lambda(x_{i,y_j})} : 1 \leq j \leq m, 1 \leq i \leq n(y_j)\}$$

and by definition $X \times Y$ is compact.

Definition 5.6.3 (Dense)
A subset $S \subseteq (X, d)$ is dense if $\bar{S} = X$.

Definition 5.6.4 (Seperable)
A subset $S \subseteq (X, d)$ is seperable if it has a countable, dense subset.

Proposition 5.6.3
If (X, d) is compact, then it is seperable.

Proof
For $n \geq 1$, get

$$x_{n,1} \dots x_{n,i(n)}$$

a 2^{-n}-net of X and take their union

$$S = \{x_{n,i} : n \geq 1, i \leq i(n)\}$$

which is certainly countable.

S is also dense by construction since every point lies in some 2^{-n} ball of some point $x_{n,k(n)}$ by the definition of a net.

Example 5.6.4
$\mathbb{R}^n$ is seperable by $\mathbb{Q}^n$.

Example 5.6.5
$l_p, 1 \leq p < \infty$ is seperable.

We approximate each sequence with one with a finite head and tail of zeros.

Then, we approximate the sequence above with Q^n.

Theorem 5.6.6 (Cantor for $\mathbb{N}$)
$|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$

Proof
Attempt to enumerate $\mathcal{P}(\mathbb{N})$ with $E_1, E_2, \ldots$ and consider

$$F = \{n : n \notin E_n\}$$

If $m \in E_m$ then $m \notin F$ and if $m \notin E_M$ then $m \in F$. So $F \neq E_m$ for any m and it is not in the enumeration.

Example 5.6.7
l_∞ is not seperable.

For $E \subseteq \mathbb{N}$ define

$$x_E(n) = \begin{cases} 1, & n \in E \\ 0, & n \notin E \end{cases}$$

Then $E \neq F \implies d(x_E, x_F) = 1$.

There are also $2^{\mathbb{N}}$ such sequences which is certainly not countable.

This means that if any $S \subseteq l_\infty$ with $\bar{S} = l_\infty$ then there are uncountably many points since there must be at least one within $\frac{1}{2}$ of every x_E.

Theorem 5.6.8
(X, d) is compact and $f : (X, d) \to (Y, \rho)$ is continuous means

(1) $f(X)$ is compact

(2) f is uniformly continuous

Proof (1)
We have $f(X) \subseteq Y$.

Let $\{V_\lambda\}$ be an open cover of X.

Define

$$U_\lambda := f^{-1}(V_\lambda)$$

and note that it is open by continuity of f.

Then $\{U_\lambda\}$ is an open cover of X.

Get a finite subcover

$$U_{\lambda_1}, \dots, U_{\lambda_n}$$

and note that

$$f(X) = \bigcup f(U_{\lambda_i}) \subseteq \bigcup V_\lambda$$

By definition, $f(X)$ is compact.

Proof (2)
Let $\epsilon > 0$. For $x \in X$, by continuity, there is $\delta_x > 0$ such that

$$d(x, x') < \delta_x \implies \rho(f(x), f(x')) < \frac{\epsilon}{2}$$

Then

$$\left\{ b_{\frac{\delta_x}{2}} : x \in X \right\}$$

is an open cover of X.

Extract a finite subcover

$$b_{\frac{\delta_{x_1}}{2}}, \dots, b_{\frac{\delta_{x_n}}{2}}$$

and define

$$\delta := \min\left(\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2}\right) > 0$$

If $z_1, z_2 \in X$ such that $d(z_1, z_2) < \delta$, by the definition of a subcover, there is an x_i with

$$d(z_1, x_i) < \frac{\delta}{2}$$

Thus we have

$$\begin{aligned} d(z_2, x_i) &\leq d(z_2, z_1) + d(z_1, x_i) \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leq \delta_{x_i} \end{aligned}$$

So

$$\begin{aligned} &\rho(f(z_1), f(x_i)) < \frac{\epsilon}{2} \\ &\rho(f(z_2), f(x_i)) < \frac{\epsilon}{2} \\ &\rho(f(z_1), f(z_2)) \leq \rho(f(z_1), f(z_i)) + \rho(f(x_i), f(z_2)) < 2 \cdot \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Corollary 5.6.8.1 (Extreme Value Theorem)
(X, d) is compact and $f : (X, d) \to (Y, \rho)$ is continuous means f attains its extrema on $f(X)$.

Proof
$f(X)$ is compact which means it is closed and bounded.

So

$$\sup f(X) = L$$

exists.

By definition

$$L \in \overline{f(X)}$$

but then by closedness

$$L \in f(X)$$

Proposition 5.6.9
Suppose (X, d) is compact and $f : (X, d) \to (Y, \rho)$ is continuous.
If f is bijective then f^{-1} is continuous (a homeomorphism).

Proof
f^{-1} is a function as f is bijective.

Show if U open in X, $(f^{-1})^{-1}(U) = f(U)$ is open in Y.

Let U be open in X, U^c is a closed subset of X and therefore compact. So $f(U^c)$ is compact and also closed.

But f is bijective so

$$f(U^c)^c = f(U)$$

is open.

By definition f^{-1} is continuous.

5.7 Completeness of $(C_b(X), \|\cdot\|_\infty)$

For (X, d), $C^b_{\mathbb{F}}(X)$ denotes the set of bounded, continuous, and real-valued functions from $X \to \mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Note that if X is compact, the its image is necessarily bounded so we can just write $C(X)$.

Definition 5.7.1 (Pointwise Convergence)
Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\{f_n : X \to Y\}$ be a sequence of functions from X to Y
We say $f_n \to f_0$ on X point-wise if for every $x \in X$

$$\lim_{n\to\infty} f_n(x) = f_0(x)$$

Definition 5.7.2 (Uniform Convergence)
We say $f_n \to f_0$ uniformly on X if for every $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$n \geq N \implies d(f_n(x), f_0(x)) < \epsilon$$

Clearly uniform convergence implies point-wise convergence. However, the converse need not hold!

Consider $f_n : [0,1] \to [0,1], f_n(x) = x^n$. We have $f_n \to f_0$ point-wise where

$$f_0(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

However since every f_n is continuous while f_0 is not (specifically at 0), we can always get infinitely close to 1 ($f(1-\delta) \approx 1$) and away from $f_0(1-\delta) = 0$ no matter the value of n.

We will now show that uniform convergence preserves continuity.

Theorem 5.7.1
Let $(X, d_X), (Y, d_Y)$ be metric spaces and $\{f_n : X \to Y\}$ be a sequence of functions from X to Y.
Suppose $f_n \to f_0$ uniformly and each f_n is continuous at $x_0 \in X$, then f_0 is continuous at x_0

Proof
Let $\epsilon > 0$.

By uniform convergence, there must be some $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$d_Y(f_n(x), f_0(x)) < \frac{\epsilon}{3}$$

By continuity, there is some $\delta > 0$ such that $x \in B(x_0, \delta)$ means we have

$$d_Y(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$$

But then for all $x \in B(x_0, \delta)$

$$d_Y(f_0(x), f_0(x_0)) \leq d_Y(f_0(x), f_N(x)) + d_Y(f_N(x), f_N(x_0)) + d_Y(f_N(x_0), f_0(x_0)) < \epsilon$$

Definition 5.7.3 ($(C_b(X), \|\cdot\|_\infty)$)
Let (X, d) be a metric space and

$$C_b(X) := \{f : X \to \mathbb{R} : f \text{ is continuous and bounded}\}$$

Define the norm on $C_b(X)$

$$\begin{aligned} \|\cdot\|_\infty : C_b(X) &\to \mathbb{R} \\ f &\mapsto \sup\{|f(x)| : x \in X\} \end{aligned}$$

Then $(C_b(X), \|\cdot\|_\infty)$ is a normed linear space.

Theorem 5.7.2 (Completeness of $(C_b(X), \|\cdot\|_\infty)$)
$(C_b(X), \|\cdot\|_\infty)$ is complete.
This is the same as saying the uniform limit of functions are continuous.

Proof
Suppose $\{f_n\}$ is cauchy in $(C_b(X), \|\cdot\|_\infty)$ and let $x_0 \in X$.

Clearly, $\{f_n(x_0)\}$ is a cauchy sequence in $\mathbb{R}$.

Now, define

$$\begin{aligned} f_0 : X &\to \mathbb{R} \\ x &\mapsto \lim_{n\to\infty} f_n(x) \in \mathbb{F} \end{aligned}$$

note the limit is in $\mathbb{F}$ by completeness of the field.

We claim $f_n \to f_0$ uniformly on X.

To see this let $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $n, m \geq N$ implies

$$\|f_n - f_m\|_\infty < \frac{\epsilon}{2}$$

Let $n \geq N$ and $x \in X$, we have

$$\begin{aligned} |f_n(x) - f_0(x)| &= \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\leq \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

This shows uniform convergence.

It only remains to show that f_0 is in the proposed space, or equivalently, it is bounded.

Since $\{f_n\}$ is cauchy, it must be bounded, say by $M \in \mathbb{R}$.

By uniform convergence there is some $N \in \mathbb{N}$ such that

$$\|f_N(x) - f_0(x)\|_\infty < 1$$

It follows that

$$f_0(x) < f_N(x) + 1 = M + 1$$

for all $x \in X$.

Note that if we let $X = \mathbb{N}$ and give it the discrete metric, then

$$(C_b(X), \|\cdot\|_\infty) \equiv (l_\infty, \|\cdot\|_\infty)$$

5.8 Space-Filling Curves

Theorem 5.8.1
Let (X, d) be compact.
There is a continuous surjection from the cantor set C onto X.

Example 5.8.2
$X = [0,1], C = \{0.(2\epsilon_1)(2\epsilon_2)(2\epsilon_3)\cdots : \epsilon_i \in \{0,1\}\}$

We can take

$$f(x) = 0.\epsilon_1\epsilon_2\ldots$$

Proof
X is compact and therefore has a finite $\frac{1}{2}$-net

$$x_1^{(1)}, \ldots, x_{n(1)}^{(1)}$$

Without loss of generality, $n(1) \geq 2$.

Split C into disjoint clopen subsets.

$$C_1^{(1)}, \ldots, C_{n(1)}^{(1)}, \operatorname{diam}(C_i^{(1)}) \leq \frac{1}{3}$$

Moreoever, define $f_1 : C \to X$ by

$$f_1\Big|_{C_i^{(1)}} = x_i^{(1)}$$

This is the base case.

Suppose at the k-th stage, we get a 2^{-k}-net of X

$$x_1^{(k)}, \dots, x_{n(k)}^{(k)}$$

a partition of C consisting of disjoint clopen sets

$$C_1^{(k)}, \dots, C_{n(k)}^{(k)}, \operatorname{diam}(C_i^{(k)}) \leq \frac{1}{3^k}$$

and $f_k : C \to X$ defined by

$$f_k\Big|_{C_i^{(k)}} = x_i^{(k)}$$

which is locally constant and therefore continuous.

Fix $1 \leq i \leq n(k)$ and note that we can find a $2^{-(k+1)}$-net of $\overline{b_{2^{-k}}(x_i^{(k)})}$

$$x_1^{(k+1,i)}, \dots, x_{n(k+1,i)}^{(k+1,i)}$$

Furthermore, we can split $C_i^{(k)}$ into disjoint clopen subsets

$$C_1^{(k+1,i)}, \dots, C_{n(k+1),i}^{(k+1,i)}, \operatorname{diam} C_j^{(k+1,i)} \leq 3^{-(k+1)}$$

Apply this to all $x_i^{(k)}, C_i^{(k)}$ and define $f_{k+1} : C \to X$ by

$$f_{k+1}\Big|_{C_j^{(k+1,i)}} = x_j^{(k+1,i)}$$

Note that f_{k+1} is again locally constant and therefore continuous.

We claim that

$$\|f_k - f_{k+1}\|_\infty \leq \frac{1}{2^k}$$

Indeed, let $y \in C_j^{(k+1,i)} \subseteq C_i^{(k)}$. We have

$$f_{k+1}(y) \in \overline{b_{2^{-k}} f_k(y)}$$

so

$$\|f_{k+1} - f_k\| \leq \frac{1}{2^k}$$

as desired.

By construction,

$$\sum_{k=1}^{\infty} \|f_{k+1} - f_k\|_\infty < \infty$$

hence for $\epsilon > 0$ there is some N such that for $n \geq m \geq N$

$$\|f_n - f_m\|_\infty \leq \sum_{k=N}^{\infty} \|f_{k+1} - f_k\|_\infty < \epsilon$$

so f_n is cauchy.

But $f_n(x)$ is then cauchy in X for any fixed $x \in X$ so

$$f_n \to f$$

point-wise.

However, for $\epsilon > 0$, there is some N such that for $m, n \geq N$

$$\|f_n - f\|_\infty = \lim_{m\to\infty} \|f_n - f_m\|_\infty \leq \frac{\epsilon}{2} < \epsilon$$

so $f_n \to f$ uniformly, demonstrating the continuity of f by our prior work.

It remains to show that f is surjective.

Let $x \in X$ there is some $x_{i_1}^{(1)}$

$$d(x, x_{i_1}^{(1)}) \leq \frac{1}{2}, \ f_1\Big|_{C_{i_1}^{(1)}} = x_{i_1}^{(1)}$$

similarly, there is some $x_{j_1}^{(2,i_1)}$ such that

$$d(x, x_{j_1}^{(2,i_1)}) \leq \frac{1}{4}, \ f_2\Big|_{C_j^{(2,i_1)}} = x_j^{(2,i_1)}$$

Recursively, find $C_{i_1}^{(1)} \supseteq C_{i_1}^{(2)} \supseteq \dots$ such that

$$f_k\Big|_{C_{i_k}^{(k)}} = x_{i_k}^{(k)}, d(x_{i_k}^{(k)}, x) \leq \frac{1}{2^k}$$

By construction

$$\lim_{n\to\infty} x_{i_k}^{(k)} = x$$

Let

$$y \in \bigcap_{n=1}^{\infty} C_{i_k}^{(k)} \neq \varnothing$$

which is possible by the FIP and compactness.

Note that diam $= 0$ so there is only one point! So

$$f_k(y) = x_{i_k}^{(k)} \implies f(y) = \lim_{k\to\infty} x_{i_k}^{(k)} = x$$

Definition 5.8.1 (path)
The continuous image of $[0,1]$ in a metric space.

Definition 5.8.2 (Peano/Space Filling Curve)
A path in $\mathbb{R}^n$ with $n \geq 2$ such that the image (path) has interior.

Theorem 5.8.3
Let X be a compact, convex subset of a normed vector space $(V, \|\cdot\|)$.
There is a continuous surjective path $\gamma : [0,1] \to X$.

Proof
By the previous theorem, $\exists f : C \to X$ which is continuous and surjective mapping the cantor set onto X.

Write

$$[0,1] \setminus C = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

as the disjoint union of intervals.

Note

$$b_n - a_n \to 0$$

since we are essentially taking the middle third that is being removed at every iteration of the construction of the Cantor set.

Define $\tilde{f} : [0,1] \to X$ by

$$\tilde{f}(x) = \begin{cases} f(x), & x \in C \\ tf(a_n) + (1-t)f(b_n), & x = ta_n + (1-t)b_n \in (a_n, b_n), t \in (0,1) \end{cases}$$

Notice that $\tilde{f}([0,1]) \subseteq X$ as X is convex and it is surjective by construction.

To see continuity let $\epsilon > 0$ there is some $\delta' > 0$ such that for $x, y \in C$

$$|x - y| < \delta' \implies \|f(x) - f(y)\| < \frac{\epsilon}{3}$$

Let $F := \{n : b_n - a_n \geq \delta'\}$ and note that it is finite by a previous observation.

Define
$$D := \max_{n\geq 1}\|f(b_n) - f(a_n)\| = \max\{\max_{n\in F}\|f(b_n) - f(a_n)\|, \frac{\epsilon}{3}\}$$

Furthermore, let
$$L := \min_{n\in F} b_n - a_n, \delta := \min\left\{\delta', \frac{\epsilon L}{3D}\right\}$$

Let $x, y \in [0,1]$ such that $|x-y| < \delta$.

Case I $x, y \in C$ so $\|\tilde{f}(x) - \tilde{f}(y)\| < \frac{\epsilon}{3}$

Case II $x, y \in [a_n, b_n], n \in F$ then

$$\|\tilde{f}(x) - \tilde{f}(y)\| = \frac{|x-y|}{b_n - a_n}\|\tilde{f}(b_n) - \tilde{f}(a_n)\| < \frac{\epsilon L}{3D} \cdot \frac{D}{L} = \frac{\epsilon}{3}$$

Case III $x, y \in [a_n, b_n], n \notin F$ so $b_n - a_n < \delta'$ so

$$\|\tilde{f}(x) - \tilde{f}(y)\| \leq \|\tilde{f}(b_n) - \tilde{f}(a_n)\| < \frac{\epsilon}{3}$$

Case IV $x \in C, y \in (a_n, b_n)$ WLOG $x \leq a_n$ (the argument is symmetric otherwise)

$$\begin{aligned}\|\tilde{f}(x) - \tilde{f}(y)\| &\leq \|\tilde{f}(x) - \tilde{f}(a_n)\| + \|\tilde{f}(a_n) - \tilde{f}(y)\| \\ &< \underbrace{\frac{\epsilon}{3}}_{x, a_n \in C, |x-a_n|<\delta} + \underbrace{\frac{\epsilon}{3}}_{a_n, y \in [a_n, b_n], |a_n - y| < \delta}\end{aligned}$$

Case V $x \in (a_n, b_n), y \in (a_m, b_m), n \geq m$ WLOG $b_n < a_m$. (relabel otherwise)

$$\begin{aligned}\|\tilde{f}(x) - \tilde{f}(y)\| &\leq \|\tilde{f}(x) - \tilde{f}(b_n)\| + \|\tilde{f}(b_n) - \tilde{f}(a_m)\| + \|\tilde{f}(a_m) - \tilde{f}(y)\| \\ &< \underbrace{\frac{\epsilon}{3}}_{x, b_n \in [a_n, b_n], \text{Case II, III}} + \underbrace{\frac{\epsilon}{3}}_{b_m, a_n \in C, \text{Case I}} + \underbrace{\frac{\epsilon}{3}}_{\text{Case II, III}}\end{aligned}$$

5.9 Compactness of $C(X)$

We ask ourselves when is $K \subseteq C(X)$ where (X, d) is compact, also compact.

(i) K must be closed (uniform convergence preserves continuity) and bounded (X is compact)

(ii) $C(X)$ is complete

This tells us that K must be complete and we only need total boundedness.

Example 5.9.1 ($C(0,1)$)

$$f_n(x) := \begin{cases} 0, & 0 \le x \le \frac{1}{n+1} \\ 1, & x = \frac{1}{n} \\ 0, & x \ge \frac{1}{n-1} \\ \text{linear}, & \text{elsewise} \end{cases}$$

f_n is continuous but

$$\|f_n - f_m\| = \left\| f_n\left(\frac{1}{n}\right) - f_m\left(\frac{1}{m}\right) \right\| = 1, n \neq m$$

so $\{f_n : n \ge 1\}$ is closed, bounded, but NOT totally bounded.

$(f_n)_{n=1}^{\infty}$ has no convergence subsequence.

Definition 5.9.1 (Equicontinuous)
$\mathcal{F} \subseteq C(X)$ is equicontinuous if for all $x \in X, \epsilon > 0$ there is some $\delta > 0$ such that

$$\forall f \in \mathcal{F}, d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon$$

Lemma 5.9.2
If $K \subseteq C(X)$ is totally bounded, then K is equicontinuous.

Proof
We know K is totally bounded.

Let $x \in X, \epsilon > 0$

Let $f_1, \dots, f_n$ be a finite $\frac{\epsilon}{3}$-net for K.

By the continuity of f_i, there is some $\delta_i > 0$ such that

$$d(x,y) < \delta_i \implies |f_i(x) - f_i(y)| < \frac{\epsilon}{3}$$

Define

$$\delta := \min\{\delta_i\} > 0$$

Let $f \in K, d(x,y) < \delta$. There is some i_0 such that $\|f - f_{i_0}\| < \frac{\epsilon}{3}$

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_{i_0}(x)| + |f_{i_0}(x) - f_{i_0}(y)| + |f_{i_0}(x) - f(y)| \\ &< \underbrace{\frac{\epsilon}{3}}_{\|f-f_{i_0}\| \leq \frac{\epsilon}{3}} + \underbrace{\frac{\epsilon}{3}}_{\text{continuity}} + \underbrace{\frac{\epsilon}{3}}_{\|f-f_{i_0}\| \leq \frac{\epsilon}{3}}\end{aligned}$$

Theorem 5.9.3 (Arzela-Ascoli)
Let (X, d) be compact.
$K \subseteq C(X)$ is compact if and only if it is closed, bounded, and equicontinuous.

Proof ($\neg \impliedby \neg$)
Straightforward.

Proof ($\impliedby$)
Suppose that K is closed, bounded, and equicontinuous.

We have $K \subseteq C(X)$, which is complete. Combined with the closedness, K is complete.

We shows that K is totally bounded.

Let $\epsilon > 0$. Since K is equicontinuous, for all $x \in X$ there is some δ_x such that

$$d(x,y) < \delta_x \implies |f(x) - f(y)| < \epsilon$$

for all $f \in K$.

Now, $\{b_{\delta_x}(x) : x \in X\}$ is an open cover, so there is a finite cover by compactness.

$$b_{\delta_{x_1}}(x_1), \dots, b_{\delta_{x_n}}(x_n)$$

Define the linear function $T : C(X) \to \mathbb{F}^n$ by

$$Tf = (f(x_1), \dots, f(x_n))$$

and note that TK is bounded in $\mathbb{F}^n$ as K is bounded in $C(X)$.

But bounded subsets of $\mathbb{F}^n$ are totally bounded! We can choose

$$f_1, \dots, f_m$$

such that

$$Tf_1, \dots, Tf_m$$

form an ϵ-net of TK.

We claim that $f_1, \dots, f_m$ are an ϵ-net for K.

Indeed, let $f \in K, y \in X$. Choose j such that

$$\|Tf - Tf_j\| < \epsilon$$

so

$$|f(x_i) - f_j(x_i)| < \epsilon, 1 \leq i \leq n$$

Choose i such that $y \in b_{x_i}(x_i)$

$$\begin{aligned} |f(y) - f_j(y)| &\leq |f(y) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(y)| \\ &< 3\epsilon \end{aligned}$$

$$\begin{aligned} \|f - f_j\|_\infty &= \sup_{y \in X} |f(y) - f_j(y)| \\ &\leq 3\epsilon \\ &< 4\epsilon \end{aligned}$$

Proof ($\Longleftarrow$ alternative)
Suppose K is closed, bounded, and equicontinuous. We show K is sequentially compact.

Let $(f_k)_{k=1}^\infty$ be a sequence in K.

Pick $\epsilon_1 > \epsilon_2 > \dots$ such that $\epsilon_i \to 0$.

Let $x \in X$, there are δ_i such that

$$d(x, y) < \delta_i \implies |f(x) - f(y)| < \epsilon_i$$

for all $f \in K$.

Since X is compact, there is some finite δ_i-net

$$x_{i,1}, \dots, x_{i,n_i}$$

Consider the sequence

$$y_1, y_2, \dots = x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}, \dots, x_{i,1}, \dots, x_{i,n_i}, \dots$$

Note that $(f_k(x_{i,j}))_{k=1}^\infty$ is a bounded sequence in $\mathbb{F}$, so it has a convergent subsequence.

$$\begin{aligned} &f_{1,1}, f_{1,2}, \dots, f_{1,k}, \lim_{k \to \infty} f_{1,k}(y_1) = a_1 \in \mathbb{F} && \\ &f_{2,1}, f_{2,2}, \dots, f_{2,k}, \lim_{k \to \infty} f_{2,k}(y_2) = a_2 \in \mathbb{F} && \text{subsequence of } f_{1,k} \\ &\dots && \\ &f_{j,1}, f_{j,2}, \dots, f_{j,k}, \lim_{k \to \infty} f_{j,k}(y_j) = a_j \in \mathbb{F} && \text{subsequence of } f_{j-1,k} \end{aligned}$$

Take the diagonal subsequence

$$(f_{j,j})_{j=1}^{\infty}$$

and note that $f_{k,k}(y_j) \to a_j$ exists for all $j \geq 1$.

We claim $f_{k,k}$ is cauchy in $C(X)$.

Fix $\epsilon > 0$, pick $\epsilon_i \leq \epsilon$, so we have equicontinuity with $\delta_i > 0$.

The δ_i-net $x_{i,1}, \ldots, x_{i,n_i}$ is in the original sequence.

But $\lim_{k\to\infty} f_{k,k}(x_{i,j})$ exists for $1 \leq j \leq n_i$. There is some N such that for all $k, l \geq N$

$$|f_{k,k}(x_{i,j}) - f_{l,l}(x_{i,j})| < \epsilon$$

Now, for $y \in X$, there is some j such that $y \in \overline{b_{\delta_i}(x_{i,j})}$

$$\begin{aligned} |f_{k,k}(y) - f_{l,l}(y)| &\leq |f_{k,k}(y) - f_{k,k}(x_{i,j})| + |f_{k,k}(x_{i,j}) - f_{l,l}(x_{i,j})| + |f_{l,l}(x_{i,j}) - f_{l,l}(y)| \\ &< 3\epsilon \end{aligned}$$

So $\|f_{k,k} - f_{l,l}\|_\infty \leq 3\epsilon$ and by definition it is cauchy.

Now, by the completeness of $C(X)$ and closedness of K,

$$f_{k,k} \to f \in K$$

Example 5.9.4
Let $f_n \in C^b(\mathbb{R})$ defined as

$$f_n(x) = \begin{cases} 0, & x \leq n \\ x - n, & n \leq x \leq n+1 \\ 1, & n+1 \leq x \end{cases}$$

and let

$$K := \{f_n : n \geq 1\}$$

Note that $\|f_n, -f_m\|_\infty = 1$ if $m \geq n$. K is discrete so it is closed.

Also, $\|f_n\|_\infty = 1$ so K is bounded.

Since f_n is either constant or linear with slope 1 it is Lipschitz. K is also equicontinuous.

But K is not compact as all subsequences are not cauchy. This construction is possible as $(\mathbb{R}, |\cdot|)$ is not bounded.

5.10 Induced Metric & the Relative Topology

Let (X, d) be a metric space and $Y \subseteq X$. (Y, d) is a metric space.

Every open set $U \subseteq Y$ has the form $V \cap Y$ for some $V \subseteq X$ open.

Let $y \in U$, there is some $r_y > 0$ such that $b^Y_{r_y}(y) \subseteq U$. We can take

$$V := \bigcup_{b_r}^{X}(y)$$

Note that in the Cantor set C

$$C_1 := C \cap \left[0, \frac{1}{3}\right] = C \cap \left(-1, \frac{1}{2}\right)$$

so C_1 is open AND closed.

Definition 5.10.1 (Induced Metric)
Let (X, d) be a metric space and $A \subseteq X$.

$$d_A : A \times A \to R$$
$$x, y \mapsto d(x, y)$$

is then a metric on A which we call the induced metric.

Definition 5.10.2 (Relative Topology)
If we let

$$\tau_A := \{W \subseteq A | W = U \cap A, \text{for some open set } U \subseteq X\}$$

this is the relative topology on A inherited from τ_d on X

We now show that the relative topology is the natural topology obtained from the induced metric d_A.

Theorem 5.10.1
Let (X, d) be a metric space and $A \subseteq X$.
Let τ_A and τ_{d_A} be the relative topology and metric topology on A respectively, then

$$\tau_A = \tau_{d_A}$$

Proof
Firstly, let $W \subseteq A$ be in τ_A. There must be some open set $U \subseteq X$ such that $W = U \cap A$.

But $x \in U \implies \exists \epsilon > 0$

$$B_d(x, \epsilon) := \{y \in X | d(x, y) < \epsilon\} \subseteq U$$
$$B_{d_A}(x, \epsilon) := \{y \in A | d_A(x, y) < \epsilon\} \subseteq A$$

So W is in fact open in A and therefore $x \in \tau_{d_A}$

Conversely, suppose $W \subseteq A$ be in τ_{d_A}. By openness, for each $x \in W$ there is some $\epsilon_x > 0$ such that

$$W := \bigcup_{x \in W} B_{d_A}(x, \epsilon_x)$$
$$U := \bigcup_{x \in W} B_d(x, \epsilon_x)$$

By our work prior, U is open in X and $W = U \cap A$ so $W \in \tau_A$

5.11 Connectedness

Definition 5.11.1 (disconnected)
$A \subseteq (X, d)$ if there are $U, V \subseteq X$ open such that

(i) $A \subseteq U \cup V$

(ii) $U \cap V = \varnothing$

(iii) $U \cap A \neq \varnothing \neq V \cap A$

Definition 5.11.2 (Connected)
Not disconnected.

Example 5.11.1
$[0, 1] \cup [2, 3]$ is disconnected.

Example 5.11.2
$\mathbb{Q}$ is a disconnected subset of $\mathbb{R}$

$$\mathbb{Q} = (-\infty, \pi) \cup (\pi, \infty)$$

Note that on the relative topology (metric restricted to a subset), $V \cap A, U \cap A$ are open in A. In addition,

$$V \cap A = A \setminus U = A \cap \underbrace{U^c}_{\text{closed}}$$

so V is closed as well in A (vice versa for U).

This means that disconnected subsets contain clopen subsubsets.

Definition 5.11.3 (Interval in $\mathbb{R}$)
$I \subseteq \mathbb{R}$ such that

$$a < b \in I \implies \forall c, a < c < b \implies c \in I$$

Theorem 5.11.3
Subsets of $\mathbb{R}$ are connected if and only if they are intervals.

Proof ($\neg \impliedby \neg$)
If I is not an interval, there is some $a < c < b$ where $a, b \in I, \notin I$.

Then

$$I = (-\infty, c) \cup (c, \infty)$$

Proof ($\impliedby$)
Let I be an interval.

Suppose there are $U, V \subseteq \mathbb{R}$ which disconnect I.

Fix $a \in I \neq \varnothing$ and suppose without loss of generality that $a \in U$.

Take

$$c := \sup\{x \in I : [a, x) \subseteq U\}, b := \inf\{x \in I : (x, a] \subseteq U\}$$

Suppose for a contradiction that $b \in I$. Then it cannot be in either U or V since neither can contain an open ball around b. This is due to the fact that we either contradict the fact c is the supremum or that $[a, x - \epsilon] \subseteq U$ for every $\epsilon > 0$. Similarly for c.

So $I \subseteq (c, b) \subseteq U$ which is a contradiction as we force $I \cap V = \varnothing$.

Theorem 5.11.4
If A is connected, $f : A \to Y$ is continuous, then $f(A)$ is connected.

Proof ($\neg \impliedby \neg$)
If $f(A)$ is not connected, there are disjoint, open, nonempty sets $U, V \subseteq Y$ which disconnect Y.

By continuity, $f^{-1}(U), f^{-1}(V)$ are both open, disjoint, and nonempty sets whose union contains A. It follows that A is disconnected which is a contradiction.

Corollary 5.11.4.1 (Intermediate Value Theorem)
If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a)f(b) < 0$ (so one is positive and one is negative), then there is some $a < c < b$ such that

$$f(c) = 0$$

Proof
$[a, b]$ is connected, and f is continuous, so $f([a, b])$ is connected and therefore an interval in $\mathbb{R}$.

This means that $0 \in f([a, b])$ as desired.

Proposition 5.11.5
If $\{X_\alpha\}_{\alpha \in A}$ are a collection of connected sets and $x_0 \in X_\alpha$ for all $\alpha \in A$,

$$X := \bigcup X_\alpha$$

is connected.

Proof
If $X \subseteq U \cup V$ with U, V disjoint, open, and having non-empty intersections with X, then without loss of generality $x_0 \in U$.

By similar reasoning as before, $X_\alpha \subseteq U$ for every α so $X \subseteq U$, which is a contradiction.

Definition 5.11.4 (Connected Component)
The connected component containing x_0 is

$$\bigcup X_\alpha$$

where each X_α contains x_0 and is connected.
This is the biggest connected subset containing x_0.

Example 5.11.6
The components of $\mathbb{Q}, C$ where C is the cantor set are points.

Definition 5.11.5 (Path-Connected)
A is path-connected if for any $a, b \in A$ there is a path

$$f : [0, 1] \to A$$

such that $f(0) = a, f(1) = b$ and

$$f[0, 1] \subseteq A$$

Proposition 5.11.7
Path connectedness implies connectedness

Proof
Fix $a \in A$ and consider for any $b \in A$

$$\gamma_b = f_b([0, 1])$$

the image of the path connecting a, b.

But the continuous image of a connected set (interval) is connected and unions preserve connectedness as long as the unions share a point so

$$A = \bigcup \gamma_b$$

is connected.

Example 5.11.8 (Topologist's Since Curve)
path-connected but not connected.

$f : \mathbb{R} \to \mathbb{R}^2$ given by

$$f(x) = \begin{cases} (x, 0), & x \in (-\infty, 0] \\ \left(x, \sin\left(\frac{1}{x}\right)\right), & x \in (0, \infty) \end{cases}$$

5.12 Total Disconnectedness

Definition 5.12.1 (Totally Disconnected)
$A \subseteq (X, d)$ is totally disconnected if all connected components are single points.

Definition 5.12.2 (Perfect)
A closed set A is perfect if it has no isolated points.

Note that perfect sets only have accumulation points.

We will show the Cantor Set is "representative" of compact, perfect, and totally disconnected metric spaces.

Theorem 5.12.1
If X is a non-empty, compact, perfect, and totally disconnected metric space, then X is homeomorphic to C.

Classifying equivalence classes of compact, countable sets up to homeomorphisms is much more complicated.

Lemma 5.12.2
If X is compact and totally disconnected, with $x \in X$ and $r > 0$, then there is a clopen set U such that $x \in U \subseteq b_r(x)$ so that the topology is generated by clopen sets.

Proof (lemma)
Let
$$y \in B := \{y : d(x,y) \geq r\}$$

Remark that B is compact, as it is a closed subset of a compact set space X.

Then x, y are contained in different components, so there exist disjoint, nonempty, open U, V such that
$$X = U \cup V, x \in U, y \in V$$

Note that U is also closed since $X \setminus U = V$ is open and vice versa for V.

Then
$$\{V_y : y \in B\}$$
covers the compact subset $B \subseteq X$.

By compactness, there is a finite subcover
$$V_{y_1}, \ldots, V_{y_n}$$

Then
$$x \in U := \bigcap V_{y_i}^c$$

where U is clopen and disjoint from $\bigcup V_{y_i} \supseteq B$.

This shows that
$$U \subseteq B^c = b_r(x)$$

Corollary 5.12.2.1
Remark if $W \subseteq X$ is open, then for each $x \in W$, there is a $r_x > 0$ such that
$$b_{r_x}(x) \subseteq W$$
so there exists U_x clopen with
$$x \in U_x \subseteq b_{r_x}(x) \subseteq W$$
so
$$W = \bigcup_{x \in W} U_x$$

Corollary 5.12.2.2
If (X, d) is compact, totally disconnected and $\epsilon > 0$, then there are finitely many clopen sets
$$U_1, \dots, U_n$$
pairwise disjoint such that
$$\text{diam}(U_i) < \epsilon, X = \bigcup U_i$$

Proof (corollary)
For each $x \in X$, take clopen U_x such that
$$x \in U_x \subseteq b_\epsilon(x)$$
by the previous lemma.

These form a clopen cover of a compact space, so a finite subcover
$$U_{x_1}, \dots, U_{x_n}$$
exists.

Then
$$\left\{ V_S := \bigcap_{i \in S} U_{x_i} \cap \bigcap_{j \notin S} U_{x_j}^c : S \subseteq [n] \right\}$$
are the clopen, disjoint, sets with $\text{diam}(V_S) < \epsilon$, and cover X.

We can view the V_S as partitions of X by "how many times" a point is covered by some U_{x_i}.

Proof (theorem)
Tile X with a finite number of nonempty, disjoint, clopen sets of $0 < \text{diam} \le \frac{1}{2}$. Say

$$X_1, \ldots, X_{n_1}, n_1 \ge 2$$

Notice that we can take subsets with positive diameter as X is perfect and therefore has no isolated points.

Split C into the same number of disjoint clopen sets (combine sets if necessary)

$$C_1, \ldots, C_{n_1}, \text{diam}(C_i) \le \frac{1}{3}$$

Repeat this argument on each X_i with finitely many disjoint clopen sets with diam $\le \frac{1}{2^2}$. In addition, split each C_i with the same number of disjoint clopen sets of diam $\le \frac{1}{3^2}$.

Again, this is possible due to the perfectness of X: any non-empty closed set has no isolated points and can therefore be split further into subsets of infinite cardinality (ie the "fractal" nature of the tree holds).

Do this recursively and build two "partition" trees of clopen sets. Each level is pariwise disjoint, and each set's diameter is less than 2^{-n}.

For each path down the X tree, this determines a unique $x \in X$ by the FIP, compactness, and diam $\to 0$ (A2). Similarly for C.

More rigorously, at the n-th level, we have

$$X_1^{(n)}, \ldots, X_{k(n)}^{(n)}, C_1^{(n)}, \ldots, C_{k(n)}^{(n)}$$

Pick $x_i^{(n)} \in X_i^{(n)}, c_i^{(n)} \in C_i^{(n)}$ and define $f_n : X \to C, g_n : C \to X$ by the following

$$f_n\Big|_{X_i^{(n)}} = c_i^{(n)},\ g_n\Big|_{C_i^{(n)}} = x_i^{(n)}$$

Note that both f_n, g_n are continuous since they are locally constant. Furthermore

$$g_n \cdot f_n\Big|_{X_i^{(n)}} = x_i^{(n)}$$

Let I_X be the identity function on X

$$\begin{aligned}\|g_n \circ f_n - I_X\|_\infty &= \max_{1\le i\le k(n)} \sup_{x\in X_i^{(n)}} d(x, x_i^{(n)}) \\ &= \max_{1\le i\le k(n)} \operatorname{diam}(X_i^{(n)}) \\ &\le 2^{-n} \to 0\end{aligned}$$

and similarly for $\|f_n \circ g_n - I_C\|$.

But in addition, we have

$$\|f_n - f_{n+1}\|_\infty = \max_{1\le i\le k(n)} \max_{1\le j\le k(n+1)} d(y_i^{(n)}, y_j^{(n+1)}) \le \operatorname{diam}(C_i^{(n)}) \le 3^{-n} \to 0$$

and similarly for $\|g_n - g_{n+1}\|$.

Since $\sum 2^{-n}, \sum 3^{-n} < \infty$ we have that

$$(f_n), (g_n)$$

are Cauchy. It follows by completeness of $\mathcal{C}(X, C), \mathcal{C}(C, X)$ that

$$f := \lim f_n, g := \lim g_n$$

exist.

In addition, we have

$$g \circ f = \lim g_n \circ f_n = I_X, f \circ g = \lim f_n \circ g_n = I_C$$

So f, g are continuous, bijective, and inverses of each other, demonstrating the homeomorphic nature of X, C.

Example 5.12.3 (Knaster Kuratowski Fan, Cantor's Triangle)
Let $x \in C$, the Cantor Set and $L(c)$ be a line segment from

$$(c, 0) \to \left(\frac{1}{2}, \frac{1}{2}\right)$$

Let $p = \left\{\left(\frac{1}{2}, \frac{1}{2}\right)\right\}$. Next, if $c \in C$ has finite ternary expasion we let

$$\chi_c\{(x, y) \in L(c) : y \in \mathbb{Q}\}$$

Otherwise, $c \in C$ has infinite ternary expansion, get

$$\chi_c = \{(x, y) \in L(c) : y \notin \mathbb{Q}\}$$

Define

$$F := p \cup \bigcup_{c \in C} \chi_c$$

F is connected but

$$F \setminus \left\{ \left(\frac{1}{2}, \frac{1}{2} \right) \right\}$$

is totally disconnected.

5.13 Baire Category Theorem

Note that this has nothing to do with Category Theory.

Definition 5.13.1 (Nowhere Dense)
$A \subseteq (X, d)$ such that $\text{int}(\bar{A}) = \varnothing$.

Example 5.13.1
The Cantor Set is nowhere dense in $\mathbb{R}$.

Definition 5.13.2 (First Category)
$A \subseteq (X, d)$ such that

$$A = \bigcup_{i=1}^{\infty} A_n$$

a countable union of nowhere dense sets A_n.

Definition 5.13.3 (Residual Sets)
$B \subseteq (X, d)$ if $X \setminus B$ is first category.

Lemma 5.13.2
Let A be a nowhere dense subset of (X, d) and $x \in X, r > 0$.

$$b_{\frac{1}{2}}(x) \setminus \bar{A} \neq \varnothing$$

and is open.

Proof
The deletion cannot be empty as $\bar{A}$ does not contain any open sets.

To see that the deletion is still open, let any y be in the ball. Then suppose every open ball around y contains a point of $\bar{A}_1$. Then $y \in \bar{A}_1$ and so it was removed.

Theorem 5.13.3 (Baire Category Theorem)
A complete metric space (X, d) is not first category.
In fact, $X \setminus \bigcup_{i=1}^{\infty} A_n$ is dense for any countable union of nowhere dense subsets A_n.

Proof
Let A_n be nowhere dense and let $x \in X, r > 0$.

By the lemma
$$B := b_{\frac{r}{2}}(x) \setminus \bar{A}_1 \neq \varnothing$$
and is open.

Pick any $x_1, 0 < r_1 < \frac{r}{2}$ such that
$$\overline{b_{r_1}(x_1)} \subseteq B \subseteq b_r(x)$$

Recursively, pick $x_n, 0 < r_n < \frac{r}{2^n}$ such that
$$\overline{b_{r_n}(x_n)} \subseteq b_{r_{n-1}}(x_{n-1}) \setminus \bar{A}_n$$

By the completeness of X, and since
$$\operatorname{diam}(\overline{b_{r_n}(x_n)}) \leq 2r_n \to 0$$
with the nestedness of the balls, we get
$$\bigcap_{n=1}^{\infty} \overline{b_{r_n}(x_n)} \neq \varnothing$$
(A2).

Take
$$x_0 = \lim_{n\to\infty} x_n \in \bigcap_{i=1}^{\infty} \overline{b_{r_n}(x_n)}$$
and note that by construction
$$x_0 \notin \bar{A}_n$$
as we escaped A_n in the n-th iteration.

So
$$x_0 \notin \bigcup_{i=1}^{n} \bar{A}_n$$

By the arbitrary choice of x, r, since there will always be some $x_0 \in b_{\frac{r}{2}}(x)$,

$$X \setminus \bigcup_{n=1}^{\infty} \bar{A}_n$$

is dense as desired.

Example 5.13.4
$\mathbb{Q} = \bigcup_{q\in\mathbb{Q}} \{q\}$ is first category (but also dense) in $\mathbb{R}$.

Corollary 5.13.4.1
If X is a complete metric space and $U_n, n \geq 1$ are dense, open subsets, then

$$\bigcap_{n=1}^{\infty} U_n$$

is dense.

Proof
U_n dense means that $A_n = U_n^c$ is closed and nowhere dense

$$x \in \operatorname{int} A_n \implies x \notin \bar{U}_n$$

Hence

$$\bigcap_{n=1}^{\infty} U_n = X \setminus \left(\bigcup_{n=1} A_n \right)$$

is dense by the BCT.

Corollary 5.13.4.2
If (X, d) is complete and $A_n, n \geq 1$ are closed sets such that

$$X = \bigcup_{n=1}^{\infty} A_n$$

there is some n_o such that

$$\operatorname{int}(A_{n_0}) \neq \varnothing$$

Proof
If all A_n are nowhere dense, its union cannot be all of X by the BCT.

Theorem 5.13.5
The set
$$\mathcal{F} := \{f \in C[a,b] : f \text{ is not differentiable at any point}\}$$
is a dense in $C[a,b]$.

Lemma 5.13.6
Differentiable functions are locally Lipschitz.

Proof (lemma)
For all $y \in [a,b]$ the Newton Quotient
$$\lim_{y \to x} \frac{f(y)-f(x)}{y-x} = f'(x)$$
exists.

So there is some $\delta > 0$ such that
$$|x-y| < \delta \implies \left|\frac{f(y)-f(x)}{y-x} - f'(x)\right| < 1$$

Hence, for $y, |y-x| < \delta$ we have
$$|f(y)-f(x)| \le (|f'(x)|+1)|y-x|$$

Elsewise, if $|y-x| \ge \delta$ then
$$|f(y)-f(x)| \le 2\|f\|_\infty \le \left(\frac{2\|f\|_\infty}{\delta}\right)|y-x|$$

Taking
$$C := \max\left\{|f'(x)|+1, \frac{2\|f\|_\infty}{\delta}\right\}$$
is a local Lipschitz consant at x.

Proof (theorem)
Let
$$A_n := \{f \in C[a,b] : \exists x \in [a,b], f \text{ is locally at Lipschitz at } x \text{ with constant at most } n\}$$
We claim that A_n is closed and nowhere dense. If we show this, then $(\bigcup_{n=1}^{\infty} A_n)^c$, the

complement of the set containing all functions that are differentiable at at least one point, is residual, which completes the proof as it is a subset of the continuous nowhere differentiable functions on $[a, b]$.

To see the claim we first show that A_n is closed. Let $f, f_k \in A_n$ such that

$$\|f_k - f\|_\infty \to 0$$

There is some $x_k \in [a, b]$ such that

$$|f_k(y) - f_k(x_k)| \leq n|y - x_k|$$

By the compactness of $[a, b]$, there is a subsequence $x_{k_i} \to x_0$.

Now

$$\begin{aligned}
|f(y) - f(x_0)| &\leq |f(y) - f_{k_i}(y)| + |f_{k_i}(y) - f_{k_i}(x_{k_i})| + |f_{k_1}(x_{k_i}) - f_{k_i}(x_0)| + |f_{k_i}(x_0) - f(x_0)| \\
&\leq \|f - f_{k_i}\|_\infty + n|y - x_{k_i}| + n|x_{k_i} - x_0| + \|f_{k_i} - f\|_\infty \\
&\to 0 + n|y - x_0| + n|x_0 - x_0| + 0 \\
&= n|y - x_0|
\end{aligned}$$

so we indeed have $f \in A_n$.

Finally, we show that A_n has empty interior. Let $f \in A_n, \epsilon > 0$. We construct a $g \notin A_n$ such that $\|f - g\|_\infty < \epsilon$.

To do this, we find h piece-wise linear such that

$$\|f - h\|_\infty < \frac{\epsilon}{2}$$

By uniform continuity, there is a $\delta > 0$ such that

$$|y - x| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{4}$$

Let $a = x_0 < x_1 < \cdots < x_p = b$ such that

$$x_{i+1} - x_i < \delta$$

Let $h(x_i) = f(x_i)$ and linear in between (check that $\|h - f\|_\infty < \frac{\epsilon}{2}$). Now, h is certainly Lipschitz with constant L equal to the maximum slope between x_i, x_{i+1}.

Let $M > \frac{4\pi}{3}(L + n)$. Define

$$g := h + \frac{\epsilon}{2} \sin(Mx)$$

and note that

$$\|g - f\|_\infty \leq \|g - h\|_\infty + \|h - f\|_\infty = \epsilon$$

It remains to show that $g \notin A_n$. Let $\bar{x} \in [a, b]$ and pick y such that

$$|y - \bar{x}| < \frac{2\pi}{M}, \text{within one period of sin}$$

and

$$\sin(My) = \begin{cases} 1, & \sin M\bar{x} < 0 \\ -1, & \sin M\bar{x} \geq 0 \end{cases}$$

We have

$$\begin{aligned} |g(y) - g(\bar{x})| &\geq \frac{\epsilon}{2}|\sin My - \sin M\bar{x}| - |h(y) - h(\bar{x})| \\ &\geq \frac{\epsilon}{2} \cdot 1 - L|y - \bar{x}| \\ &= \frac{\epsilon M}{2 \cdot 2\pi}|y - \bar{x}| - L|y - \bar{x}| \\ &= \left(\frac{\epsilon m}{4\pi}\right)|y - \bar{x}| \\ &> n|y - \bar{x}| \end{aligned}$$

and hence $g \notin A_n$, completing the proof.

5.14 Weierstrauss' Nowhere Differentiable Function

Theorem 5.14.1 (Weierstrauss M-test)
If $f_n \in C(X)$ and $\sum_{i=0}^\infty \|f_n\| < \infty$ then

$$\sum_{i=1}^\infty f_n$$

converges uniformly.

Proof
Let $S_N(s) := \sum_{n=1}^N f_n(x)$ and $\epsilon > 0$.

There is some N_0 such that $\sum_{i=N_0}^{\infty}\|f_i\|_\infty < \epsilon$ by convergence. For $M, N \geq N_0$

$$\begin{aligned} |S_M(x) - S_N(x)| &= \left|\sum_{k=N+1}^{M} f_k(x)\right| \\ &\leq \sum_{k=N+1}^{\infty} \|f_k\|_\infty \end{aligned}$$

So f_k is cauchy, and by the completeness of $C^b(\mathbb{R})$, it converges uniformly to some continuous bounded function.

Our construction is a specific case of Weierstrauss' original construction and avoids checking the general case for simplicity sake.

Definition 5.14.1 (Weierstrauss Function)
$f \in C^b(\mathbb{R})$ given by

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \sin(10^n \pi x)$$

where $f_n(x) = 2^{-n}\sin(10^n \pi x)$ is continuous and 1-periodic.

Notice that $\|f_n\|_\infty = 2^{-n}$ and so

$$\sum_{i=1}^{\infty} \|f_n\|_\infty < \infty$$

Proposition 5.14.2
f is nowhere differentiable.

Proof
Since f is 1-periodic, it suffices to check that it is not differentiable on $[0, 1]$.

Let $x = 0.x_1x_2 \cdots \in [0, 1]$, and consider the decimal expansion.

For $n \geq 1$, let

$$\begin{aligned} y_n &:= 0.x_1x_2 \ldots x_n \\ z_n &:= y_n + 10^{-n} \end{aligned}$$

Now,

$$\left.\begin{aligned} f_n(y_n) &= 2^{-n}\sin(10^n \pi y_n) = 2^{-n}(-1)^{10^n y_n \in \mathbb{N}} = \pm 2^{-n} \\ f_n(z_n) &= 2^{-n}\sin(10^n \pi y_n) = 2^{-n}(-1)^{10^n y_n + 1 \in \mathbb{N}} = \mp 2^{-n} \end{aligned}\right\} \text{opposite sign}$$

hence $|f_n(y_n) - f_n(z_n)| = 2^{-n+1}$.

If $k > n$ then

$$\begin{aligned} f_k(y_n) &= 2^{-k}\sin\bigl(10^k\pi y_n\bigr) \\ &= 2^{-k}(-1)^{10^k y_n \in 2\mathbb{N}} \\ &= 2^{-k} \end{aligned}$$

$$\begin{aligned} f_k(z_n) &= 2^{-k}(10^k\pi z_n) \\ &= 2^{-k}(-1)^{10^k z_n \in 2\mathbb{N}} \\ &= 2^{-k} \end{aligned}$$

and so $f_k(y_n) - f_k(z_n) = 0$.

For $1 \le k < n$ we have

$$\begin{aligned} |f_k(y_n) - f_k(z_n)| &= |f_k'(c) \cdot (y_n - z_n)| && \text{Mean Value Theorem} \\ &\le \|f_k'\|_\infty |y_n - z_n| \\ &= \|2^{-k} \cdot 10^k \pi \cos\bigl(10^k \pi x\bigr)\|_\infty \cdot 10^{-n} \\ &= \frac{5^k \pi}{10^n} \\ &= 2^{-n} \frac{\pi}{5^{n-k}} \end{aligned}$$

Overall

$$\begin{aligned} |f(y_n) - f(z_n)| &= \left| \sum_{k=1}^{\infty} f_k(y_n) - f_k(z_n) \right| \\ &= \left| f_k(y_n) - f_k(z_n) + \sum_{k=1}^{k-1} f_k(y_n) - f_k(z_n) \right| \\ &\ge |f_k(y_n) - f_k(z_n)| - \left| \sum_{k=1}^{k-1} f_k(y_n) - f_k(z_n) \right| && \text{LS} > \text{RS} \\ &= 2^{-n+1} - \sum_{k=1}^{n-1} 2^{-n}\pi \cdot \frac{1}{5^{n-k}} \\ &> 2^{-n+1} - 2^{-n}\pi \cdot \frac{1/5}{1 - 1/5} && \text{geometric series} \\ &= 2^{-n}\left(2 - \frac{\pi}{4}\right) \\ &> 2^{-n} \end{aligned}$$

By the Pigeonhole Principle, we can take $w_n \in \{y_n, z_n\}$ such that

$$|f(w_n) - f(x)| > 2^{-n-1}$$

The Newton Quotient at x is given by

$$\begin{aligned}\left|\frac{f(x) - f(w_n)}{x - w_n}\right| &> \frac{2^{-n-1}}{10^{-n}} \\ &= \frac{5^n}{2} \\ &\to \infty\end{aligned}$$

showing that the derivative at x does not exist. Hence, by the choice of x, f is nowhere differentiable.

5.15 Oscillations

Definition 5.15.1 (Oscillation)
Let $f : (X, d) \to (Y, \rho)$.
For $x \in X, \delta > 0$, define

$$\omega_f(x, \delta) := \sup_{y,z \in b_\delta(x)} \rho(f(y), f(z))$$

The oscillation of f at x is defined as

$$\omega_f(x) := \inf_{\delta > 0} w_f(x, \delta)$$

Lemma 5.15.1
f is continuous at x if and only if $\omega_f(x) = 0$.

Lemma 5.15.2
For $\epsilon > 0$ and

$$U := \{x \in X : \omega_f(x) < \epsilon\}$$

is open.

Proof
Let $x \in U, \delta > 0$ such that $\omega_f(x, \delta) < \epsilon$.

If $y \in b_\delta(x), d(x, y) = r < \delta$.

So

$$b_{\delta-r}(x) \subseteq b_\delta(x)$$

since $\omega_f(y, \delta - r) \leq w_f(x, \delta)$ and $w_f(y) < \epsilon$, hence

$$b_\delta(x) \subseteq U$$

By definition, U is open.

Definition 5.15.2 (G_δ-set)
A countable intersection of open sets.

Definition 5.15.3 (F_σ-set)
A countable union of closed sets.

Recall that point-wise discontinuity does not necessarily preserve continuity. We now show that it preserves continuity at many points.

Theorem 5.15.3
If $f_n \in C[a, b]$ and $f(x) = \lim_{n\to\infty} f_n(x)$ exists point-wise.
Then the set of points of continuity of f is a residual, dense G_δ-set.

Proof
Notice that

$$\begin{aligned}\{x : f \text{ is continuous at } x\} &= \{x : \omega_f(x) = 0\} \\ &= \bigcap_{n\geq 1} \underbrace{\left\{x : w_f(x) < \frac{1}{n}\right\}}_{\text{open by lemma}}\end{aligned}$$

which is a G_δ-set.

Now, the complement is

$$A := \bigcup_{n\geq 1} A_n$$

where

$$A_n := \left\{x : \omega_f(x) \geq \frac{1}{n}\right\}$$

and notice each A_n is closed by the lemma.

We claim that $\operatorname{int} A_n = \varnothing$ so A_n is nowhere dense, and so A is first category by definition. Then A^c is a residual set, and in particular, dense.

Suppose there is an open interval $I \subseteq A_n$.

Take $\epsilon = \frac{1}{3n}$ and let

$$E_N := \bigcap_{i,j \geq N} \{x \in I : |f_i(x) - f_j(x)| \leq \epsilon\}$$

and notice that it is an intersection of closed sets and is therefore closed.

Since $\lim_{n\to\infty} f_i(x) = f(x)$, the sequence $(f_i(x))_{i=1}^{\infty}$ is cauchy. In other words, for all $x \in I$, there is some $N(x)$ such that

$$x \in E_{N(x)}$$

It follows that $I = \bigcup_{N=1}^{\infty} E_N$ and by the BCT there is some N_0 such that we have some open interval

$$J \subseteq \operatorname{int} E_{N_0} \neq \varnothing$$

Fix $x \in J$. By uniform continuity of f_{N_0} on $[0, 1]$, there is some $\delta > 0$ such that

$$|x - y| < \delta \implies |f_{N_0}(x) - f_{N_0}(y)| < \epsilon$$

WLOG, $(x - \delta, x + \delta) \subseteq J$ so

$$\begin{aligned}
|y - x| < \delta \implies |f(x) - f(y)| \\
&\leq |f(x) - f_{N_0}(x)| + |f_{N_0}(x) - f_{N_0}(y)| + |f_{N_0}(y) - f(y)| \\
&< \epsilon + \epsilon + \epsilon \\
&= 3\epsilon \\
&= \frac{1}{n}
\end{aligned}$$

This is due to the fact that

$$\forall i \geq N_0, |f_i(x) - f_{N_0}(x)| < \epsilon \implies |f(x) - f_{N_0}(x)| \leq \epsilon$$

So $\omega_f(x, \delta) < \frac{1}{n}$ meaning

$$\omega_f(x) < \frac{1}{n}$$

and so

$$J \cap A_n = \varnothing$$

which contradicts the assumption that $I \subseteq A_n$.

So $\operatorname{int} A_n = \varnothing$ and by BCT, A^c is a residual set as desired.

5.16 Contraction Mapping

Definition 5.16.1 (Contraction Mapping)
$T : (X, d) \to (X, d)$ is a contraction mapping if T is Lipschitz with a constant $c < 1$

Example 5.16.1
If $x < y \in [-1, 1]$ then there is some $\theta \in (x, y)$ such that

$$\left|\frac{Tx - Ty}{x - y}\right| = \left|\frac{\cos x - \cos y}{x - y}\right| = |\sin\theta| \le \sin 1 < 1$$

by the Mean Value Theorem.

So cos is indeed a contraction mapping on $[-1, 1]$.

Let $x_0 \in [0, 1]$ and define a sequence $x_n := \cos(x_{n-1}), n \ge 1$.

Notice that there is some $x^* \in [-1, 1]$ such that $\cos(x^*) = x^*$ (consider the graph of $y = x, y = \cos(x)$).

$$\begin{aligned}|x_n - x^*| &= |\sin 1| \cdot |x_{n-1} - x^*| \\ &\dots \\ &\le |\sin 1|^n |x_0 - x^*| \\ &\to 0\end{aligned}$$

Hence, any arbitrary point $x_0 \in [-1, 1]$ converges to x^*, the fixed point.

Theorem 5.16.2
Let (X, d) be a complete metric space and $T : X \to X$ be a contraction with Lipschitz costant $c < 1$.
T has a unique fixed point x_* such that

$$x_* = Tx_*$$

In particular, if $x_0 \in X$ is arbitrary and $x_{n+1} := T(x_n), n \ge 0$, then $x_n \to x_*$ and

$$d(x_n, x_*) \le c^n d(x_0, x_*) \le \frac{c^n}{1 - c} d(x_1, x_0)$$

Proof
We first note that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T(x_{n-1}), T(x_n)) \\ &\leq cd(x_{n-1}, x_n) \\ &\leq \dots \\ &\leq c^n d(x_0, x_1) \end{aligned}$$

We claim that (x_n) is a Cauchy sequence. Indeed, let $m \geq n \geq N$, then

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} c^k d(x_0, x_1) \\ &< \left(\sum_{k=N}^{\infty} c^k \right) d(x_0, x_1) \\ &= \frac{c^N}{1-c} d(x_0, x_1) \end{aligned}$$

For sufficiently large N, we will have $d(x_n, x_m) < \epsilon$ for any fixed $\epsilon > 0$.

By the completeness of X, we can define

$$x_* := \lim_{n \to \infty} x_n$$

By continuity

$$x_n \to x_* \implies T(x_n) = x_{n+1} \to x_*$$

gives us the fixed point.

Observe that

$$\begin{aligned} d(x_n, x_*) &= \lim_{m \to \infty} d(x_n, x_m) \\ &\leq \lim_{m \to \infty} \left(\sum_{k=n}^{m-1} c^k \right) d(x_0, x_1) \\ &= \frac{c^n}{1-c} d(x_0, x_1) \end{aligned}$$

Finally, we show that it is unique. Suppose $T(y) = y$, then we have

$$\begin{aligned} d(x_*, y) &= d(T(x_*), T(y)) \\ &\leq cd(x_*, y) \\ &= cd(T(x_*), T(y)) \\ &\leq c^2 d(x_*, y) \\ &= \dots \\ &\to 0 \end{aligned}$$

Example 5.16.3
Consider the function $T(x) = 108(x - x^3)$.

Clearly, it is not a contraction. As we would have 3 fixed points

$$0, \pm \frac{\sqrt{107/3}}{6}$$

However, if we select a point close enough to one of the three fixed points, we would still converge to that fixed point.

Theorem 5.16.4
Suppose $T \in C^1([a, b], [a, b]), T(x_*) = x_*$ and $|T'(x_*)| < 1$.
Then, there is a $\delta > 0$ such that

$$T : b_\delta(x_*) \to b_\delta(x_*)$$

is a contraction.

Proof
By continuity and compactness, there is some $\delta > 0$ such that

$$\sup_{|x - x_*| < \delta} |T'(x)| = c < 1$$

If $|x - x_*| \leq \delta$ then we have

$$\begin{aligned} |T(x) - x_*| &= |T(x) - T(x_*)| \\ &= |T'(\xi)| \cdot |x - x_*| && \text{MVT} \\ &\leq c|x - x_*| \end{aligned}$$

Definition 5.16.2 (Affine Map)
$T(x) : \mathbb{R}^m \to \mathbb{R}^n$ such that
$$T_(x) = L(x) + y$$
where L is linear and y is fixed.

Example 5.16.5 (Fractals)
Let $T_1, \ldots, T_n : \mathbb{R}^d \to \mathbb{R}^d$ be affine mappings. We wish to find a set $X \subseteq \mathbb{R}^d$ such that
$$X = T_1(X) \cup \cdots \cup T_n(X)$$

Assume T_i are contractions, ie
$$\|L_i(x)\|_2 \leq c_i \|x\|_2, c_i < 1$$

If there is such an X, then we have that X is similar to $T_i(X)$.

We provide a concrete example. Consider $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined by
$$T_0(x) = \frac{1}{2}x$$
$$T_1(x) = (2, 0)$$
$$T_2(x) = (1, \sqrt{3}) + \frac{1}{2}x$$
then
$$(0,), (4, 0), (2, 2\sqrt{3})$$
are fixed for T_0, T_1, T_2 respectively.

Let $X \subseteq \mathbb{R}^2$ be compact and we define the mapping
$$T(X) = T_1(X) \cup T_2(X) \cup T_3(X)$$

Let X be the solid triangle.
$$T^n(X)$$
gives us the Sierpinski Triangle under the limit.

Lemma 5.16.6
$d_H(\bigcup_{i=1}^n A_i, \bigcup_{i=1}^n B_i) \leq \max\{d_H(A_i, B_i) : 1 \leq i \leq n\}$

Proof
Let the RHS be r, then $d_H(A_i, B_i) \le r$ means that

$$A_i \subseteq E_r(B_i) := \{x : d(x, B_i) \le r\}$$
$$B_i \subseteq E_r(A_i)$$

So we have

$$\bigcup A_i \subseteq E_r(\bigcup B_i)$$
$$\bigcup B_i \subseteq E_r(\bigcup A_i)$$

$$d_H(\bigcup A_i, \bigcup B_i) \le r$$

Theorem 5.16.7
Let (X, d) be complete and $T_1, \ldots, T_n : X \to X$ be contractions. Recall that

$$H(X) := \{K \subseteq X : K \text{ is compact}\}$$

is a metric space with d_H.
Define $T : H(X) \to H(X)$ by

$$T(Y) := \bigcup T_1(Y) \bigcup \cdots \bigcup T_n(Y)$$

Then, T is a contraction mapping, ie there is a unique compact set $K_* \subseteq X$ such that

$$T(K_*) = K_*$$

Proof
We claim that

$$d_H(T_i(A), T_i(B)) \le c_i d_H(A, B)$$

Let $a \in A$ and note that there is some $b_a \in B$ such that

$$d(a, b_a) \le d_H(A, B)$$
$$d(T_i(a), T_i(b_a)) \le c_i d(a, b_a) \le c_i d_H(A, B)$$
$$T_i(A) \subseteq E_{c_i d_H(A,B)}(T_i(B))$$

and vice versa for $T_i(B) \subseteq E_{c_i d_H(A,B)}(T_i(A))$. Thus

$$d_H(T_i(A), T_i(B)) \le c_i d_H(A, B)$$

So T_i is a contraction in $H(X)$, and thus by the lemma, T is a contraction and so we have our desired unique K_*.

Example 5.16.8 (A Counterexample)
$T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = x + 1$$

has Lipschitz constant 1 and no fixed point.

So we require Lipschitz constant less than 1.

Example 5.16.9
$S : [1, \infty) \to [1, \infty)$ defined by

$$Sx = x + \frac{1}{x}$$

We have

$$\begin{aligned} Sx - Sy &= x + \frac{1}{x} - y - \frac{1}{y} \\ &= (x - y) + \frac{y - x}{xy} \\ &= (x - y)\left(1 - \frac{1}{xy}\right) \end{aligned}$$

So $|Sx - Sy| < |x - y|$ but S has no fixed points. This is due to the fact that

$$\begin{aligned} \operatorname{Lip} S &:= \sup_{1 \le x < y \le \infty} \left|1 - \frac{1}{xy}\right| \\ &= 1 \end{aligned}$$

5.16.1 Newton's Method

Let $f \in C^2$ with $f(x_*) = 0$

We want

$$0 \approx f(x_1) + f'(x_1)(x_2 - x_1)$$

by taking

$$x_2 \approx x_1 - \frac{f(x_1)}{f'(x_1)}$$

Definition 5.16.3 (Quadratic Convergence)
$x_n \to x_0$ converges quadratically if

$$|x_{n+1} - x_*| \leq M|x_n - x_*|^2$$

for some constant M.

Theorem 5.16.10 (Newton's Method)
If $f \in C^2$ such that $f(x_*) = 0$ and $f'(x_*) \neq 0$, then there is some $R > 0$ such that on $[x_* - R, x_* + R]$

$$Tx = x - \frac{f(x)}{f'(x)}$$

is a contraction.
Moreoever, for $x_0 \in [x_* - R, x_* + R]$

$$x_n := T^n x_0 \to x_*$$

defines a convergent sequence converging to x_*
If fact, (x_n) converges quadratically.

Proof
First, we compute the derivative of T

$$\begin{aligned} T'x &= 1 - \frac{f'(x)f'(x) - f(x)f^{(2)}(x)}{f'(x)^2} \\ &= \frac{f'(x)^2 - f'(x)^2 + f(x)f^{(2)}(x)}{f'(x)^2} \\ &= \frac{f(x)f^{(2)}(x)}{f'(x)^2} \end{aligned}$$

Notice that $T'x_* = 0$.

Choose R such that

$$|T'x| \leq \frac{1}{2}, f'(x) \neq 0$$

on $b_R(x_*)$. This is possible by the continuity of $f, f', f^{(2)}$.

We claim T is a contraction with $\operatorname{Lip} T \leq \frac{1}{2}$. Indeed

$$\begin{aligned} |Tx - Ty| &= |T'\xi||x - y| && \text{MVT} \\ &\leq \frac{1}{2}|x - y| \end{aligned}$$

and hence there is a unique fixed point

$$Tx_* = x_* - \frac{f(x_*)}{f'(x_*)} = x_*$$

Let

$$\begin{aligned} A &:= \sup_{|x-x_*|\le R} |f^{(2)}(x)| \\ B &:= \inf_{|x-x_*|\le R} |f'(x)| \end{aligned}$$

and

$$M := \frac{A}{B}$$

Notice that

$$\begin{aligned} f(x_n) - f(x_*) &= f'(\xi_n)(x_n - x_*) && \xi_n \in (x_n - x_*) \\ f(x_n) &= f'(\xi_n)(x_n - x_*) \\ \frac{f(x_n)}{f'(\xi_n)} &= x_n - x_* \end{aligned}$$

By definition

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}$$

So

$$\begin{aligned} x_{n+1} - x_* &= (x_{n+1} - x_n) + (x_n - x_*) \\ &= -\frac{f(x_n)}{f'(x_n)} + \frac{f(x_n)}{f'(\xi_n)} \\ &= \boxed{\frac{f(x_n)}{f'(\xi_n)}} \cdot \frac{f'(\xi_n) - f'(x_n)}{f'(x_n)} \\ &= \frac{(x_n - x_*)}{f'(x_n)}(f'(\xi_n) - f'(x_n)) \\ &= \frac{x_n - x_*}{f'(x)} f^{(2)}(\zeta_n)(\xi_n - x_n) && \text{MVT again} \end{aligned}$$

$$\begin{aligned} |x_{n+1} - x_*| &\le \left|\frac{f^{(2)}(\zeta_n)}{f'(x_n)}\right| \cdot |x_n - x_*|^2 && \xi_n \in (x_n, x_*)\text{or vice versa} \\ &\le \frac{A}{B}|x_n - x_*|^2 \\ &= M|x_n - x_*|^2 \end{aligned}$$

Example 5.16.11
Find $\sqrt{a}$ where

$$f(x) = x^2 - a$$

and

$$Tx = x - \frac{x^2 - a}{2x}$$

6 Metric Completions

6.1 Metric Completion

Definition 6.1.1 (Completion)
The completion of a metric space (X, d) is a complete metric space (Y, ρ) with an isometry

$$J : X \to Y$$

such that JX is dense in Y.

Theorem 6.1.1
Every metric space has a completion.

Proof (slick)
Define $J : X \to C_b^{\mathbb{R}}(X)$ as follows.

Pick $x_0 \in X$ and define

$$f_x(y) = d(x, y) - d(x_0, y)$$

for all $x \in X$.

Then define

$$Jx = f_x$$

Bounded

We have

$$\begin{aligned}
f_x(y_1) - f_x(y_2) &= d(x, y_1) - d(x_0, y_1) - d(x, y_2) + d(x_0, y_2) \\
d(y, x) &\leq d(y, x_0) + d(x_0, y) && \implies d(y, x) - d(y, x_0) \leq d(x_0, x) \\
d(y, x_0) &\leq d(y, x) + d(x, x_0) && \implies d(y, x_0) - d(y, x) \leq d(x_0, x) \\
|d(y, x) - d(y, x_0)| &\leq d(x, x_0) \\
\|f_x\|_\infty &\leq d(x, x_0)
\end{aligned}$$

Continuous

We actually claim f_x are Lipschitz.

$$\begin{aligned}
|f_x(y_1) - f_x(y_2)| &\leq |d(x, y_1) - d(x, y_2)| + |d(x, y_1) - d(x, y_2)| \\
&\leq 2d(y_1, y_2)
\end{aligned}$$

Isometry

$$\begin{aligned}\|f_{x_1} - f_{x_2}\|_\infty &= \sup_{y\in X}|f_{x_1}(y) - f_{x_2}(y)| \\ &= \sup_{y\in X}|d(x_1,y) - d(x_0,y) - d(x_2,y) + d(x_0,y)| \\ &= \sup_{y\in X}|d(x_1,y) - d(x_2,y)| \\ &= d(x_1, xx_2) && y = x_2\end{aligned}$$

Define $Y := \overline{JX}$ and notice that it is a closed subset of $C_b(X)$ and therefore complete. JX is also dense in Y by definition so we are done!

Proof (intuitive)
Take

$$\mathcal{C} := \{(x_n) : (x_n) \text{ is Cauchy}\}$$

Let $\sim$ denote an equivalent relationship so that

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0$$

(Check that $\sim$ is an equivalent relationship)

Let

$$Y := \mathcal{C}/\sim$$

be the quotient space and define

$$\rho(\vec{x}, \vec{y}) := \lim_{n\to\infty} d(x_n, y_n)$$

ρ is well-defined as

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

and vice versa which shows

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_n, y_m)$$

so $d(x_n, y_m)$ is Cauchy in $\mathbb{R}$ so the limit always exists.

(Check that ρ is well-defined on the equivalent class)

(Check that ρ is a metric)

Define $J : X \to Y$ by the equivalence class of the constant sequence

$$Jx := [(x)_{n=1}^\infty]$$

Isometry

$$\rho(J\vec{x}, J\vec{y}) = \lim d(x_n, y_n) = d(x, y)$$

Dense Image

Let $\vec{y} = [(y_n)]$ be equivalence class of a not necessarily constant sequence in Y and $\epsilon > 0$.

There is an N such that for $m, n \geq N$ we get

$$d(y_m, y_n) < \epsilon$$

Consider

$$\vec{x} = Jy_N = [(y_N)]$$

and notice that

$$\rho(\vec{y}, \vec{x}) = \lim d(y_n, y_N) \leq \epsilon$$

Completeness

Let $(\vec{y}_k)_{k=1}^{\infty}$ be a sequence of equivalence classes of cauchy sequences in Y.

Choose $x_k \in X$ such that

$$\rho(Jx_k, \vec{y}_k) < 2^{-k}, k \geq 1$$

and notice this is possible by the density of JX.

Let $\vec{x} = [(x_k)]$ (Check it is Cauchy).

For $\epsilon > 0$ There is some K, such that for $k, l \geq K$

$$\begin{aligned} \rho(\vec{y}_k, \vec{y}_l) &< \epsilon \\ 2^{-k} &< \epsilon \end{aligned}$$

$$\begin{aligned} d(x_k, x_l) &= \rho(Jx_k, Jx_l) && \text{isometry} \\ &\leq \rho(Jx_k, \vec{y}_k) + \rho(\vec{y}_k, \vec{y}_l) + \rho(\vec{y}_l, Jx_l) \\ &< 2^{-k} + \epsilon + 2^{-l} \\ &< 3\epsilon \end{aligned}$$

We claim that $\lim \vec{y}_k = \vec{x} = [(x_k)]$.

First, note that

$$\lim Jx_k = \vec{x}$$

since

$$\rho(Jx_k, \bar{x}) = \lim d(x_k, x_n)$$

and for all $\epsilon > 0$ there is an N such that for $k, n \geq N$

$$d(x_k, x_n) < \epsilon$$

For $k \geq N$

$$\rho(Jx_k, \vec{x}) \leq \epsilon$$

and hence

$$Jx_k \to \vec{x}$$

by definition.

Finally

$$\begin{aligned}\rho(\vec{y}_k, \vec{x}) &\leq \rho(\vec{y}_k, Jx_k) + \rho(Jx_k, \vec{x}) \\ &< 2^{-k} + \epsilon \\ &\to 0 && k \to \infty\end{aligned}$$

so $\vec{y}_k \to \vec{x}$ and we are done.

Theorem 6.1.2 (Extension Theorem)
Let (X, d) be a metric space with completion (Y, ρ).
Let (Z, σ) be another complete metric space.
Suppose that $f : X \to Z$ is uniformly continuous, then f extends uniquely to a uniformly continuous $\tilde{f} : Y \to Z$.

Example 6.1.3
$f : (0, 1) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}$$

is not uniformly continuous and thus does not extend to $[0, 1]$.

Example 6.1.4
$g(x) = \sin x$ also does not extend.

Example 6.1.5
However, $h(x) := x^3$ does in fact extend to $[0, 1]$.

Proof
Let $\epsilon > 0$. By uniform continuity, there is some $\delta > 0$ such that for $d(x, y) < \delta$, we have

$$\sigma(f(x), f(y)) < \epsilon$$

By Cauchyness, there is some N such that for $n, m \geq N$

$$d(x_n, x_m) < \delta$$

Hence

$$\sigma(f(x_n), f(x_m)) < \epsilon$$

and $(f(x_n))_{n=1}^{\infty}$ is Cauchy and

$$\lim f(x_n)$$

exists (by Completeness).

Define

$$\tilde{f}([\vec{x}]) := \lim_{n\to\infty} f(x_n)$$

<u>Well-Defined</u>

Let $(x_n) \sim (x'_n)$ and note that

$$(x_1, x'_1, x_2, x'_2, \dots)$$

is then Cauchy (by $\sim$).

For $\epsilon > 0$ there is N such that for $n, m \geq N$

$$d(x_n, x_m), d(x'_n, x'_m), d(x'_n, x_n) < \epsilon$$

so $d(x_n, x'_m) < 2\epsilon$ gives that

$$\lim f(x_n) = \lim f(x'_n) =: \tilde{f}([\vec{x}])$$

is well defined.

<u>Uniform Continuity</u> Let $\epsilon > 0$ and get $\delta > 0$ such that $d(x, x') < \delta$ means

$$\sigma(f(x), f(x')) < \epsilon$$

If $\vec{y}, \vec{y}' \in Y$ such that $\rho(\vec{y}, \vec{y}') = r < \delta$. Let

$$\vec{y} = \lim Jx_n$$
$$\vec{y}' = \lim Jx'_n$$

which is possible by density of JX.

We have

$$\rho(\vec{y}, \vec{y}') = \lim \rho(Jx_n, Jx'_n)$$

so if $\rho(\vec{y}, \vec{y}') < \delta$ then there is some N so that $n \geq N$ means

$$d(x, x_n') = \rho(Jx_n, Jx_n') < \delta$$

and hence

$$\sigma(f(x_n), f(x_n')) < \epsilon$$

By cauchyness of (x_n) and $(x_n) \to \vec{y}$ we have that

$$\tilde{f}(\vec{y}) = \lim f(x_n)$$
$$\tilde{f}(\vec{y}') = \lim f(x_n')$$

and so

$$\sigma(\tilde{f}(\vec{y}), \tilde{f}(\vec{y}')) = \lim \sigma(f(x_n), f(x_n')) \leq \epsilon$$

showing us that $\tilde{f}$ is uniformly continuous.

Theorem 6.1.6 (Uniquess of the Metric Completion)
If (X, d) is a metric space and $(Y, \rho), (Z, \sigma)$ are both completions with JX, KX dense images of an isometry in Y, Z respectively, then there is a unique $h : Y \to Z$ such that

$$h(Jx) = Kx$$

which is an isometric homeomorphism.

Proof
Define $h_0 : JX \to KX$ by

$$h_0(Jx) := Kx$$

and notice by definition that h_0 is an isometry and is therefore uniformly continuous (Lipschitz with constant 1).

By the extension theorem, h_0 extends to $Y \to Z$ uniformly continuous.

To see that h is an isometry, notice that

$$\begin{aligned}\sigma(h(\vec{y}), h(\vec{y}')) &= \lim \sigma(h_0(Jx_n), h_0(Jx_n')) \\ &= \lim \rho(Jx_n, Jx_n') \\ &= \rho(\vec{y}, \vec{y}')\end{aligned}$$

To see that this is a surjection, let $\vec{z} \in Z$ with

$$\vec{z} = \lim Kx_n$$

and (x_n) cauchy.

This means (Jx_n) is also cauchy with

$$\vec{y} = \lim Jx_n \in Y$$

in fact

$$h(\vec{y}) = \lim h_0 Jx_n = \lim Kx_n = \vec{z}$$

Since isometries are bi-Lipschitz, they are therefore homeomorphic as well.

6.2 p-adics

Recall the p-adic norm

$$\left\| p^n \frac{a}{b} \right\|_p := p^{-n}$$

Remark that we have

(1) $\|x\| = 0 \iff x = 0$

(2) $\|xy\| = \|x\|\|y\|$

(3) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$

this $d_p := \|x - y\|_p$ is a metric.

Also, recall that by a previous assignment, the rationals are not complete under the p-adic norm. Let $\mathbb{Q}_p$ be the completion of $(\mathbb{Q}, d_p)$, then $\mathbb{Q}_p$ is the set (field) of p-adic numbers.

Remark that for any ball in $\mathbb{Q}_p$, any point in the ball is in the center.

Proposition 6.2.1
If (x_n) is a d_p-Cauchy sequence in $\mathbb{Q}$ and

$$x = \lim x_n \neq 0$$

in $\mathbb{Q}_p$, then $\|x_n\|_p$ is eventually constant.

Proof
We claim

$$|\|x_n\|_p - \|x_m\|_p| \leq \|x_n - x_m\|_p \to 0$$

Indeed, we have

$$\|x_n\|_p \in \{\|r\|_p : r \in \mathbb{Q}\} = \{p^n : n \in \mathbb{Z}\} \cup \{0\}$$

Since x_n is cauchy for $\epsilon > 0$ there is N such that $n, m \geq N$ means

$$\|x_n - x_m\|_p < \epsilon$$

so $|\|x_m\|_p - \|x_n\|_p| < \epsilon$ and

$$\lim_{n\to\infty} \|x_n\|_p$$

exists by the completeness of $\mathbb{R}$.

Suppose $x \not\to 0$ so $\|x_n\|_p \not\to 0$. We have

$$\lim \|x_n\|_p = p^k$$

and $|\|x_n\|_p - p^k| < \epsilon$.

Choose $\epsilon > \min\{p^k - p^{k+1}, p^{k+1} - p^k\}$ so

$$\|x\|_p = \lim \|x_n\|_p$$

as desired.

Proposition 6.2.2
If $x, y \in \mathbb{Q}_p$ with $(x_n), (y_n)$ cauchy sequences in $(\mathbb{Q}, d_p)$ such that

$$\lim x_n = x, \lim y_n = y$$

then we may define

(1) $x + y = \lim x_n + y_n$

(2) $xy = \lim x_n y_n$

(3) $-x = \lim -x_n$

which is all well-defined and makes $\mathbb{Q}_p$ into a commutative ring.
Moreoever,

(i) $\|xy\|_p = \|x\|_p \|y\|_p$

(ii) $\|x + y\|_p < \max\{\|x\|_p, \|y\|_p\}$

(iii) $\|x\|_p = 0 \iff x = 0$

Proof
Claim: $(x_n + y_n)$ is cauchy.

We have

$$\begin{aligned} \|(x_n + y_n) - (x_m - y_m)\|_p &= \|(x_n - x_m) + (y_n - y_m)\|_p \\ &\leq \max\{\|x_n - x_m\|_p, \|y_n - y_m\|_p\} \\ &< \epsilon \end{aligned}$$

for any fixed $\epsilon > 0$ and sufficiently large $n \geq N$.

To see well-definedness suppose $x_n' \to x, y_n' \to y$, then

$$\begin{aligned}\lim x_n' + y_n' &= \lim x_n + y_n \\ &\leq \max\{\|x_n - x_n'\|_p, \|y_n - y_n'\|_p\} \\ &\to 0\end{aligned}$$

<u>Claim</u>: $(x_n y_n)$ is cauchy.

We have

$$\begin{aligned}\|x_n y_n - x_m y_m\|_p &= \|(x_n - x_n')y_n + x_m(y_n - y_n')\|_p \\ &\leq \max\{\|x_n - x_m\|_p\|y_n\|_p, \|x_m\|_p\|y_n - y_n'\|_p\} \end{aligned} \tag{*}$$

so either $x, y \neq 0$ and thus

$$\|y_n\|_p = \|y\|_p, \|x_n\|_p = \|x\|_p, \forall n \geq N_o$$

for some N_0 by our work earlier, or $x = 0, y = 0$ and

$$\|x_n\|_p \to 0 \vee \|y_n\|_p \to 0$$

In both cases, $\|x_n\|_p, \|y_n\|_p$ are bounded, say by p^{M_0} then

$$(*) \leq p^{M_0} \max\{\|x_n - x_m\|_p, \|y_n - y_m\|_p\}$$

giving us cauchyness.

So

$$xy := \lim x_n y_n$$

makes sense.

Well-definedness is similar to the above.

If $xy \neq 0$ we have

$$\begin{aligned}\|xy\|_p &= \lim\|x_n y_n\|_p \\ &= \|x_n y_n\|_p & n \geq N_0 \\ &= \|x\|_p\|y\|_p\end{aligned}$$

as well as

$$\begin{aligned}\|x + y\|_p &= \lim\|x_n + y_n\|_p \\ &\leq \lim \max\{\|x_n\|_p, \|y_n\|_p\} \\ &= \max\{\|x\|_p m\|y\|_p\}\end{aligned}$$

We can check all the facts for a commutative ring.

Theorem 6.2.3
$\mathbb{Q}_p$ is a topologically complete field that contains $\mathbb{Q}$ as a dense subfield.

Proof
$\mathbb{Q}_p$ is complete and $\mathbb{Q}$ is dense by construction.

It remains to check for inverses. Let $x \in \mathbb{Q}, x \neq 0$.

Write

$$x = \lim x_n, x_n \in \mathbb{Q} \wedge \forall n \geq N, \|x_n\|_p = \|x\|_p \neq 0$$

So $x_n \neq 0, n \geq N$.

Let $y = \lim \frac{1}{x_n}$.

$$\begin{aligned} \left\| \frac{1}{x_n} - \frac{1}{x_m} \right\| &= \left\| \frac{x_m - x_n}{x_m x_n} \right\|_p \\ &= \frac{\|x_m - x_n\|_p}{\|x_m\|_p \|x_n\|_p} \\ &= \frac{\|x_m - x_n\|_p}{\|x_p\|^2} \end{aligned}$$

and hence the sequence is cauchy and makes sense.

In addition

$$xy = \lim x_n \frac{1}{x_n} = 1$$

as desired.

Proposition 6.2.4
$\mathbb{Z}_p := \overline{\mathbb{Z}}^{d_p} = \{x \in \mathbb{Q}_p : \|x\|_p \leq 1\}$

Proof
RHS is closed and equal to

$$\overline{b_1(0)} \supseteq \mathbb{Z}$$

hence

$$\text{RHS} \supseteq \overline{\mathbb{Z}}^{d_p}$$

To see inclusion in the other direction, let

$$x \in \mathbb{Q}_p, \|x\|_p \leq 1$$

Let $n \geq 0$. There is some $r_n \in \mathbb{Q}$ such that

$$\|x - r_n\| \leq p^{-n}$$

We have

$$\begin{aligned}\|r_n\|_p &= \|x - (x - r_n)\|_p \\ &\leq \max\{\|x\|_p, \|x - r_n\|_p\} \\ &\leq 1\end{aligned}$$

and hence

$$r_n = p^k \frac{a}{b}$$

where a, b, p are relatively prime and $k \geq 0$.

Replace a with $a + p^n c \equiv 0 \mod b$ and solve for $c \in \mathbb{Z}$ since $\gcd(p^n, b) = 1$.

Let

$$s_n := p^k \left(\frac{a}{b} + \frac{p^n c}{b}\right) = p^k \left(\frac{a + p^n c}{b}\right) \in \mathbb{Z}$$

We have

$$\begin{aligned}\|r_n - s_n\|_p &= \left\|p^k \frac{p^n c}{b}\right\|_p \\ &= p^{-k-n} \\ &\leq p^{-n}\end{aligned}$$

Also

$$\begin{aligned}\|x - s_n\|_p &\leq \max\{\|x - r_n\|_p, \|r_n - s_n\|_p\} \\ &\leq p^{-n}\end{aligned}$$

We may then conclude

$$\begin{aligned}\lim s_n &= x \\ &\implies \\ x &\in \overline{\mathbb{Z}}^{d_p} \\ &= \mathbb{Z}_p\end{aligned}$$

6.2.1 p-adic Expansion

Proposition 6.2.5
Let $x \in \mathbb{Z}$, we claim there is a unique $\alpha_0 \in \{0, 1, \ldots, p-1\}$ such that

$$\|x - x_0\|_p \leq \frac{1}{p}$$

Proof
Existence

Pick $k \in \mathbb{Z}, \|x - k\|_p \leq \frac{1}{p}$ We have

$$k = \alpha_0 \mod p, \alpha_0 \in \{0, 1, \ldots, p-1\}$$

and $\|k - \alpha_0\| \leq \frac{1}{p}$.

Hence

$$\begin{aligned}\|x - \alpha_0\|_p &\leq \max\{\|x - k\|_p, \|k - \alpha_0\|_p\} \\ &\leq \frac{1}{p}\end{aligned}$$

Uniqueness

Suppose there is some $\beta_0 \in \{0, 1, \ldots, p-1\}$ with $\beta_0 \neq \alpha_0$.

So $\|\alpha_0 - \beta_0\|_p = 1$ as $p \not| \alpha_0 - \beta_0$.

It follows that

$$\begin{aligned}1 &= \|\alpha_0 - \beta_0\|_p \\ &= \|(\alpha_0 - x) + (x - \beta_0)\| \\ &\leq \max\{\underbrace{\|x - \alpha_0\|_p}_{\leq \frac{1}{p}}, \|x - \beta_0\|_p\}\end{aligned}$$

then

$$\|x - \beta_0\| \geq 1 \quad (= 1)$$

Proposition 6.2.6
for $n \geq 0$ there is a unique $\alpha_n \in \{0, 1, \ldots, p-1\}$ such that

$$\left\| x - \sum_{i=0}^{n} \alpha_i p^i \right\|_p \leq \frac{1}{p^{n+1}}$$

Proof
Establish this for $n = 0$.

Suppose this is true up to $n-1$.

$$y = x - \sum_{i=0}^{n-1} \alpha_i p^i$$

has

$$\|y\|_p \leq \frac{1}{p_n}$$

Then $\|p^{-n}y\|_p = \|p^{-n}\|_p \|y\|_p = p^n \|y\|_p \leq 1$. By the $n = 0$ case, we are done.

By construction

$$x = \lim \sum_{i=0}^{n} \alpha_i p^i = \sum_{i=0}^{\infty} \alpha_i p^i$$

To see uniqueness, if $x \in \mathbb{Q}_p$ with $\|x\|_p = p^n$ then

$$\begin{aligned} \|p^n x\|_p &= p^{-n} \cdot p^n \\ &= 1 \end{aligned}$$

$$\begin{aligned} p^n x &= \sum \beta_i p^i \\ x &= \sum \beta_i p^{i-n} \\ &= \sum \alpha_i p^i \end{aligned}$$

Proposition 6.2.7
$\mathbb{Z}_p$ is compact.

Proof
From an assignment, $\mathbb{Z}$ is totally bounded and

$$\{0, 1, \ldots, p-1\}$$

is a $\frac{1}{p^n}$-net.

Hence $\mathbb{Z}_p$ is totally bounded and complete.

It follows that $\mathbb{Z}_p$ is compact.

6.3 Construction of $\mathbb{R}$

6.3.1 Definitions

Definition 6.3.1 (Ordered)
A field is ordered if there is a subset $P \subseteq \mathbb{F}$ (positive) such that

(1) $\mathbb{F} = -P \cup \{0\} \cup P$ (disjoint union)

(2) $P + P \subseteq P$

(3) $P \cdot P \subseteq P$

(4) $x < y$ if $y - x \in P$

Definition 6.3.2 (Upper Bound)
$\varnothing \neq S \subseteq \mathbb{F}$ has an upper bound if there is $x \in \mathbb{F}$ such that

$$s \in S \implies s \leq x$$

Definition 6.3.3 (Lowest Upper Bound, LUB)
$\varnothing S \subseteq \mathbb{F}$ has a LUB $y := \sup S$ if every upper bound x is such that

$$y \leq x$$

Definition 6.3.4 (Lowest Upper Bound Property, LUBP)
A field has this property if for all $\varnothing \neq S \subseteq \mathbb{F}$ with an upper bound, S has a LUB.

Definition 6.3.5 (Archimedean)
A field is said to be Archidemean if for every $x > 0$ there is $n \in \mathbb{N}$ such that

$$\frac{1}{n} < x$$

Definition 6.3.6 (Complete)
A field is complete if every cauchy sequences converges.

Definition 6.3.7 (Cauchy)
A sequence is cauchy if for all $0 < r \in \mathbb{Q}$, there is N such that $n, m \geq N$ implies

$$|x_n - x_m| < r$$

6.3.2 Results

Proposition 6.3.1
If $\mathbb{F}$ is an ordered field

(1) $\mathbb{Q} \subseteq \mathbb{F}$

(2) $\mathbb{F}$ has the LUBP if and only if $\mathbb{F}$ is complete

(3) $\mathbb{F}$ has the LUBP implies $\mathbb{F}$ is archimedean

(4) if $\mathbb{F}$ is archimedean, then $x < y \implies \exists r \in \mathbb{Q}, x < r < y$

Proof (1)
We know $0, 1 \in \mathbb{F}$ so

$$0 < 1 < 1 + 1 < +_{i=1}^{n} 1 =: n \in \mathbb{F}$$

and so $-n \in \mathbb{F}$.

It follows that $\mathbb{Z} \subseteq \mathbb{F}$.

But then $\frac{p}{q} := p \cdot q^{-1} \in \mathbb{F}$ and so $\mathbb{Q} \subseteq \mathbb{F}$. This means the field has characteristic 0.

Proof (2)
($\Rightarrow$) Notice that the monotonic subsequence version of the Bolzano-Weierstrass Theorem gives us completeness assuming the LUBP as all cauchy sequences are bounded and converge if there it as a convergent subsequences.

($\Leftarrow$) Our previous proof of the existence of LUBD in $\mathbb{R}$ can be repeated with some sort of base d expansions.

Proof (3)
Let $J := \{x : x > 0 \wedge \forall n \geq 1, nx < 1\}$ (infinitessimals).

Suppose $J \neq \varnothing$. Notice that 1 is an UB for J and if $x, y \in J$ then $nx + my \in J$ for $n, m \geq 1$.

If $\mathbb{F}$ has the LUBP, let $y := \sup J$. Let $x_0 \in J$ and notice that $\forall x_0 \in J, x + x_0 \in J$ so $x + x_0 \leq y \implies x \leq y - x_0$, a clear contradiction.

So $J = \varnothing$ as desired.

Proof (4)
If $x < y$, then $y - x > 0$ so there is some n such that

$$0 < \frac{1}{n} < y - x$$

It follows that there is some $k \in \mathbb{Z}$ with

$$\frac{k}{n} \leq x < \underbrace{\frac{k+1}{n}}_{\in \mathbb{Q}} \leq x + \frac{1}{n} < y$$

as desired.

6.3.3 Order Embedding

Definition 6.3.8 (Order Embedding)
The order embedding of an ordered field $\mathbb{K}$ into an ordered field $\mathbb{F}$ means there is an order preserving (ring) homomorphism.

Proposition 6.3.2
If $\mathbb{F}$ is an ordered field and $\mathbb{K}$ is a complete ordered field. There is an ordered embedding

$$\gamma : \mathbb{F} \to \mathbb{K}$$

Proof
Define $\gamma(0) = 0, \gamma(1) = 1$ and thus

$$\gamma(n) = n, \gamma(r) = r$$

for all $n \in \mathbb{Z}, r \in \mathbb{Q}$.

If $x \in \mathbb{F}$ let

$$S_x := \{r \in \mathbb{Q} : r < x\}$$

So there is some $n \in \mathbb{N}$ such that $x < n$ by the Archemedean and so S_x is bounded above. Thus, we have $\gamma(S_x)$ is bounded above in $\mathbb{K}$ by $\gamma(n)$.

Define

$$\gamma(x) := \sup \gamma(S_x)$$

Notice that

$$S_x + S_y = S_{x+y}$$

since for $r, s \in \mathbb{Q}, r < x, s < y$ meaning

$$r + s < x + y$$

If $t \in \mathbb{Q}, t < x + y$ we can choose $r \in \mathbb{Q}, x - \frac{1}{2n} < r < x$ and $s \in \mathbb{Q}, y - \frac{1}{2n} < s < x$. Remark

$$t < x + y - \frac{1}{n} < r + s < x + y$$

and so $t = r + (t - r)$ with $r \in S_x$ and $t - r < s \implies t - r \in S_y$

$$S_y = \{r \leq 0\} \cup \{0 < s < y\}$$

It follows that $\gamma(x) + \gamma(y) = \gamma(x + y)$

We can also check

$$S_{xy} = \{r \leq 0\} \cup \{rs : 0 < r < x, 0 < s < y\}$$

meaning $\gamma(xy) = \gamma(x)\gamma(y)$.

Theorem 6.3.3 (Uniquess of $\mathbb{R}$)
There is a unique complete ordered field $\mathbb{R}$.

Proof
Let $\mathbb{K}, \mathbb{L}$ be two complete ordered fields, then they are Archimedean.

By the second proposition, there is an ordered embedding $\gamma : \mathbb{K} \to \mathbb{L}$ (order homomorphism). Also, there is $\delta : \mathbb{L} \to \mathbb{K}$ order homomorphism.

The composition of both necessarily act as the identity on $\mathbb{Q}$. Hence

$$\delta\gamma(x) = \sup S_x = x$$

and similarly for the other side.

This shows that δ, γ are inverses of each other and that $\mathbb{K}, \mathbb{L}$ are isomorphic by definition.

6.3.4 Actual Constructions of the Real Numbers

Example 6.3.4 (Cantor)
equivalence classes of cauchy sequences of $\mathbb{Q}$

Example 6.3.5 (Dedikind Cut)
We say S is a cut of $\mathbb{Q}$ if $\varnothing \neq S \neq \mathbb{Q}$ and

$$x \in S \implies \forall r \leq x, x \in S$$

Notice that S has an upperbound as the complement is not the emptyset, no biggest element and, and we define $S > 0$ if there is $r \in S$ such that $r > 0$.

Example 6.3.6 (Base *d* Expansion)
We can define

$$\mathbb{R} := \{x := a_0 a_1 \cdots : a_i \in \{0, 1, \dots, d-1\}\}$$

By our work prior, it has the LUBP. We can then take equivalence clases by $\sim$ so

$$0.4\bar{9} \sim 0.5$$

for example.

7 Approximation Theorem

7.1 Polynomial Approximation

We attempt to approximate an arbitrary continuous function f on $[a, b]$. This means finding $g \in \mathbb{R}[x]$ such that

$$\|f - g\|_\infty < \epsilon$$

where $\epsilon > 0$ is arbitrary.

7.1.1 Interpolation

We first consider polynomial interpolation.

Let $f \in C[a, b]$ and defines the points

$$x_i := a + i\frac{b-a}{n}$$

We wish to find a polynomial p such that $p(x_i) = f(x_i)$ for $0 \leq i \leq n$.

Let

$$q_i(x) := \frac{\prod_{j \neq i} x - x_j}{\prod_{j \neq i} x_i - x_j}$$

and note that q_i has degree n with $q_i(x_j) = \delta_{ij}$.

Furthermore, deifne

$$p(x) = \sum_{i=0}^{n} f(x_i) q_i(x)$$

with degree at most n and $p(x_i) = f(x_i)$.

If $r(x)$ is another polynomial with degree $r \leq n$ and $r(x_i) = f(x_i)$ then

$$(r - p)(x_i) = 0$$

for all i and so the degree of $r - p$ is at most n with $n + 1$ roots.

So $r - p = 0$ and $r = p$.

However, Runge showed in 1901 that these interpolations do not always converge uniformly to f on $[a, b]$.

7.1.2 Taylor Polynomials

But how about Taylor Polynomials?

We do not wish to assume that derivatives exist.

In fact, we showed that the set of functions which are not differentiable are a residual subset of $C[a, b]$.

7.1.3 Weierstrauss Approximation

Theorem 7.1.1 (Weierstrauss)
$\mathbb{R}[x]$ is dense in $C[a, b]$.

Notice that if the theorem holds, we can apply it to $\mathbb{R}^n, \mathbb{C}^n \sim \mathbb{R}^{2n}$ by approximating components.

It suffices to prove $[a, b] = [0, 1]$ by the (continuous) change of variables

$$x \mapsto \frac{x-a}{b-a}$$

We have

$$1 = (x + (1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

If we let $P_{n,k} := \binom{n}{k} x^k (1-x)^{n-k}$, we then have

$$\begin{aligned} P'_{n,k} &= \binom{n}{k} (kx^{k-1}(1-x)^{n-k} + (n-k)x^k(1-x)^{n-k-1}) \\ &= \binom{n}{k} x^{k-1}(1-x)^{n-k-1}(k(1-x) - (n-k)x) \\ &= \binom{n}{k} x^{k-1}(1-x)^{n-k-1}(k - nx) \end{aligned}$$

Notice that

$$x = \frac{k}{n} \implies k - nk = 0 \implies P'_{n,k}(x) = 0$$

Define

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x)$$

and remark that $\deg B_n f \leq n$ is a polynomial.

Lemma 7.1.2
We claim that $B_n : C[0,1] \to \mathbb{R}[x]$ is such that
(I) $B_n(sf + tg) = sB_n f + tB_n g$ (linearity)
(II) $f \geq 0 \implies B_n f \geq 0$
(III) $f \leq g \implies B_n f \leq B_n g$ and $|f| \leq g \implies |B_n f| \leq B_n g$.

Proof
Claim I: To see linearity, note that

$$\begin{aligned} B_n(sf + tg) &:= \sum_{k=0}^{n} \left(sf\left(\frac{k}{n}\right) + tg\left(\frac{k}{n}\right)\right) P_{n,k}(x) \\ &= sB_n f + tB_n g \end{aligned}$$

Claim II: To see positivity, notice

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \geq 0$$

so if $f \geq 0$, then $f\left(\frac{k}{n}\right) \geq 0$ and so

$$B_n f \geq 0$$

by definition.

Claim III: Since $f \leq g \implies g - f \leq 0$ then

$$B_n(g - f) \geq 0 \implies B_n g \geq B_n f$$

by linearity.

In addition, if $|f| \leq g$ then $-g \leq f \leq g$ so

$$-B_n g \leq B_n f \leq B_n g \implies |B_n f| \leq B_n g$$

Lemma 7.1.3
We claim
(I) $B_n(1) = 1$
(II) $B_n x = x$
(III) $B_n(x^2) = \frac{n-1}{n}x^2 + \frac{x}{n}$ and hence $\|B_n(x^2) - x^2\|_\infty = \left\|\frac{x-x^2}{n}\right\|_\infty = \frac{1}{4n}$

Proof
By computation.

Claim I:

$$\begin{aligned}(B_n 1)(x) &= \sum_{k=0}^{n} 1 \cdot \binom{n}{k} x^k (1-x)^{n-k} \\ &= (x + (1-x))^n \\ &= 1\end{aligned}$$

Claim II:

$$\begin{aligned}(B_n x)(x) &= \sum_{k=0}^{n} \frac{k}{n} \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\ &= \dots && \text{start summation at } k=1 \\ &= \dots && \text{cancel fraction } 1/n \\ &= \dots && \text{pull out } x \text{ from } x^k \\ &= x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1-x)^{n-1-j} \\ &= x B_{n-1}(1) \\ &= x\end{aligned}$$

Claim III:

$$\begin{aligned}
(B_n x^2)(x) &= \sum_{k=0}^{n} \frac{k^2}{n^2} \cdot \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \\
&= \dots && \text{starting at } k=1 \\
&= \dots && \text{cancel } k/n \\
&= \frac{1}{n} \sum_{k=2}^{n} (k-1) \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&+ \frac{1}{n} \sum_{k=1}^{n} 1 \cdot \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} \\
&= \frac{1}{n} \sum_{k=2}^{n} (k-1) \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} + \frac{1}{n} B_{n-1}(x) \\
&= \frac{1}{n} \sum_{k=2}^{n} (k-1) \frac{(n-1)!}{(k-1)!(n-k)!} x^k (1-x)^{n-k} + \frac{x}{n} \\
&= \dots && \text{cancel } k-1 \\
&= \dots && \text{pull out } n-1 \\
&= \dots && \text{pull out } x \\
&= \frac{n-1}{n} x^2 \sum_{j=0}^{n-2} \frac{(n-2)!}{j!(n-2-j)!} x^j (1-x)^{n-2-j} + \frac{x}{n} \\
&= \frac{n-1}{n} x^2 B_{n-2}(1) + \frac{x}{n} \\
&= x^2 + \frac{x - x^2}{n}
\end{aligned}$$

Notice that $B_n x^2 \to x^2$ as $n \to \infty$.

Proof (theorem, Bernstein)
For $\epsilon > 0$ by continuity on compact domain (uniform continuity) there is $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

For $a \in [0, 1]$ we have

$$|x - a| < \delta \implies |f(x) - f(a)| \leq \epsilon$$

On the other hand, for $|x - a| \geq \delta$

$$|f(x)| - f(a) \leq 2\|f\|_\infty \leq \frac{2\|f\|_\infty}{\delta^2}(x-a)^2$$

and therefore

$$f(x) - f(a) \leq \epsilon \cdot 1 + \frac{2\|f\|_\infty}{\delta^2}(x-a)^2$$

hence

$$\begin{aligned}
|(B_n f)(x) - f(a)| &= |(B_n f)(x) - f(a) \cdot (B_n 1)(x)| \\
&= |B_n (f - f(a) \cdot 1)(x)| \\
&\leq B_n \left(\epsilon \cdot 1 + \frac{2\|f\|_\infty}{\delta^2}(x-a)^2 \right) \\
&\leq \epsilon B_n 1 + \frac{1\|f\|_\infty}{\delta^2} B_n (x-a)^2 \\
&= \epsilon + \frac{2\|f\|_\infty}{\delta^2} \left((x-a)^2 + \frac{x - x^2}{n} \right)
\end{aligned}$$

Plug in $x = a$ and we have

$$\begin{aligned}
|B_n f(a) - f(a)| &\leq \epsilon + \frac{2\|f\|_\infty}{\delta^2}(0 + \frac{a - a^2}{n}) \\
&\leq \epsilon + \frac{\|f\|_\infty}{2\delta^2 n}
\end{aligned}$$

If we choose $n \geq \left\lceil \frac{\|f\|_\infty}{2\delta^2 \epsilon} \right\rceil$ then

$$|B_n f(a) - f(a)| \leq 2\epsilon$$

and by the choice of a

$$\|B_n f - f\|_\infty \leq 2\epsilon$$

7.1.4 More

Definition 7.1.1
$\operatorname{dist}(f, \mathcal{F}) := \inf_{g \in \mathcal{F}} \|f - g\|_\infty$ where $\mathcal{F}$ is a collection of functions.

Proposition 7.1.4
If $f \in C[a,b], n \in \mathbb{N}$ then there is a polynomial $p \in P_n[a,b]$ such that

$$\|f - p\|_\infty = \operatorname{dist}(f, P_n[a,b])$$

Proof
Notice that $P_n[a,b]$ is subspace with $\dim P_n[a,b] = n+1 < \infty$. So the closest polynomials

are at LEAST as close as the zero polynomial.

This means the closest polynomial lies in

$$P_n[a,b] \cap \overline{b_{\|f\|_\infty}(f)} =: K_n$$

which is a closed subspace of $\mathcal{P}_n$.

It follows that it is homeomorphic to $\mathbb{F}^{n+1}$ and hence K_n is compact (already bdd).

Define $\rho : K_n \to [0, \infty)$ by

$$\rho(p) := \|f - p\|_\infty$$

and note that it is continuous.

By the EVT, ρ attains its minimum at some $p \in K_n$ as desired.

Example 7.1.5
$S := \{f \in C[a,b] : f(0) = 0\}$ then

$$\operatorname{dist}(1, S) = \inf_{f \in S} \|1 - f\|_\infty \leq 1$$

Example 7.1.6
$(\mathbb{R}^n, \|\cdot\|_2)$ is a convex open ball so the distance to a point is unique.

But with $\|\cdot\|_\infty$ an entire line attains the distance.

Example 7.1.7
$T := \{f \in C[0,1] : f(0) = 0, \int_0^1 f(t)dt = 0\}$

Let $g(x) = x, f \in T$

$$\begin{aligned}
\underbrace{\|g - f\|_\infty}_{} &= \int_0^1 \|g - f\|_\infty \\
&\underbrace{\geq}_{=} \underbrace{\int_0^1 (t - f(t))dt}_{=\frac{1}{2}} \\
&= \frac{t^2}{2}\Big|_0^1 - \int_0^1 f(t)dt \\
&= \frac{1}{2}
\end{aligned}$$

Equality forces $f(x) = x - \frac{1}{2}$ but then $f \notin T$. Hence $\operatorname{dist}(f, T)$ is NOT attained.

Theorem 7.1.8 (Chebychev)
If $f \in C_{\mathbb{R}}[a,b], n \in \mathbb{N}$, then there is a unique polynomial $p \in \mathcal{P}_n[a,b]$ such that

$$\|f-g\|_\infty = \operatorname{dist}(f, \mathcal{P}_n)$$

Definition 7.1.2 (Equioscillation)
$g \in C_{\mathbb{R}}[a,b]$ has equioscillation of degree n if there are

$$a \leq x_1 \leq x_2 \leq \ldots, \leq x_{n+2} \leq b$$

such that

$$g(x_i) = (-1)^i \|g\|_\infty \vee g(x_i) = (-1)^{i+1} \|g\|_\infty$$

Lemma 7.1.9
If $f \in C_{\mathbb{R}}[a,b], p \in \mathcal{P}_n[a,b]$ and $f-p$ satisfies equioscillation of degree n, then

$$\|f-p\|_\infty = \operatorname{dist}(f, \mathcal{P}_n)$$

Proof
If there is $q \in \mathcal{P}_n[a,b]$ such that

$$\|g-q\|_\infty = \|f-p-q\| = \|f-p\|_\infty - \delta = \|g\|_\infty - \delta$$

then let

$$a \leq x_i \leq b$$

exhibit the equioscillation.

$$\|g\|_\infty - \delta \geq |g(x_i) - q(x_i)| = \pm\|g\|_\infty - q(x_i)$$

hence $q(x_i)$ has the same sign as $g(x_i)$.

It follows that $q(x)$ changes signs on $[x_i, x_{i+1}]$ for $1 \leq i \leq n+1$.

By IVT, it has $n+1$ roots but $\deg q \leq n$ so this is the desired contradiction.

Lemma 7.1.10
Let $f \in C_{\mathbb{R}}[a,b], p \in \mathcal{P}_n$ such that

$$\|r\|_\infty = \|f - p\|_\infty = d(f, \mathcal{P}_n[a,b])$$

then r satisfies equioscillation of degree n.

Proof
Without loss of generality, $r \neq 0$.

By uniform continuity, there is $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\|r\|_\infty}{2}$$

Divide $[a,b]$ into intervals of length less than δ. Label intervals I_i upon which r attains $\pm\|r\|_\infty$ on $\overline{I_i}$.

If $r(x) = \|r\|_\infty$ on I_i

$$\begin{aligned} |I_i| < \delta &\implies \forall y \in I_i, |r(x) - r(y)| < \frac{\|r\|_\infty}{2} \\ &\implies r(y) \geq \frac{\|r\|_\infty}{2} \end{aligned}$$

and similarly if $r(x) = -\|r\|_\infty$ on I_i, there $y \in I_i$

$$r(y) \leq -\frac{\|r\|_\infty}{2}$$

Pick $x_i \in I_i$ such that $r(x_i) = \pm\|r\|_\infty$ and define

$$\epsilon_i := \operatorname{sgn} r(x_i) \in \{\pm 1\}$$

Group I_i's into adjacent groups with common sign, say $J_1, \dots, J_k$. If $k \geq n+2$, then get $x_j \in J_j$ and we are done.

Suppose $k \leq n+1$ and pick $a_1, \dots, a_{k-1}$ such that

$$J_j < a_j < J_{j+1}$$

(which exist as r alternates between $\pm\|r\|_\infty$ on J_i, J_{i+1}).

Let $q(x) := \prod_{j=1}^{k-1}(x - a_j)$ and notice that $\deg q = k - 1 \leq n$ and $q \in \mathcal{P}_n$. Without loss of generality, by multiplication of -1 if necessary, $\operatorname{sgn} q(x) = \epsilon_i$ on I_i.

As $q \neq 0$ on J_i, let

$$m := \min_{x \in \bigcup J_j} |q(x)| > 0$$

Moreoever, let

$$L := \bigcup \overline{I_i}, M := \overline{[a,b] \setminus L}$$

be the union of closed intervals on which r does NOT attain $\pm\|r\|_\infty$ so

$$\sup_{x \in M} |r(x)| = \|r\|_\infty - d < \|r\|_\infty$$

Let

$$s := \frac{q}{\|q\|_\infty} \cdot \frac{d}{2} \in \mathcal{P}_n$$

and consider

$$\begin{aligned}
\|f - (p+s)\|_\infty &= \|f - s\|_\infty \\
&= \max\{\sup_{x \in L}|r(x) - s(x)|, \sup_{x \in M}|r(x) - s(x)|\} \\
&\leq \max\left\{\|r\|_\infty - \frac{dm}{2\|q\|_\infty}, \|r\|_\infty - d + \overbrace{\|s\|_\infty}^{=\frac{d}{2}}\right\} \\
&\leq \max\left\{\|r\|_\infty - \frac{dm}{2\|q\|_\infty}, \|r\|_\infty - \frac{d}{2}\right\} \\
&< \|r\|_\infty
\end{aligned}$$

which is a contradiction.

Theorem 7.1.11 (Chebychev)
For all $f \in C_\mathbb{R}[a,b]$ there is a unique $p \in \mathcal{P}_n$ with

$$\|f - p\|_\infty = d(f, \mathcal{P}_n)$$

characterized by equioscillation of degree n.

Proof
The two lemma above show that $p \in \mathcal{P}_n$ is a closest polynomial if and only if $f - p$ satisfies equioscillation of degree n.

The existence of p is guaranteed by the compactness.

Suppose that p, q both attain the minimum distance to f.

$$\begin{aligned}\left\|f-\frac{p+q}{2}\right\|_\infty &\le \left\|\frac{1}{2}(f-p)\right\|_\infty + \left\|\frac{1}{2}(f-q)\right\|_\infty \\ &= \operatorname{dist}(f, \mathcal{P}_n) \\ &= D\end{aligned}$$

So $\frac{1}{2}(p+q)$ also attains the minimum distance. If follows that $\frac{1}{2}(p+q)$ satisfies equioscillation of degree n, so there are

$$a \le x_1 < x_2 < \cdots < x_{n+1} \le b$$

and

$$\begin{aligned}D &= \left(f-\frac{p+q}{2}\right)(x_i) \\ &= \frac{1}{2}(f-p)(x_i) + \frac{1}{2}(f-q)(x_i) \\ &\le \frac{1}{2}D + \frac{1}{2}D \\ &= D\end{aligned}$$

$$\begin{aligned}-D &\ge -\frac{1}{2}D - \frac{1}{2}D \\ &= -D\end{aligned}$$

and hence $f(x_i) - p(x_i) = \pm D = f(x_i) - q(x_i)$ for $1 \le i \le n+2$.

It follows that $(p-q)(x_i) = 0$ and

$$1 \le i \le n+2 \implies p - q = 0$$

by $\deg p - g \le n$.

Example 7.1.12
$f(x) = \cos x \in C_{\mathbb{R}}\left[\pm\frac{\pi}{2}\right]$.

The best polynomial approximation of degree n must be even.

7.2 Stone-Weierstrauss Theorem

7.2.1 Definitions

Let (X, d) be a compact metric space.

Definition 7.2.1 (Subalgebra)
A subalgebra of $C(X), C_{\mathbb{R}}(X)$ is a subspace closed under multiplication.

Definition 7.2.2 (Vector Sublattice)
A Vector Sublattice of $C(X)$ is a subspace A such that for all $f, g \in A$ then

$$f \vee g = \max(f, g)$$
$$f \wedge g = \min(f, g)$$

are both in A.

Example 7.2.1
$A := \{f(\pi) = a_0 + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx\} \subseteq C_{\mathbb{R}}(X)[\pi, \pi]$ is a subalgebra but

$$f \in A \implies f(-\pi) = f(\pi)$$

so A is NOT dense in $C_{\mathbb{R}}[-\pi, \pi]$ (does not separate points).

Definition 7.2.3 (Seperates Points)
$A \subseteq C(X)$ separates points if

$$\forall x \neq y \in X, \exists f \in A, f(x) \neq f(y)$$

Example 7.2.2
$A = \{p(x) = \sum_{k=1}^{n} a_k x^k : a_k \in \mathbb{R}\} \subseteq C[0, 1]$ is not dense (vanishes).

Definition 7.2.4 (Vanishes)
$A \subseteq C(X)$ vanishes at $x_0 \in X$ if $f(x_0) = 0$ for every $f \in A$.

7.2.2 The Theorem

Lemma 7.2.3
If A is a subalgebra of $C_{\mathbb{R}}(X)$, then $\bar{A}$ is a subalgebra and and a vector sublattice.

Proof
We have shown that $\bar{A}$ is a vector space (assignment).

Let $f, g \in \bar{A}$ there are $(f_n), (g_n) \in A$ such that

$$f_n \to f, g_n \to g$$

uniformly.

Then $f_n g_n \in A$ by the definition of a subalgebra. Moreoever

$$f_n g_n \to fg$$

uniformly and hence $fg \in \bar{A}$. So $\bar{A}$ is a subalgebra.

Now, let $f, g \in \bar{A}$ and remark that

$$f \vee g = \frac{f+g}{2} + \left|\frac{f-g}{2}\right|, f \wedge g = \frac{f+g}{2} - \left|\frac{f-g}{2}\right|$$

So we need only show $f \in A \implies |f| \in \bar{A}$.

By the Weierstrauss Approximation Theorem, there is some sequence of polynomials $p_n(x)$ such that

$$p_n(x) \to |x|$$

uniformly on $[-\|f\|_\infty, \|f\|_\infty]$.

Remark that $p_n(0) \to 0$. Let

$$q_n(x) := p_n(x) - p_n(0) = \sum_{k=1}^{k_n} a_{n,k} x^k$$

and notice that

$$\begin{aligned} \||x| - q_n\|_\infty &\leq \||x| - p_n\|_\infty + \underbrace{\|p_n - q_n\|_\infty}_{|p_n(0)|} \\ &\leq 2\||x| - p_n\|_\infty \end{aligned}$$

Look at $q_n(f)(x) = \sum_{k=1}^{k_n} a_{n,k} f(x)^k \in A$ by definitions

$$\begin{aligned}\||f| - q_n(f)\|_\infty &= \sup_{x\in X} ||f(x)| - g_n(f(x))| \\ &\leq \sup_{t\in[\pm\|f\|_\infty]} ||t| - q_n(t)| \\ &= \||x| - q_n\| \to 0\end{aligned}$$

hence $|f| \in \bar{A}$, $f \vee g, f \wedge g \in \bar{A}$.

Lemma 7.2.4
Let (X, d) be compact and $A \subseteq C_{\mathbb{R}}(X)$ be a subalgebra which separates points and does not vanish anywhere.
If $x \neq y \in X, \alpha, \beta \in \mathbb{R}$ then there is $h \in A$ such that

$$h(x) = \alpha, h(y) = \beta$$

Proof
Since A separates points, there is $f \in A$ such that

$$f(x) = a \neq f(y) = b$$

Without loss of generality, we may assume $b \neq 0$ (or else take swap $b = f(x)$ as they cannot both be 0).

Case I: $a \neq 0$.

Then

$$\det \begin{bmatrix} a & a^2 \\ b & b^2 \end{bmatrix} \neq 0$$

and we may find $h = uf + vf^2$ such that

$$\begin{aligned} h(x) &= ua + va^2 = \alpha \\ h(y) &= ub + vb^2 = \beta \end{aligned}$$

as desired.

Case II: $a = 0$.

Since A does NOT vanish, there is some $g \in A$ such that $g(x) \neq 0$. We can take $h = uf + vg$ so

$$\begin{aligned} h(x) &= vg(x) = \alpha \\ h(y) &= ub + vg(y) = \beta \end{aligned}$$

This concludes the proof.

Theorem 7.2.5 (Stone-Weierstrauss)
Let (X, d) be compact and $A \subseteq C_{\mathbb{R}}(X)$ be a subalgebra which separates points and does NOT vanish at any point, then A is dense in $C_{\mathbb{R}}(X)$.

Proof (Stone Weierstrauss)
Fix $f \in C_{\mathbb{R}}(X)$ and let $\epsilon > 0$.

Fix $a \in X$ and let $a \neq x \in X$. By our second lemma, we can choose $h_x \in A$ such that

$$h_x(a) = f(a)$$
$$h_x(x) = f(x)$$

Let

$$U_x := \{y \in X : h_x(y) > f(y) - \epsilon\} = (h_x - f)^{-1}(-\epsilon, \infty)$$

We remark that U_x is open as f, h_x are continuous. Moreoever, $\forall x \in X, a, x \in U_x$.

Thus $\{U_x : x \neq a\}$ is an open cover of X. Choose a finite subcover

$$U_{x_1}, \dots, U_{x_n}$$

of X

Let

$$g_a := h_{x_1} \wedge \dots \wedge h_{x_n} \in \bar{A}$$

as $\bar{A}$ is a vector sublattice. Notice that $g_a(a) = f(a)$ and if $x \neq a$ then for some $i, x \in U_{x_i}$ so

$$h_{x_i}(x) > f(x) - \epsilon \implies g_a(x) > f(x) - \epsilon$$

Now, define

$$V_a = \{x \in X : g_a(x) < f(x) + \epsilon\} = (g_a - f)^{-1}(-\infty, \epsilon)$$

and remark that $a \in V_a$ is open so $\{V_a : a \in X\}$ is an open cover. Retrieve an open cover

$$V_{a_1}, \dots, V_{a_m}$$

and define

$$g := g_{a_1} \vee \dots \vee g_{a_m} \in \bar{A}$$

for the same reason as above. Again, notice that for all $x \in X$ there is j such that $x \in V_{a_j}$ and

$$g_{a_j}(x) < f(x) + \epsilon \implies g(x) < f(x) + \epsilon$$

But $\forall x \in X, g_a(x) > f(x) - \epsilon$ so all in all

$$f(x) - \epsilon < g(x) < f(x) + \epsilon \implies \|f - g\|_\infty \leq \epsilon$$

so by the choice of $f, \bar{A} = C_{\mathbb{R}}(X)$ as desired.

Remark that complex polynomials are dense in $C(X)$ since for $f \in C(X)$, we can write

$$f = \mathrm{Re}(f) + i\,\mathrm{Im}(f)$$

and estimate the components with real valued polynomials.

Corollary 7.2.5.1
Let X be a compact subset of $\mathbb{R}^n$. Then, A, the subalgebra of polynomials in

$$x_1, \ldots, x_n$$

is dense in $C_{\mathbb{R}}(X)$.

Proof
Notice that $1 \in A$ so it does not vanish. In addition, $x \neq y \in X$ then there is $i, x_i \neq y_i$ and thus

$$p = x_i \in A$$

so A indeed separates points.

By the Stone-Weierstrauss Theorem, $\bar{A} = C_{\mathbb{R}}(X)$.

Corollary 7.2.5.2
Let X, Y be compact metric spaces. Then

$$A := \left\{ \sum_{i=1}^{n} f_i(x) g_i(y) : f_i \in C_{\mathbb{R}}(X), g_i \in C_{\mathbb{R}}(Y), n \in \mathbb{N} \right\}$$

is dense on $C_{\mathbb{R}}(X \times Y)$.

Proof
Put a metric on $X \times Y$ to be

$$\rho((a,b),(c,d)) = d_X(a,c) + d_Y(b,d)$$

The identity mapping is in A so A does not vanish anywhere.

To see A separates points, suppose

$$(x_1, y_1) \neq (x_2, y_2) \in A$$

Case I: if $x_1 \neq x_2$.

Let

$$f(x) = d_X(x_1, x)$$
$$g(y) = 1$$

so

$$f(x_1)g(y_1) = 0$$
$$f(x_2)g(y_2) = d_X(x_1, x) \neq 0$$

Case II if $y_1 \neq y_2$.

Similar.

So A indeed satisfies the conditions to apply the Stone-Weierstrauss Theorem and $\bar{A} = C_{\mathbb{R}}(X)$ as desired.

7.2.3 Complex Stone-Weierstrauss Theorem

Definition 7.2.5 (Self-Adjoint)
A subalgebra $A \subseteq C(X)$ is self-adjoint if

$$f \in A \implies \bar{f} \in A$$

where

$$\bar{f}(x) = \underbrace{\overline{f(x)}}_{\text{complex conjugate}}$$

Remark that

$$\operatorname{Re} f(x) = \frac{1}{2}(f(x) + \bar{f}(x))$$
$$\operatorname{Im} f(x) = \frac{1}{2}(f(x) - \bar{f}(x))$$

so if A is self-adjoint, then $\operatorname{Re} f, \operatorname{Im} f \in A \cap C_{\mathbb{R}}(X)$.

Theorem 7.2.6 (Complex Stone-Weierstrauss Theorem)
If A is a subalgebra of $C(X)$ where X is a compact metric space and

(i) A does not vanish anywhere

(ii) A separates points

(iii) A is self-adjoint

then $\bar{A}^{\|\cdot\|_\infty} = C(X)$.

Proof (sketch)
Basically, $\overline{\operatorname{Re} A} = C_{\mathbb{R}}(X)$ hence

$$\begin{aligned}\bar{A} &= \overline{\operatorname{Re} A} + i\overline{\operatorname{Re} A} \\ &= C_{\mathbb{R}}(X) + iC_{\mathbb{R}}(X) \\ &= C(X)\end{aligned}$$

8 Ordinary Differential Equations

8.0.1 Introduction

Consider the following motivating example

$$y'(x) = \varphi(x, y), y(a) = y_0$$

We wish to find $y = f(x)$ which solves this.

It is useful to picture this as a vector field and an initial point. We need only follow along according to the directions.

Example 8.0.1
$y' = \varphi(x), y(a) = y_0$

Compute

$$y(x) = y(0) + \int_0^x y'(t)dt = a + \int_0^x \varphi(t)dt$$

and remark that by FTC

$$y'(x) = \varphi(x), y(0) = 0$$

as desired.

Example 8.0.2 (Level Lines)
$y' = xy, y(0) = 3$ apply separation of variables to get

$$\frac{y'}{y} = x$$

and integrate with respect to x

$$\log y = \frac{x^2}{2} + C$$
$$y = c_1 e^{\frac{x^2}{2}}$$
$$y(0) = c_1 = 3$$
$$y(x) = 3e^{\frac{x^2}{2}}$$

Definition 8.0.1 (Standard Form)
Write the highest derivative as a function of

$$x, y, y', \dots, y^{(n-1)}$$

Let $u(x,y) \in C^1$ and recall the level lines ODE

$$u(x,y) = c$$

Since

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = y' = 0$$

by the Implicit Function Theorem, if $\frac{\partial u}{\partial y}(x_0, y_0) \neq 0$ then there is $\delta > 0$ such that

$$f : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$$

such that

$$u(x, f(x)) = u(x_0, y_0) = c$$

> **Example 8.0.3**
> $y' = 1 + x - y, |x| \leq \frac{1}{2}, y(0) = 1.$
>
> We have
>
> $$\begin{aligned} y(x) &= y(0) + \int_0^x y'(t)dt \\ &= 1 + \int_0^x 1 + t - y(t)dt \end{aligned}$$
>
> $$y(x) = 1 + x + \frac{x^2}{2} - \int_0^x y(t)dt$$
>
> Define $T : C\left[\pm\frac{1}{2}\right] \to C\left[\pm\frac{1}{2}\right]$ given by
>
> $$Tf(x) := 1 + x + \frac{x^2}{2} - \int_0^x g(t)dt$$
>
> and notice that a solution $Tf = f$ solves the ODE and is a fixed point of T.
>
> Furthermore, T is a contraction mapping by an assignment.
>
> By computation
>
> $$f(x) = e^{-x} + x$$
>
> solves this ODE and we can arrive at this conclusion by iteratively computing the fixed point and noticing it is the power series of $e^{-x} + x$.

8.0.2 General Setup

For an n-th order ODE we have

(1) relationship between $x, f(x), f'(x), \dots, f^{(n)}(x)$

(2) initial conditions, usually a point x_0.

Recall the standard form

$$\begin{aligned} f^{(n)}(x) &= \Phi(x, f(x), \dots, f^{(n-1)}(x)) \\ \Phi &: [a,b] \times \mathbb{R}^n \to \mathbb{R} \\ f(a) &= \gamma_0 \\ f'(a) &= \gamma_1 \\ &\dots \\ f^{(n-1)}(a) &= \gamma_{n-1} \end{aligned}$$

Φ MUST be continuous, but perhaps even better.

Convert this to a 1st order vector-vaued ODE. Set

$$F(x) := (f(x), f'(x), \dots, f^{(n-1)}(x))$$

and remark that

$$F'(x) = \begin{pmatrix} f_0'(x) \\ \dots \\ f_{n-1}'(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \dots \\ \Phi(x, f_0(x), f_0'(x), \dots, f_0^{(n-1)}(x)) \end{pmatrix} = \Psi(x, F(x))$$

so $f_i' = f_0^{(i)}$ and

$$f_{n-1}'(x) = f^{(n)}(x) = \Phi(x, f_0(x), f_0'(x), \dots, f_0^{(n-1)}(x))$$

with

$$F(a) = (f_0(a), \dots, f_{n-1}(a)) = \Gamma = (\gamma_0, \dots, \gamma_{n-1})$$

Now, convert this to an integral equation.

$$F(x) = F(a) + \int_a^x F'(t)dt = \Gamma + \int_a^x \Psi(t, F(t))dt$$

Define $T : C([a,b], \mathbb{R}^n) \to C([a,b], \mathbb{R}^n)$ with

$$TF(x) := \Gamma + \int_a^x \Psi(t, F(t))dt$$

Clearly, F is a solution if and only if $TF = F$ (ie a fixed point)

Example 8.0.4
$(1+(y')^2)y^{(3)} - y^{(2)} + xyy' = \sin x$ with $x \in [\pm 1]$.

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \\ y^{(2)}(0) &= 2 \\ y^{(3)} &= \frac{y^{(2)} - xyy' + \sin x}{1+(y')^2} \\ &= \varphi(x, y, y', y^{(2)}) \end{aligned}$$

$$\varphi(x, y, y', y^{(2)}) = \frac{y_2 - xy_0y_1 + \sin x}{1+y_1^2}$$

We have

$$\begin{aligned} F &= (f_0, f_1, f_2) \\ F'(x) &= \Psi(x, F(x)) \\ \Psi(x, y_0, y_1, y_2) &= (y_1, y_2, \varphi(x, y_0, y_1, y_2)) \\ F(a) &= \Gamma = (1, 0, 2) \end{aligned}$$

We wish to find $F \in C\left([\pm 1], \mathbb{R}^3\right)$ such that

$$TF = \Gamma + \int_0^x \Psi(t, f_0(t), \ldots, f_2(t))dt$$

8.0.3 Basic Results

Definition 8.0.2 (Lipschitz)
$\Phi(x, y_0, \ldots, y_{n-1}), \Phi : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz in y if there is $L < \infty$ such that

$$\|\Phi(x, \vec{y}) - \Phi(x, \vec{z})\|_2 \leq L\|\vec{y} - \vec{z}\|_2$$

for all $x \in [a,b]$ and $\vec{y}, \vec{z} \in \mathbb{R}^n$.

Example 8.0.5
If $\Phi \in C^1$ in y variables with

$$\nabla_y \Phi(\frac{\partial \Phi}{\partial y_0}, \ldots, \frac{\partial \Phi}{\partial y_{n-1}})$$

By MVT

$$\Phi(x,\vec{y}) - \phi(x,\vec{z}) = \nabla_y \Phi(\xi) \cdot (\vec{y} - \vec{z})$$

for some $\xi \in [\vec{y}, \vec{z})$ (line).

So by the Cauchy-Schwartz Inequality

$$\|\Phi(x,\vec{y}) - \Phi(x,\vec{z})\|_2 \le \|\nabla_y \Phi\|_\infty \|\vec{y} - \vec{z}\|_2$$

if

$$\|\nabla_y \Phi\|_\infty := \sup_{y\in\mathbb{R}^n, x\in[a,b]} \|\nabla_y \Phi(x,y)\|_2 = L < \infty$$

and Φ is Lipschitz in y.

Example 8.0.6 (Linear ODE)
$y^{(n)}(x) = a_0(x)y(x) + a_1(x)y'(x) + \cdots + a_n(x)y^{(n-1)}(x) + b(x)$ (linear in y, NOT necessarily in x).

$$\begin{aligned}
\Phi(x, y_0, \ldots, y_{n-1}) &= (y_1, \ldots, y_{n-1}, \sum_{i=0}^{n-1} a_i(x) y_i + b(x)) \\
&= \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \ldots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \ldots \\ b(x) \end{pmatrix} \\
&= A(x)\vec{y} + B(x) \qquad A \in M_n(C[a,b]), B(x) \in C([a,b], \mathbb{R}^n)
\end{aligned}$$

Notice that

$$\nabla_y \Phi(x,\vec{y}) = (A(x)e_i)$$

and it is actually independent of y.

$$\begin{aligned}
\|\nabla\Phi\|_\infty &= \sup_{x\in[a,b], y\in\mathbb{R}^n} \|\nabla\Phi(x,\vec{y})\|_2 \\
&= \sup_{x\in[a,b]} \|\nabla\Phi(x,\vec{y})\| \\
&< \infty \qquad \text{EVT}
\end{aligned}$$

Indeed,

$$\Phi(x,\vec{y}) - \Phi(x,\vec{z}) = \begin{pmatrix} y_1 - z_1 \\ \dots \\ y_{n-1} - z_{n-1} \\ \sum_{i=0}^{n} a_i(x)(y_i - z_i) \end{pmatrix}$$

and so

$$\begin{aligned} \|\Phi(x,\vec{y})\|_2 &= \sum_{i=1}^{n-1} |y_i - z_i|^2 + \left|\sum a_i(x)(y_i - z_i)\right|^2 \\ &\leq \|y - z\|_2^2 + \left[\left(\sum a_i(x)^2\right)^{\frac{1}{2}} \left(\sum |y_i - z_i|^2\right)^{\frac{1}{2}}\right]^2 \end{aligned}$$

8.0.4 Picard's Theorem

Lemma 8.0.7
Suppose Φ is Lipschitz in y with constant L and let

$$TF(x) = \Gamma + \int_c^x \Phi(t, F(t))dt$$

If $F, G \in C([a,b], \mathbb{R}^n)$ satisfies

$$\|F(x) - G(x)\|_2 \leq \frac{M|x-c|^k}{k!}$$

then

$$\|TF(x) - TG(x)\|_2 \leq \frac{LM|x-c|^{k+1}}{(k+1)!}$$

Proof
We have

$$\begin{aligned} \|TF(x) - TG(x)\|_2 &= \left\|\Gamma + \int_c^x \Phi(t, F(t))dt - \Gamma - \int_c^x \Phi(t, G(t))dt\right\|_2 \\ &\leq \left|\int_c^x \|\Phi(t, F(t)) - \Phi(t, G(t))\|_2 dt\right| \\ &\leq \left|\int_c^x LM\frac{|t-c|^k}{k!}dt\right| \\ &= \frac{LM|x-c|^{k+1}}{(k+1)!} \end{aligned}$$

Theorem 8.0.8 (Global Picard Theorem)
Let the DE

$$y^{(n)}(x) = \Phi(x, y(x), \ldots, y^{(n-1)}(x)), x \in [a, b]$$

with $c \in [a, b]$

$$y(c) = \gamma_0, \ldots, y^{(n-1)}(c) = \gamma_{n-1}$$

Assume

$$\Phi(x, \vec{y}) = (y_1, \ldots, y_n, \varphi(x, y_0, \ldots, y_{n-1}))$$

is Lipschitz in y.
Then the ODE has a unique solution on $[a, b]$. Moreover, define

$$TF(x) := \Gamma + \int_c^x \Phi(t, F(t))dt$$

If F is a solution, then

$$F(x) = \lim T^n \Gamma$$

uniformly.

Proof
Set $F_0(x) = \Gamma$ to be the constant function and define

$$F_k(x) = T^k F_0$$

Notice that

$$\begin{aligned}\|F_1(x) - F_0(x)\|_2 &= \left\| \int_c^x \Phi(t, \Gamma)dt \right\|_2 \\ &\leq |x - c| \underbrace{\sup_{t \in [a,b]} \|\Phi(t, \Gamma)\|_2}_{=M<\infty} \\ &= M \frac{|x - c|^1}{1!}\end{aligned}$$

We claim $\|F_{k-1}(x) - F_k(x)\|_2 \leq \frac{ML^{k-1}|x-c|^k}{k!}$ for $k \geq 1$.

The base case is shown above. Suppose the claim holds for k, then

$$\begin{aligned}\|F_k(x) - F_{k+1}(x)\|_2 &= \|TF_{k-1}(x) - TF_k(x)\|_2 \\ &\leq \frac{L(ML^{k-1})|x - c|^{k+1}}{(k+1)!}\end{aligned}$$

and hence

$$\begin{aligned}\|F_{k-1} - F_k\|_\infty &= \sup_{x\in[a,b]} \|F_{k-1}(x) - F_k(x)\|_2 \\ &\leq \frac{ML^k(b-a)^{k+1}}{(k+1)!}\end{aligned}$$

But then

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{ML^k(b-a)^{k+1}}{(k+1)!} &= \frac{M}{L}\sum_{k=1}^{\infty} \frac{[L(b-a)]^{k+1}}{(k+1)!} \\ &= \frac{M}{L}(e^{L(b-a)} - 1 - L(b-a)) \\ &< \infty\end{aligned}$$

This shows that for $m > n \geq N$ gives

$$\|F_m - F_n\|_\infty \leq \sum_{k=n}^{m-1} \|F_{k+1} - F_k\|_\infty \to 0$$

as $N \to \infty$.

So (F_n) is cauchy and $F_n \to F_*$ exists. But then

$$F_* = \lim F_n = \lim TF_{n-1} = TF_*$$

is a fixed point and

$$F_*'(x) = \Phi(x, F_*(x))$$

so F is a solution by the FTC.

To see uniqueness, let G be another solution. So $TG = G$ with $G(c) = \Gamma$.

We have that

$$\begin{aligned}\|F(x) - G(x)\|_2 &\leq \sup_{x\in[a,b]} \|F(x) - G(x)\|_2 \\ &\leq \|F - G\|_\infty \frac{|x-c|^0}{0!}\end{aligned}$$

Inductively, by the lemma

$$\begin{aligned}\|F(x) - G(x)\|_2 &= \|T^n F(x) - T^n G(x)\|_2 \\ &\leq \|F - G\|_\infty \frac{[L|x-c|]^n}{n!} \\ &\to 0\end{aligned}$$

so $F = G$.

Definition 8.0.3 (Locally Lipschitz)
We say Φ is locally Lipschitz in y if for all $(x, y) \in [a, b] \times \mathbb{R}^n$, there is $\epsilon > 0$ such that Φ is Lipschitz in y on
$$[a, b] \cap [x \pm \epsilon] \times \overline{b_R(y)}$$
for any $0 < R < \infty$.

Lemma 8.0.9
If $\Phi \in C^1$ in y then Φ is Lipschitz in y on
$$[a, b] \times \overline{b_R(\Gamma)}$$
for any $0 < R < \infty$

Proof
Notice that $\nabla_y \Phi$ is continuous so
$$\|\nabla_y \Phi\|_2 =: L < \infty$$
by EVT on $[a, b] \times \overline{b_R(\Gamma)}$.

Then by MVT for $y, z \in \overline{b_R(\Gamma)}$ there is $\xi \in [y, z]$ such that
$$\|\Phi(x, y) - \Phi(x, z)\|_2 = \|\nabla_y \Phi(\xi) \cdot (y - z)\|_2 \leq L\|y - z\|_2$$

Lemma 8.0.10
If Φ is locally Lipschitz in y on $[a, b] \times \mathbb{R}^n$, then Φ is Lipschitz in y on $[a, b] \times K$ for any compact $K \subseteq \mathbb{R}^n$.

Proof
WLOG, by replacing K with $\overline{\operatorname{conv} K}$, we can assume K is convex.

For each $(x, y) \in [a, b] \times K$, find $\epsilon > 0$ and $L_{x,y}$ such that
$$\|\Phi(x', y') - \Phi(x', z')\|_2 \leq L_{x,y}\|y' - z'\|_2$$
for $x' \in [x - \epsilon, x + \epsilon] \cap [a, b]$ and $y', z' \in \overline{b_\epsilon(y)}$.

Then $U_{x,y} := (x - \epsilon, x + \epsilon) \times b_\epsilon(y)$ covers $[a, b] \times K$. By compactness, there is a finite subcover $\{U_{x_i,y_i}\}$.

Let
$$L := \max\{L_{x_i,y_i}\}$$

and for $x \in [a,b], y, z \in K$ we can pick

$$\vec{y} = z_0, z_1, \dots, z_m = \vec{z}, [z_{i-1}, z_i] \subseteq U_{x_{j(i)}, y_{j(i)}}$$

(straight line).

So

$$\begin{aligned} \|\Phi(x, \vec{y}) - \Phi(x, \vec{z})\|_2 &\leq \sum_{i=1}^{m} \|\Phi(x, z_{i-1}) - \Phi(x, z_i)\|_2 \\ &\leq \sum_{i=1}^{m} L\|z_{i-1} - z_i\|_2 \\ &= L\|x - y\|_2 && \text{since all straight lines} \end{aligned}$$

Example 8.0.11
Consider $y' = y^2, y(0) = 1, x \in [0, 2]$. Then

$$\phi(x, y_0) = y_0^2$$

and $y'(x) = \phi(x, y)$. So $\frac{\partial \phi}{\partial y} = 2y$ and

$$|\phi(x, y) - \phi(x, z)| = |y^2 - z^2| = |y - z| \cdot |y + z|$$

and ϕ clearly is not globally Lipschitz but is indeed locally Lipschitz.

On $[0, 2] \times [\pm R]$, we have Lipschitz constant $2R$.

Start with $y(0) = 1$ and consider

$$\begin{aligned} \frac{y'}{y^2} &= 1 \\ \int_0^x \frac{y'(t)}{y^2(t)} dt &= \int_0^x 1 dt \\ &= x \\ &= \left. \frac{-1}{y(t)} \right|_0^x \\ &= \frac{1}{y(0)} - \frac{1}{y(x)} \\ &= 1 - \frac{1}{y(x)} \end{aligned}$$

$$y(x) = \frac{1}{1 - x}$$

which blows on $[0,2]$ as there is an asymptote at $x=1$ so this is NOT a solution.

Start with $y(2)=c$

$$\begin{aligned}\int_2^x \frac{y'}{y^2}dt &= \int_2^x 1dt \\ &= x-2 \\ &= \left.\frac{-1}{y(t)}\right|_2^x \\ &= \frac{-1}{y(x)}+\frac{1}{y(2)} \\ &= \frac{1}{c}-\frac{1}{y(x)}\end{aligned}$$

$$y(x)=\frac{c}{1+2(c-x)}$$

which still blows up at $x=\frac{1+2x}{2}$ but has nothing to do with the solution on $[0,1)$.

Theorem 8.0.12 (Local Picard Theorem)
Suppose

$$F'(x)=\phi(x,F(x))$$

for some $\phi:[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ and $F(c)=\Gamma$ where $c\in[a,b]$. In addition, assume ϕ is locally Lipschitz on y. Then, there is $h>0$ so that the ODE has a solution on $[c\pm h]\cap[a,b]$.

Proof
Let $R>0$ and remark that Φ has a Lipschitz condition on y on $K:=[a,b]\times\overline{b_R(\Gamma)}$. Let

$$M:=\|\Phi\|_K$$

we will show that

$$h:=\frac{R}{M}$$

satisfies the statement of the theorem.

Take $F_0(x)=\Gamma$ and

$$F_{n+1}:=TF_n$$

for

$$Tf(x):=\Gamma+\int_c^x \Phi(t,f(t))dt$$

The proof proceeds exactly the same way as the Global Picard Theorem and works assuming $F_n(x) \in \overline{b_R(\Gamma)}$. Indeed

$$\begin{aligned}\|F_{n+1}(x) - \Gamma\| &= \left\| \int_c^x \Phi(t, F_n(t)) dt \right\| \\ &\le \left| \int_c^x \|\Phi(t, F_n(t))\| dt \right| \\ &\le |x - c| M \\ &\le hM \\ &= \frac{R}{M} M \\ &= R\end{aligned}$$

so the assumption above holds and a solution exists.

Proposition 8.0.13
The solution given by the Local Picard Theorem is Unique.

Proof
Suppose F, G are solutions and let

$$d := \sup \left\{ t \in [c, c+h] : G\Big|_{[c,c+t]} = F\Big|_{[c,c+t]} \right\}$$

Say $F(d) = G(d) = \Delta$. Since both F, G are solutions we have

$$\begin{aligned}F'(x) = \Phi(x, F(x)), F(d) = \Delta \\ G'(x) = \Phi(x, G(x)), G(d) = \Delta\end{aligned}$$

Find $r > 0$ such that both F, G stays within

$$[d, d+k] \times \overline{b_r(\Delta)} \subseteq [c, c+h] \times \overline{b_R(\Gamma)}$$

The argument for Global Picard applies as the solution stays inside the specified compact subset.

$$\begin{aligned}\|F(x) - G(x)\| &= \|T^n F(x) - T^n G(x)\| \\ &\le \|F - G\|_{[d,d+k]} \frac{(x-d)^n}{n!} \to 0\end{aligned}$$

hence $G = F$ on $[d, d+k]$ which contradicts that d is the supremum. The proof is identical on $[c-h, c]$.

8.0.5 Corollaries of the Picard's Theorem

Theorem 8.0.14 (Continuation)
Let $F'(x) = \Phi(x, F(x)), F(c) = \Gamma, \Phi : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in y where $c \in [a,b]$. Either

(i) $F(x)$ extends to a solution on $[c,b]$ OR

(ii) $F(x)$ extends to a solution on $[c,d)$ for some $d \leq b$ and $\|F(x)\| \to \infty$ as $x \to d$ (similarly going left towards a).

Proof
Let

$$d := \sup\{t \geq c : \text{DE has a solution on } [c,t]\}$$

and remark that $d > c$ by the Local Picard Theorem.

If F_1, F_2 are solutions on $[c, c+t_1], [c, c+t_2]$ with $t_1 \leq t_2$, then we claim

$$F_2\Big|_{[c,c+t_1]} = F_1$$

Indeed, let

$$d_1 := \left\{t : F_2\Big|_{[c,c+t]} = F_1\Big|_{[c,c+t]}\right\}$$

and

$$F_1(d_1) = F_2(d_1) =: \Delta$$

Both F_1, F_2 are solutions of $F'(x) = \Phi(x, F(x)), F(d_1) = \Delta$. By the Local Picard Theorem, there is $h > 0$ such that

$$F_1\Big|_{[d_1,d_1+h]} = F_2\Big|_{[d_1,d_1+h]}$$

which contradicts that d_1 is the supremum.

It follows that $d_1 = t_1$ as desired.

Let F^* be the unique solution on $[c,d)$ obtained by piecing together solutions on $[c,t]$ for $t < d$. If $d = b$ and F is continuous then we are in case (1) as F^* is a solution on $[c,d]$.

Else, if $\|F(x)\| \to \infty$ as $x \to d$ then we are in case (2).

It remains to consider the scenario where F^* fails to continue to d, yet

$$\exists x_n \to d, \|F^*(x_n)\| \leq K$$

But Φ is locally Lipschitz in y and thus is Lipschitz in y on

$$[c,d] \times \overline{b_{K+1}(0)} \supseteq [x_n, b] \times \overline{b_1(F^*(x_n))}$$

But the Local Picard Theorem provides a solution on $[x_n, x_n + h]$ where

$$h := \min\left\{b - x_n, \frac{1}{\|\Phi\|_\infty}\right\} = \min\left\{b - x_n, \frac{1}{M}\right\}$$

and

$$\|\Phi\|_\infty := \sup_{x \in [c,d] \times \overline{b_{k+1}(0)}} \|\Phi(x,y)\| = M < 0$$

We can choose x_n such that

$$d - x_n < \frac{1}{2M}$$

by convergence so the solution extends to

$$\left[x_n, x_n + \frac{1}{M}\right] \cap [c,b] \supseteq \left[x_n, d + \frac{1}{2M}\right] \cap [c,b]$$

and contradicts the definition of d.

If $d = b$ and F is bounded on $x_n \to b$, then the solution extends to be continuous at b. All in all, we have shown either (1) or (2) hold.

Corollary 8.0.14.1
If $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, and $F'(x) = \Phi(x, F(x))$ with $F(c) = \Gamma$ for some $c \in [a,b]$. Then this ODE has a solution ...

(i) on all $\mathbb{R}$

(ii) on $(-\infty, d)$ and $\|F(x)\| \to \infty$ as $x \to d^-$

(iii) on (d, ∞) and $\|F(x)\| \to \infty$ as $x \to d^+$

(iv) on (d_1, d_2) with $\|F(x)\| \to \infty$ as $x \to d_i^{(-1)^{1-i}}$

Example 8.0.15
A counterexample is

$$x^4 y^{(2)} + 2x^3 y' + y = 0$$

with $y\left(\frac{2}{\pi}\right) = 1, y'\left(\frac{2}{\pi}\right) = 0$.

In standard form $F(x) = (f_0(x), f_1(x))$ we get

$$y^{(2)} = \frac{-2x^3 y' - y}{x^4}$$

Furthermore

$$F'(x) = \Phi(x, y_0, y_1) = \left(y_1, \frac{-2x^3y_1 - y_0}{x^4}\right)$$

on (ϵ, ∞) as it is a linear ODE with continuous coefficients.

So Φ is globally Lipschitz on $[\epsilon, R]$ and there is a solution on $(0, \infty)$ by the Continuation Theorem. It can be computed that $f(x) = \sin\left(\frac{1}{x}\right)$.

However, the soution does NOT extend to 0 as there is no local Lipschitz condition on $\left(0, \frac{1}{\pi}\right)$. This is due to the fact that

$$f\left(\frac{1}{(2n+1)\frac{2}{\pi}}\right) \in (\pm 1, 0)$$

so $n \to \infty$ gives

$$\|f\| = \|(\pm 1, 0)\| = 1$$

8.0.6 Existence of Solutions without Lipschitz Condition

Example 8.0.16
$y' = y^{\frac{2}{3}}, y(0) = 0$ Write

$$\frac{y'}{y^{\frac{2}{3}}} = 1$$

and integrate with respect to x

$$\begin{aligned}\int_0^x \frac{y'(t)}{y(t)^{\frac{2}{3}}} dt = \int_0^x 1 dt = x \\ = 3y(t)^{\frac{1}{3}}\Big|_0^x \\ = 3y(x)^{\frac{1}{3}} - 0 = x\end{aligned}$$

gives

$$y(x) = \frac{x^3}{27}$$

But $y = 0$ is also a solution!

In fact

$$f(x) = \begin{cases} \frac{(x-a)^3}{27}, & x \geq a \\ 0, & b \leq x \leq a \\ \frac{(x-b)^3}{27}, & x \leq b \end{cases}$$

is a solution.

$$y' = \varphi(x,y) = y^{\frac{2}{3}}, \frac{\partial \varphi}{\partial y} = \frac{2}{3} y^{-\frac{1}{3}}$$

is discontinuous at $y = 0$.

There is a local Lipschitz condition on

$$\mathbb{R} \times ([\epsilon, \infty) \cup (-\infty, -\epsilon])$$

8.0.7 Peano's Theorem

Theorem 8.0.17 (Peano)
$\Phi : [a,b] \times \overline{b_R(\Gamma)} \to \mathbb{R}^n$ continuous.
The DE given by

$$F'(x) = \Phi(x, F(x)), F(a) = \Gamma$$

has at least 1 solution on $[a, a+h]$ where

$$h := \min \left(b - a, \frac{R}{\|\Phi\|_\infty} \right)$$

Lemma 8.0.18 (Arzela-Ascoli)
Recall that if (X, d) is compact, then $K \subseteq C(X)$ is compact if and only if K is closed, bounded, and equicontinuous.

Proof (theorem)
We are looking for a fixed point of $T : C[a,b] \to C[a,b]$ given by

$$Tf(x) := \Gamma + \int_a^x \Phi(t, F(t))dt$$

Define $F_n(x)$ on $[a, a+h]$ by

$$F_n(x) := \begin{cases} \Gamma, & a \leq x \leq a + \frac{1}{n} \\ \Gamma + \int_a^{x-\frac{1}{n}} \Phi(t, F_n(t))dt, & a + \frac{1}{n} \leq x \leq a + h \end{cases}$$

Remark that this makes sense as F_n is already defined on

$$\left[a, a + \frac{1}{n}\right], \left[a, a + \frac{2}{n}\right], \ldots, \left[a, x - \frac{1}{n}\right]$$

so for $x \in \left[a + \frac{i-1}{n}, a + \frac{i}{n}\right]$, the function is well-defined as it depends on F_n on $\left[a, \frac{i-1}{n}\right]$.

We have

$$\|F_n(x) - \Gamma\| \begin{cases} = 0, & a \le x \le a + \frac{1}{n} \\ \le \int_a^{x-\frac{1}{n}} \Phi(t, F_n(t))dt \le \|\Phi\|_\infty \cdot h \le R, & a + \frac{1}{n} \le x \le a + h \end{cases}$$

which is needed for F_n to be well-defined.

In addition

$$\begin{aligned} \|TF_n(x) - F_n(x)\| &= \begin{cases} \left\|\Gamma + \int_a^x \Phi(t, F_n(t))dt - \Gamma\right\|, & a \le x \le a + \frac{1}{n} \\ \left\|\Gamma + \int_a^x \Phi(t, F_n(t))dt - \Gamma - \int_a^{x-\frac{1}{n}} \Phi(t, F_n(t))dt\right\|, & a + \frac{1}{n} \le x \le a + h \end{cases} \\ &\le \begin{cases} \int_a^x \|\Phi(t, F_n(t))\|dt \\ \int_{x-\frac{1}{n}}^x \|\Phi(t, F_n(t))\|dt \end{cases} \\ &\le \frac{1}{n}\|\Phi\|_\infty \end{aligned}$$

Consider the set $\{F_n : n \ge 1\}$

<u>Closed</u>:

It is clearly bounded as

$$\|F_n\|_\infty = \sup \underbrace{F_n(x)}_{\in \overline{b_R(\Gamma)}} \le \|\Gamma\| + R$$

<u>Equicontinuous</u>:

For $a \le x_1 \le x_2 \le a + h$ we have

$$\begin{aligned} F_n(x_2) - F_n(x_1) &= \Gamma - \Gamma + \int_{x_1-\frac{1}{n}}^{x_2-\frac{1}{n}} \Phi(t, F_n(t))dt \\ \|F_n(x_2) - F_n(x_1)\| &\le \int_{x_1-\frac{1}{n}}^{x_2-\frac{1}{n}} \|\Phi(t, F_n(t))\|dt \\ &\le \|\Phi\|_\infty |x_2 - x_1| \end{aligned}$$

For $\epsilon > 0$ simply take

$$\delta := \frac{\epsilon}{\|\Phi\|_\infty}$$

for $\{F_n\}$ to exhibit equicontinuity.

This shows that $\mathcal{F} := \overline{\{F_n\}}$ is compact by the Arzela-Ascoli Theorem.

It follows that there is some subsequence indexed by $\{n_i\}$ such that

$$F_{n_i} \to F$$

uniformly.

Clearly

$$\|F - \Gamma\|_\infty \leq R$$

since $\|F_n - \Gamma\|_\infty \leq R$.

In addition

$$TF - F = TF - TF_{n_i} + \underbrace{TF_{n_i} - F_{n_i}}_{\leq_{\|\cdot\|} \frac{\|\Phi\|_\infty}{n_i}} + \overbrace{F_{n_i} - F}^{\to 0}$$

$$\begin{aligned}
\|TF - F\|_\infty &\leq \frac{\|\Phi\|_\infty}{n_i} + \|F_i - F\|_\infty + \left\|\Gamma + \int_a^x \Phi(t, F(t))dt - \Gamma - \int_a^x \Phi(t, F_{n_i}(t))dt\right\| \\
&\leq \frac{\|\Phi\|_\infty}{n_i} + \|F_i - F\|_\infty + \int_a^x \|\Phi(t, F(t))dt - \Phi(t, F_{n_i}(t))\|dt
\end{aligned}$$

But Φ is continuous and thus uniformly continuous on $[a, b] \times \overline{b_R(\Gamma)}$.

For $\epsilon > 0$ take $\delta > 0$ such that $\|y_1 - y_2\| < \delta$ gives

$$\|\Phi(t, y_1) - \Phi(t, y_2)\| < \epsilon$$

for all $t \in [a, b]$.

Choose n_i sufficiently large so that

$$\|F_{n_i} - F\|_\infty < \min(\delta, \epsilon)$$
$$\frac{\|\Phi\|_\infty}{n_i} < \epsilon$$

We have

$$\begin{aligned}
\int_a^x \|\Phi(t, F(t)) - \Phi(t, F_{n_i}(t))\|dt &\leq \int_a^x \epsilon dt \\
&\leq \epsilon h
\end{aligned}$$

$$\begin{aligned}
\|TF - F\| &\leq \frac{\|\Phi\|_\infty}{n_i} + \|F_{n_i} - F\|_\infty + \epsilon h \\
&< \epsilon + \epsilon + \epsilon h \\
&= (2 + h)\epsilon
\end{aligned}$$

and hence F is indeed a solution.

8.0.8 Stability of Solutions

We close this section by presenting a result that if two DEs have close initial values or functions, then their solutions are also close.

Theorem 8.0.19 (Perturbation)
Let $\Phi : [a,b] \times B[\Gamma, \mathbb{R}] \to \mathbb{R}^n$ be continuous with Lipschitz condition on the $\vec{y}$ variable and Lipschitz constant L.
Suppose Ψ is another continuous function (not necessarily Lipschitz) on $[a,b] \times B[\Gamma, R]$ such that

$$\|\Phi - \Psi\|_\infty \leq \epsilon$$

for some $\epsilon > 0$.
Suppose F, G are solutions on $[a, a+h]$ by the Picard Theorems

$$F'(x) = \Phi(x, F(x)), F(a) = \Gamma$$
$$G'(x) = \Psi(x, G(x)), G(a) = \Delta$$

such that

$$(x, F(x)), (x, G(x)) \in [a,b] \times B[\Gamma, R]$$

for each $x \in [a,b]$ and $\|\Delta - \Gamma\|_2 \leq \delta$.
Then, for all $x \in (a,b)$

$$\|G(x) - F(x)\|_2 \leq \delta e^{2|x-a|} + \frac{\epsilon}{L}(e^{2|x-a|} - 1)$$

so

$$\|G - F\|_\infty \leq \delta e^{2h} + \frac{\epsilon}{L}(e^{2h} - 1)$$

Proof
Define $\tau : [a,b] \to \mathbb{R}$

$$x \mapsto \|F(x) - G(x)\|_2$$

Then $\tau^2(x) = \sum_{i=0}^{n-1} |f_i(x) - g_i(x)|^2$ and we have by the Chain Rule

$$\begin{aligned}
(\tau^2(x))' &= 2\tau(x)\tau'(x) \\
&= \left(\sum_{i=0}^{n-1} |f_i(x) - g_i(x)|^2\right)' \\
&= \sum_{i=0}^{n-1} [f_i(x) - g_i(x)][f_i'(x) - g_i'(x)] \\
&\leq 2\left(\sum [f_i(x) - g_i(x)]^2\right)^{\frac{1}{2}} \left(\sum [f_i'(x) - g_i'(x)]^2\right)^{\frac{1}{2}} \\
&= 2\|F(x) - F(x)\|_2 \|F'(x) - G'(x)\|_2 \\
&= 2\tau(x)\|F'(x) - G'(x)\|_2
\end{aligned}$$

If $\tau(x) = 0$, then there is nothing to prove so assume $\tau(x) \neq 0$ and the computation above yields

$$\tau'(x) \leq \|F'(x) - G'(x)\|_2$$

Consider $\|F'(x) - G'(x)\|_2$. Make use of the DE to get

$$\begin{aligned}
\|F'(x) - G'(x)\|_2 &= \|\Phi(x, F(x)) - \Psi(x, G(x))\|_2 \\
&= \|\Phi(x, F(x)) - \Phi(x, G(x))\|_2 + \|\Phi(x, G(x)) - \Psi(x, G(x))\|_2 \\
&\leq L\|F(x) - G(x)\|_2 + \epsilon \\
&= L\tau(x) + \epsilon
\end{aligned}$$

So

$$\tau'(x) \leq L\tau(x) + \epsilon \implies 1 \geq \frac{\tau'(x)}{L\tau(x) + \epsilon}$$

Notice that we assume $\tau(x) \neq 0$. To make this happen in the final analysis, we fix $x \in [a, b]$ and define

$$d := \sup\{t \in [a, x] : \tau(t) = 0\}$$

so $\tau(y) \neq 0$ for all $y \in (d, x]$.

Now, $a \leq d \leq x$ so

$$\begin{aligned} x - a &\geq x - d \\ &= \int_d^x 1 \cdot dt \\ &\geq \int_d^x \frac{\tau'(1)}{L\tau(t) + \epsilon} dt \\ &= \frac{1}{L} \ln[L\tau(t) + \epsilon] \Big|_{t=d}^{x} \\ &= \frac{1}{L} \ln \left[\frac{L\tau(x) + \epsilon}{L\tau(d) + \epsilon} \right] \end{aligned}$$

We either have $d = a$ or $\tau(d) = 0$. If $d = a$, then

$$\begin{aligned} \tau(a) &= \tau(d) \\ &= \|F(a) - G(a)\|_2 \\ &= \|\Gamma - \Delta\|_2 \\ &< \delta \end{aligned}$$

In both cases

$$L(x - a) \geq \ln \frac{L\tau(x) + \epsilon}{L\tau(d) + \epsilon}$$

$$\begin{aligned} e^{L(x-a)} &\geq \frac{L\tau(x) + \epsilon}{L\tau(d) + \epsilon} \\ &\geq \frac{L\tau(x) + \epsilon}{L\delta + \epsilon} \end{aligned}$$

$$\begin{aligned} \tau(x) &\leq \frac{(2\delta + \epsilon)e^{L(x-a)} - \epsilon}{L} \\ &= \delta e^{L(x-a)} + \frac{\epsilon}{L}(e^{L(x-a)} - 1) \end{aligned}$$

$$\|F(x) - G(x)\|_2 = \delta e^{L(x-a)} + \frac{\epsilon}{L}(e^{L(x-a)} - 1)$$

which yields

$$\|F - G\|_\infty \leq \delta e^{Lh} + \frac{\epsilon}{L}(e^{Lh} - 1)$$

as required.

Corollary 8.0.19.1 (Continuous Dependence on Parameters)
Suppose $\Phi : [a, b] \times B[R, \Gamma] \to \mathbb{R}^n$ is a locally Lipschitz function. Then, the solutions F_Δ to the DE

$$F'(x) = \Phi(x, F(x)), F(c) = \Delta, c \in [a, b]$$

are a continuous function of Δ.

Proof
We wish to show that for $\Delta \in B[\Gamma, R]$, if $\Delta - \Gamma$ is small, then $\|F_\Delta - F_\Gamma\|$ is also small.

If $\Delta \in B[\Gamma, R]$ with

$$\|\Delta - \Gamma\|_2 < \delta$$

for some $\delta > 0$, then by the Local Picard Theorem, the solution to

$$F'(x) = \Phi(x, F(x)), F(c) = \Delta$$

exists on $[c - h, c + h] \cap [a, b]$ where

$$h := \frac{R - \delta}{\|\Phi\|_\infty}$$

given that $\Delta \in B[\Gamma, R - \delta]$.

Apply the Perturbation Theorem and take $\Psi = \Phi$ and consequently

$$\epsilon = \|\Phi - \Psi\|_\infty = 0$$

This shows that for all $x \in [c - h, c + h] \cap [a, b]$ we have

$$\begin{aligned} \|F_\Gamma(x) - F_\Delta(x)\|_2 &\leq \|\Delta - \Gamma\|_2 e^{L|c-x|} \\ &\leq \|\Delta - \Gamma\|_2 e^{Lh} \end{aligned}$$

where $\operatorname{Lip} \Phi = L$.

This shows that $\Delta \mapsto F_\Delta$ is Lipschitz with Lipschitz constant e^{Lh} as desired.