# Introduction to Smooth Manifolds

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# Contents

**1 Euclidean Spaces** **11**

1.1 Smooth Functions on a Euclidean Space . . . . . . . . . . . . . . . . . . . . 11

1.1.1 Smooth Functions . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11

1.1.2 Taylor's Theorem with Remainder . . . . . . . . . . . . . . . . . . . 12

1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations . . . . . . . . . . . . . . . . . . . 13

1.2.1 The Directional Derivative . . . . . . . . . . . . . . . . . . . . . . . 14

1.2.2 Germs of Functions . . . . . . . . . . . . . . . . . . . . . . . . . . . 14

1.2.3 Derivations at a Point . . . . . . . . . . . . . . . . . . . . . . . . . 15

1.2.4 Vector Fields . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17

1.2.5 Vector Fields as Derivations . . . . . . . . . . . . . . . . . . . . . . 19

1.3 The Exterior Algebra of Multicovectors . . . . . . . . . . . . . . . . . . . . 20

1.3.1 Dual Space . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20

1.3.2 Permutations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

1.3.3 Multilinear Functions . . . . . . . . . . . . . . . . . . . . . . . . . . 21

1.3.4 The Permutation Action on Multilinear Functions . . . . . . . . . . . 22

1.3.5 The Symmetrizing and Alternating Operators . . . . . . . . . . . . . 23

1.3.6 The Tensor Product . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

1.3.7 The Wedge Product . . . . . . . . . . . . . . . . . . . . . . . . . . . 25

1.3.8 Anticommutativity of the Wedge Product . . . . . . . . . . . . . . . . 26

1.3.9 Associativity of the Wedge Product . . . . . . . . . . . . . . . . . . . 27

1.3.10 A Basis for $k$-Covectors . . . 30
1.3.11 Useful Results . . . 32
1.4 Differential Forms on $\mathbb{R}^n$ . . . 33
1.4.1 Differential 1-Forms and the Differential of a Function . . . 33
1.4.2 Differential $k$-Forms . . . 35
1.4.3 Differential Forms as Multilinear Functions on Vector Fields . . . 37
1.4.4 The Exterior Derivative . . . 38
1.4.5 Closed & Exact Forms . . . 40
1.4.6 Applications to Vector Calculus . . . 41
1.4.7 Convention on Subscripts and Superscripts . . . 44
1.4.8 Miscellaneous Results . . . 44

**2 Manifolds** **47**
2.1 Manifolds . . . 47
2.1.1 Topological Manifolds . . . 47
2.1.2 Compatible Charts . . . 49
2.1.3 Smooth Manifolds . . . 50
2.1.4 Examples of Smooth Manifolds . . . 51
2.2 Smooth Maps on a Manifold . . . 53
2.2.1 Smooth Functions on a Manifold . . . 54
2.2.2 Smooth Maps between Manifolds . . . 55
2.2.3 Diffeomorphisms . . . 56
2.2.4 Smoothness in Terms of Components . . . 57
2.2.5 Examples of Smooth Maps . . . 58
2.2.6 Partial Derivatives . . . 61
2.2.7 Inverse Function Theorem . . . 62
2.3 Quotients . . . 64
2.3.1 The Quotient Topology . . . 64

2.3.2 Continuity of a Map on a Quotient . . . 64
2.3.3 Identification of Subset to a Point . . . 64
2.3.4 A Necessary Condition for a Hausdorff Quotient . . . 65
2.3.5 Open Equivalence Relations . . . 65
2.3.6 The Real Projective Space . . . 67
2.3.7 The Standard Smooth Atlas on a Real Projective Space . . . 69
2.3.8 The Grassmannian Manifold . . . 70

**3 The Tangent Space** **75**

3.1 The Tangent Space . . . 75
3.1.1 The Tangent Space at a Point . . . 75
3.1.2 The Differential of a Map . . . 76
3.1.3 The Chain Rule . . . 78
3.1.4 Bases of Tangent Space at a Point . . . 79
3.1.5 A Local Expression for the Differential . . . 80
3.1.6 Curves in a Manifold . . . 81
3.1.7 Computing the Differential Using Curves . . . 84
3.1.8 Immersions and Submersions . . . 85
3.1.9 Rank, Critical & Regular Points . . . 86
3.1.10 Useful Results . . . 87
3.2 Submanifolds . . . 90
3.2.1 Submanifolds . . . 90
3.2.2 Level Sets of a Function . . . 92
3.2.3 The Regular Level Set Theorem . . . 94
3.2.4 Examples of Regular Manifolds . . . 95
3.2.5 Transversality . . . 96
3.2.6 Useful Results . . . 99
3.3 Categories and Functors . . . 102

3.3.1 Categories . . . 103
3.3.2 Functors . . . 104
3.3.3 The Dual and Multicovector Functors . . . 105
3.4 The Rank of a Smooth Map . . . 106
3.4.1 Constant Rank Theorem . . . 107
3.4.2 The Immersion & Submersion Theorems . . . 110
3.4.3 Images of Smooth Maps . . . 112
3.4.4 Smooth Maps into Submanifold . . . 114
3.4.5 The Tangent Plane to a Surface in $\mathbb{R}^3$ . . . 115
3.4.6 The Differential of an Inclusion Map . . . 117
3.4.7 Useful Results . . . 118
3.5 The Tangent Bundle . . . 119
3.5.1 The Topology of the Tangent Bundle . . . 119
3.5.2 The Manifold Structure on the Tangent Bundle . . . 122
3.5.3 Vector Bundles . . . 123
3.5.4 Smooth Sections . . . 125
3.5.5 Smooth Frames . . . 127
3.6 Bump Functions and Partitions of Unity . . . 129
3.6.1 Smooth Bump Functions . . . 130
3.6.2 Partitions of Unity . . . 132
3.6.3 Existence of a Partition of Unity . . . 132
3.7 Vector Fields . . . 134
3.7.1 Smoothness of Vector Field . . . 135
3.7.2 Integral Curves . . . 137
3.7.3 Local Flows . . . 138
3.7.4 The Lie Bracket . . . 142
3.7.5 The Pushforward of Vector Fields . . . 144
3.7.6 Related Vector Fields . . . 144

**4 Lie Groups and Lie Algebras** **147**

4.1 Lie Groups . . . 147

4.1.1 Lie Groups & Examples . . . 147

4.1.2 Lie Subgroups . . . 150

4.1.3 The Matrix Exponential . . . 151

4.1.4 The Trace of a Matrix . . . 153

4.1.5 The Differential of Det at the Identity . . . 154

4.2 Lie Algebras . . . 155

4.2.1 Tangent Space at the Identity of a Lie Group . . . 155

4.2.2 Left-Invariant Vector Fields on a Lie Group . . . 157

4.2.3 The Lie Algebra of a Lie Group . . . 159

4.2.4 The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$ . . . 160

4.2.5 The Pushforward of Left-Invariant Vector Fields . . . 162

4.2.6 The Differential as a Lie Algebra Homomorphism . . . 163

**5 Differential Forms** **165**

5.1 Differential 1-Forms . . . 165

5.1.1 The Differential of a Function . . . 165

5.1.2 Local Expression for a Differential 1-Form . . . 166

5.1.3 The Cotangent Bundle . . . 167

5.1.4 Characterization of Smooth 1-Forms . . . 168

5.1.5 Pullback of 1-Forms . . . 171

5.1.6 Restriction of 1-Forms to Immersed Submanifolds . . . 173

5.2 Differential $k$-Forms . . . 175

5.2.1 Differential Forms . . . 175

5.2.2 Local Expression for a $k$-Form . . . 177

5.2.3 The Bundle Point of View . . . 178

5.2.4 Smooth $k$-Forms . . . 179

5.2.5 Pullback of $k$-Forms . . . 180
5.2.6 The Wedge Product . . . 181
5.2.7 Differential Forms on a Circle . . . 184
5.2.8 Invariant Forms on a Lie Group . . . 184
5.3 The Exterior Derivative . . . 186
5.3.1 Exterior Derivative on a Coordinate Chart . . . 187
5.3.2 Local Operators . . . 187
5.3.3 Existence of an Exterior Derivative on a Manifold . . . 189
5.3.4 Uniqueness of the Exterior Derivative . . . 189
5.3.5 Exterior Differentiation Under a Pullback . . . 191
5.3.6 Restriction of $k$-Forms to a Submanifold . . . 193
5.3.7 A Nowhere-Vanishing 1-Form on the Circle . . . 194
5.4 The Lie Derivative & Interior Multiplication . . . 195
5.4.1 Families of Vector Fields and Differential Forms . . . 195
5.4.2 The Lie Derivative of a Vector Field . . . 198
5.4.3 The Lie Derivative of a Differential Form . . . 201
5.4.4 Interior Multiplication . . . 202
5.4.5 Properties of the Lie Derivative . . . 205
5.4.6 Global Formulas for the Lie and Exterior Derivatives . . . 208

**6 Integration** **211**
6.1 Orientations . . . 211
6.1.1 Orientations of a Vector Space . . . 211
6.1.2 Orientations & $n$-Covectors . . . 212
6.1.3 Orientations on a Manifold . . . 213
6.1.4 Orientations & Differential Forms . . . 215
6.1.5 Orientations & Atlases . . . 218
6.2 Manifolds with Boundary . . . 219

6.2.1 Smooth Invariance of Domain in $\mathbb{R}^n$ . . . . . 220
6.2.2 Manifolds with Boundary . . . . . 222
6.2.3 The Boundary of a Manifold with Boundary . . . . . 224
6.2.4 Tangent Vectors, Differential Forms, and Orientations . . . . . 224
6.2.5 Outward-Pointing Vector Fields . . . . . 225
6.2.6 Boundary Orientation . . . . . 226
6.3 Integration on Manifolds . . . . . 227
6.3.1 The Riemann Integral of a Function on $\mathbb{R}^n$ . . . . . 228
6.3.2 Integrability Conditions . . . . . 230
6.3.3 The Integral of an $n$-Form on $\mathbb{R}^n$ . . . . . 231
6.3.4 Integral of a Differential Form over a Manifold . . . . . 232
6.3.5 Integration over a Zero-Dimensional Manifold . . . . . 237
6.3.6 Stokes' Theorem . . . . . 237
6.3.7 Line Integrals & Green's Theorem . . . . . 239

# Chapter 1

# Euclidean Spaces

Our goal in this section is to develop calculus in $\mathbb{R}^n$ that is independent of coordinates. This allows us to transition to the setting of manifolds where there is no global coordinate system.

## 1.1 Smooth Functions on a Euclidean Space

### 1.1.1 Smooth Functions

In keeping with the conventions of differentiaal geometry, the indices on coordinates are superscripts and not subscripts.

**Definition 1.1.1 ($C^k$)**
Let $k \geq 0$. A function $f : U \to \mathbb{R}$ is said to be $C^k$ at $p \in U$ if its parptial derivatives or all orders $j \leq k$ exist and are continuous at $p$.
A vector-valued function $f : U \to \mathbb{R}^m$ is $C^k$ at $p \in U$ if all its component functions are $C^k$ at $p$.
We say $f : U \to \mathbb{R}$ is $C^k$ on $U$ if it is $C^k$ at every point $p \in U$.

We treat the terms $C^\infty$ and *smooth* as synonymous.

**Definition 1.1.2 (Real-Analytic)**
We say the function $f$ is *real-analytic* at a point $p$ if in some neighborhood of $p$ it is equal to its Taylor series at $p$:

$$f(x) = f(p) + \sum_{k\geq 1} \frac{1}{k!} \sum_{i_1,\dots,i_k} \frac{\partial^k}{\partial x^{i_1} \dots \partial x^{i_k}}(p)(x^{i_1} - p^{i_1}) \dots (x^{i_k} - p^{i_k}).$$

Recall that a convergent power series can be differentiated term by term in its domain of convergence. Hence a real-analytic function is necessarily $C^\infty$. However, the converse need not hold.

**Example 1.1.1**
The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$x \mapsto \begin{cases} \exp(-1/x) & x > 0 \\ 0, & x = 0 \end{cases}$$

is $C^\infty$ on $\mathbb{R}$ but $f^{(k)}(0) = 0$ for all $k$. Hence it cannot be real-analytic about the point $x = 0$.

### 1.1.2 Taylor's Theorem with Remainder

Although a $C^\infty$ function need not be equal to its Taylor series, there is a version of Taylor's theorem for $C^\infty$ functions that is often good enough.

We say $S \subseteq \mathbb{R}^n$ is *star-shaped* with respect to a point $p \in S$ if for every $x \in S$, the line segment $[p, x]$ lies in $S$.

**Lemma 1.1.2**
Let $f$ be $C^\infty$ on an open subset $U \subseteq \mathbb{R}^n$ that is star-shaped with respect to some $p \in U$. There are functions $g_1, \dots, g_n \in C^\infty(U)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n} (x^i - p^i) g_i(x)$$
$$g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

**Proof**
Since $U$ is star-shaped with respect to $p$, for any $x \in U$ the line segment $y_t := p + t(x - p), t \in [0, 1]$ lies in $U$. Thus $f(y_t)$ is well-defined for all $t \in [0, 1]$.

By the chain rule,
$$\frac{d}{dt}f(y_t) = \sum_i (x^i - p^i)\frac{\partial f}{\partial x^i}(y_t).$$
Integrating both sides with respect to $t$ from 0 to 1,
$$f(x) - f(p) = \sum_i (x^i - p^i)\int_0^t \frac{\partial f}{\partial x^i}(y_t)dt.$$
Taking
$$g_i(x) := \int_0^t \frac{\partial f}{\partial x^i}(y_t)dt$$
suffices. Indeed, $g_i \in C^\infty$ since $f \in C^\infty$ and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$.

Let $n = 1, p = 0$. The lemma above states that
$$f(x) = f(0) + xg_1(x)$$
for some $g_1 \in C^\infty$. Applying the lemma repeatedly gives
$$\begin{aligned} f(x) &= f(0) + x(g_1(0) + xg_2(x)) \\ &= \dots \\ &= f(0) + g_1(0)x + g_2(0)x^2 + \cdots + g_k(0)x^k. \end{aligned}$$
Differentiating the expression above and evaluationg at 0 yields
$$g_k(0) = \frac{1}{k!}f^{(k)}(0).$$
Thus the above is a polynomial expansion of $f(x)$ whose terms up to the last term agree with the Taylor series of $f$ at 0.

## 1.2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

A *secant plane* to a surface in $\mathbb{R}^n$ is a plane determined by $n$ ppoints of the surface. As the points approach a point $p$ on the surface, if the corresponding secant planes approach a limiting position, then the plane that is the limiting position of the secant planes is called the *tangent plane* to the surface at $p$. A vector $p$ is tangent to a surface if it lies in the tangent plane at $p$.

The notions above presupposes that the surface is embedded in an Euclidean space, and so does not permit to surfaces such as the projective plane. Our goal is to find a characterization of tangent vectors that generalize to manifolds.

### 1.2.1 The Directional Derivative

In order to distinguish between points and vectors (directions), we write a point $p \in \mathbb{R}^n$ as $p = (p^1, \dots, p^n)$ and a vector in the tangent space $T_p(\mathbb{R}^n)$ as $v = \langle v^1, \dots, v^n \rangle$.

The line through a point $p$ with direction $v$ in $\mathbb{R}^n$ has parametrization

$$c(t) = (p^i + tv^i)_i.$$

If $f \in C^\infty$ in a neighborhood of $p$ and $v \in T_p(\mathbb{R}^n)$, recall the *directional derivative* of $f$ in the direction $v$ at $p$ is defined as

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left. \frac{d}{dt} \right|_{t=0} f(c(t)).$$

By the chain rule,

$$D_v f = \sum_i \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum_i v^i \frac{\partial f}{\partial x^i}(p).$$

The notation $D_v f$ is understood to be an evaluation at $p$ and is thus a number and not a function. We explicitly write

$$D_v = \sum_i v^i \left. \frac{\partial}{\partial v^i} \right|_p = \left\langle v, \left. \frac{\partial}{\partial x^i} \right|_p \right\rangle.$$

for the map that sends a function $f$ to the number $D_v f$. We often omit the subscript $p$ for simplicity of the meaning is clear from context. The association $v \mapsto D_v$ offers a way to characterize tangent vectors as certain operators on functions. We study this in greater detail in the next subsections.

### 1.2.2 Germs of Functions

As long as two functions agree on some neighborhood of a point $p$, they will have the same directional derivatives at $p$. This suggests that we introduce an equivalence relation on the $C^\infty$ functions defined in some neighborhood of $p$.

**Definition 1.2.1 (Germ)**
Consider the set of all pairs $(f, U)$ where $U \ni p$ is a neighborhood of $p$ and $f : U \to \mathbb{R}$ is a $C^\infty$ function. We say that $(f, U)$ is equivalent to $(g, V)$ if there is an open set $W \subseteq U \cap V$ containing $p$ such that $f = g$ when restricted to $W$.
The equivalence class of $(f, U)$ is called the *germ* of $f$ at $p$.

We write $C_p^\infty(\mathbb{R}^n)$ or simply $C_p^\infty$ if there is no possibility of confusion for the set of all germs of $C^\infty$ functions on $\mathbb{R}^n$ at $p$.

**Example 1.2.1**
The functions $f(x) = {}^1\!/_{1-x}$ with domain $\mathbb{R} \setminus \{1\}$ and $g(x) = \sum_{k \geq 0} x^k$ with domain $(-1, 1)$ have the same germ at any point $p$ in the open interval $(-1, 1)$.

Recall the following definition.

**Definition 1.2.2 (Algebra)**
An *algebra* over a field $K$ is a vector space $A$ over $K$ equiped with a multiplication map

$$\mu : A \times A \to A$$

usually written $\mu(a, b) = a \cdot b$ such that for all $a, b, c \in A$ and $r \in K$,

(i) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ [associativity]

(ii) $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$ [distributivity]

(iii) $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$ [homogeneity]

Equivalently, an algebra over a field $\mathbb{K}$ is a ring $A$ (with or without multiplicative identity) that is also a vector space over $\mathbb{K}$ such that the ring multiplication satisfies the homogeneity condition (iii). So an algebra has three operations: ring addition and multiplication as well as the scalar multiplication of a vector space.

Recall a map $L : V \to W$ between vector spaces over a field $\mathbb{K}$ is called a $\mathbb{K}$-*linear map/operator* if for any $r \in \mathbb{K}$ and $u, v \in V$,

(i) $L(u + v) = L(u) + L(v)$

(ii) $L(rv) = rL(v)$

In the case that $V, W$ are algebras over a field $\mathbb{K}$ and $L$ satisfies the additional property for all $u, v \in V$, we say that $L$ is an *algebra homomorphism.*

(iii) $L(uv) = L(u)L(v)$

The addition and multiplication of functions induce corresponding operations on $C_p^\infty$, making it an algebra over $\mathbb{R}$. Note that $C_p^\infty$ is an equivalence class over functions, so the previous statement is not trivial.

### 1.2.3 Derivations at a Point

For each tangent vector $v$ at a point $p \in \mathbb{R}^n$, the directional derivative at $p$ gives a map of real vector spaces

$$D_v : C_p^\infty \to \mathbb{R}.$$

By the chain rule, $D_v$ is $\mathbb{R}$-linear and satisfies the Leibneiz rule

$$D_v(fg) = (D_v f)g(p) + f(p)D_v g$$

since the partial derivatives have these properties.

**Definition 1.2.3 (Point-Derivation)**
A linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule is said to be a *derivation at $p$* or a *point-derivation* of $C_p^\infty$.

Denote the set of all derivations at $p$ by $\mathfrak{D}_p(\mathbb{R}^n)$. This is a real vector space since the sum of two derivations at $p$ and a scalar multiple of a derivation at $p$ are again derivations at $p$.

We know that directional derivatives at $p$ are all derivations at $p$, so the map $\phi : T_p(\mathbb{R}^n) \to \mathfrak{D}_p(\mathbb{R}^n)$ from tangent vectors to derivations given by

$$v \mapsto D_v = \left\langle v, \frac{\partial}{\partial x^i}\bigg|_p \right\rangle.$$

is a linear map of vector spaces.

**Lemma 1.2.2**
If $D$ is a point-derivation of $C_p^\infty$, then $D(c) = 0$ for any constant function $c$.

**Proof**
By $\mathbb{R}$-linearity, $D(c) = cD(1)$ and it suffices to prove that $D(1) = 0$. By the Leibniz rule,

$$D(1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

This is only possible if $D(1) = 0$.

**Theorem 1.2.3**
The linear map $\phi(v) := D_v$ is an isomorphism of vector spaces from the tangent space $T_p(\mathbb{R}^n)$ to the space of point-derivations of $\mathfrak{D}_p(\mathbb{R}^n)$.

**Proof**
Injectivity. Suppose $D_v \equiv 0$ for some $v \in T_p(\mathbb{R}^n)$. We claim that $v = 0$. To see this, apply $D_v$ to the coordinate function $x^j$ to see that

$$0 = D_v(x^j) = \langle v, \delta_i^j \rangle = v^j.$$

Surjectivity. Let $D$ be a derivation at $p$ and $(f, V)$ a representative of a germ in $C_p^\infty$. Making $V$ smaller if necessary, we may assume that $V$ is an open ball and hence star shaped at $p$. By Taylor's theorem with remainder, there are $g_i \in C^\infty(V)$ such that

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x)$$

and $g_i(p) = \frac{\partial f}{\partial x^i}(p)$. Applying $D$ to both sides and recalling $D$ annihilates constant functions, we get by the Leibniz rule

$$\begin{aligned} D(f) &= \sum_i [D(x^i) g_i(p) + p^i D(g_i)] - \sum_i p^i D(g_i) \\ &= \sum_i D(x^i) g_i(p) \\ &= \sum_i D(x^i) \frac{\partial f}{\partial x^i}(p). \end{aligned}$$

Thus $D = D_v$ for $v = \langle Dx^1, \ldots, Dx^n \rangle$.

This theorem shows that one may identify the tangent vectors at $p$ with the derivations at $p$. Under the vector space isomorphism $T_p(\mathbb{R}^n) \cong \mathfrak{D}_p(\mathbb{R}^n)$, the standard basis correspondes to the coordinate partial derivatives From now on, we make this identification and write a tangent vector $v = \sum_i v^i e_i$ as

$$v = \sum_i v^i \left. \frac{\partial}{\partial x^i} \right|_p .$$

The vector space $\mathfrak{D}_p(\mathbb{R}^n)$ of derivations turns out to be more suitable for generalization to manifolds.

### 1.2.4 Vector Fields

**Definition 1.2.4 (Vector Field)**
A *vector field* $X$ on an open subset $U \subseteq \mathbb{R}^n$ is a function that assigns to each point $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n) \cong \mathfrak{D}_p(\mathbb{R}^n)$.

Since $T_p(\mathbb{R}^n)$ has basis $\left\{ \partial/\partial x^i \big|_p \right\}$, the vector $X_p$ is a linear combination

$$X_p = \sum_i a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p$$

for some $p \in U$ and $a^i(p) \in \mathbb{R}$. Omitting $p$, we may write $X = \sum_i a^i \partial/\partial x^i$, where the $a^i$'s are now functions on $U$. We say that the vector field $X$ is $C^\infty$ on $U$ if the coefficient functions

$a^i \in C^\infty(U)$. One can identify vector fields on $U$ with column vectors of functions on $U$

$$X = \sum_i a^i \frac{\partial}{\partial x^i} \leftrightarrow \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}.$$

This is the same identification from tangent vectors to derivations, except we now allow the point $p$ to move in $U$.

The ring of $C^\infty$ functions on an open set $U$ is commonly denoted by $C^\infty(U)$ or $\mathfrak{F}(U)$. The multiplication of vector fields by functions on $U$ is defined pointwise:

$$(fX)_p := f(p)X_p.$$

If $X = \sum_i a^i \partial/\partial x^i$ is a $C^\infty$ vector field and $f \in C^\infty(U)$, then

$$fX = \sum_i (fa^i)\partial/\partial x^i$$

is a $C^\infty$ vector field on $U$. Thus the set of all $C^\infty$ vector fields on $U$, denoted $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a *module* over the ring $C^\infty(U)$.

**Definition 1.2.5 (Module)**
Let $R$ be a commutative ring with identity. A *(left) $R$-module* is an abelian group $A$ with a scalar multiplication map $\mu : R \times A \to A$, usually written $\mu(r, a) = ra$, such that for all $r, s \in R$ and $a, b \in A$,

(i) $(rs)a = r(sa)$ [associativity]

(ii) $1a = a$ if $1 \in R$ is the [identity]

(iii) $(r + s)a = ra + sa, r(a + b) = ra + rb$ [distributivity]

If $R$ is a field, then an $R$-module is precisely a vector space over $R$. Thus modules generalize a vector space by allowing scalars over a ring rather than a field.

**Definition 1.2.6 ($R$-Module Homomorhism)**
Let $A, A'$ be $R$-modules. An *$R$-module homomorphism* from $A$ to $A'$ is a map $f : A \to A'$ that preserves both addition and scalar multiplication: for all $a, b \in A$ and $r \in R$,

(i) $f(a + b) = f(a) + f(b)$,

(ii) $f(ra) = rf(a)$.

### 1.2.5 Vector Fields as Derivations

If $X$ is a $C^\infty$ vector field on an open subset $U$ of $\mathbb{R}^n$ and $f \in C^\infty(U)$, we can define a new function $Xf$ on $U$ by

$$(Xf)(p) := X_p f$$

for any $p \in U$. Writing $X = \sum_i a^i \partial/\partial x^i$, we get that

$$\begin{aligned}(Xf)(p) &= \sum_i a^i(p) \frac{\partial f}{\partial x^i}(p) \\ Xf &= \sum_i a^i \frac{\partial f}{\partial x^i}.\end{aligned}$$

This shows that $Xf \in C^\infty(U)$. So a $C^\infty$ vector field $X$ gives rise to an $\mathbb{R}$-linear map

$$\begin{aligned}C^\infty(U) &\to C^\infty(U) \\ f &\mapsto Xf.\end{aligned}$$

**Proposition 1.2.4 (Leibniz Rule for a Vector Field)**
If $X$ is a $C^\infty$ vector field and $f, g \in C^\infty(U)$ for some open subset $U \subseteq \mathbb{R}^n$, then $X(fg)$ satisfies the product (Leibniz) rule

$$X(fg) = (Xf)g + fXg.$$

**Proof**
At each point $p \in U$, the vector $X_p$ satisfies the Leibniz rule by definition of a derivation:

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g).$$

We now define a *derivation* as opposed to a point-derivation.

**Definition 1.2.7 (Derivation)**
If $A$ is an algebra over a field $\mathbb{K}$, a *derivation* of $A$ is a $\mathbb{K}$-linear map $D : A \to A$ such that

$$D(ab) = (Da)b + aDb$$

for all $a, b \in A$.

Note that a derivation at $p$ is not a derivation of the algebra $C_p^\infty$. A derivation at $p$ is a map from $C_p^\infty \to \mathbb{R}$, while a derivation is a map from $C_p^\infty \to C_p^\infty$.

The set of all derivations is closed under addition and scalar multiplication and forms a vector space denoted by $\operatorname{Der}(A)$. From the discussion above, a $C^\infty$ vector field on an open set $U$ gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \mathrm{Der}(C^\infty(U)) \\ X &\mapsto (f \mapsto Xf). \end{aligned}$$

Just as tangent vectors at a point $p$ can be identified with point-derivations of $C_p^\infty$, the vector fields on an open set $U$ can be identified with the derivations of the algebra $C^\infty(U)$, ie the map $\varphi$ is an isomorphism of vector spaces. The injectivity property is not too hard to show but the surjectivity is non-trivial.

## 1.3 The Exterior Algebra of Multicovectors

Our goal in this section is to generalize parts of vector calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$, such as the cross product. a key insight of Grassman, the author of the the multivector, is to work in the dual space of linear functionals. This provides more flexibility than the viewpoint of tangent vectors.

### 1.3.1 Dual Space

If $V, W$ are real vector spaces, we denote by $\mathrm{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$. Recall the *dual space* $V^\vee$ of $V$ is the vector space of all real-valued linear functionals on $V$

$$V^\vee = \mathrm{Hom}(V, \mathbb{R}).$$

The elements of $V^\vee$ are known as *covectors* or 1-*covectors* on $V$.

In the rest of this section, $V$ is some finite-dimensional vector space. If $\{e_i : i \in [n]\}$ is some basis for $V$, then every $v \in V$ is a unique linear combination of the basis vectors. Let $\alpha^i : V \to \mathbb{R}$ denote the $i$-th coordinate functional that picks out the $i$-th coordinate, $\alpha^i(v) = v^i$.

**Proposition 1.3.1**
The functions $\{\alpha^i : i \in [n]\}$ form a basis for $V^\vee$.

**Proof**
Any $f \in V^\vee$ can be expressed as

$$f(v) = \sum_i f(e_i)\alpha^i(v).$$

Linear independence can be checked using the basis vectors $e_i$'s.

We say the $\alpha^i$'s form the *dual basis* of the $e_i$'s.

**Corollary 1.3.1.1**
$\dim V^\vee = \dim V$ for any finite-dimensional vector space $V$.

### 1.3.2 Permutations

Fix a positive integer $k$. A permutation of $[k]$ is a bijection $\sigma : A \to A$. An *r-cycle* is a permutation that is cyclic on some $r$ elements while fixing the others. A *transposition* is a 2-cycle. Two cycles $(a_1 \dots a_r)$ and $(b_1 \dots b_s)$ are said to be *disjoint* if the sets $\{a_i\}$ and $\{b_j\}$ have empty intersection. The *product* $\tau\sigma$ of two permutations $\tau, \sigma$ of $A$ is the composition $\tau \circ \sigma$. We write $S_k$ to denote the set of all permutations on $[k]$.

Recall from elementary group theory that any permutation is the product of disjoint cycles. Moreover, the *sign* of a permutation, denoted $\operatorname{sgn}(\sigma)$, takes on value $\pm 1$ depending on whether the permutation is the product of even or odd number of transpositions. This function is well-defined and satisfies

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau).$$

An *inversion* in a permutation $\sigma$ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. A second way to compute the sign of a permutation is to count the bnumber of inversions.

**Proposition 1.3.2**
A permutation is even if and only if it has an even number of inversions.

**Proof**
The proof is algorithmic and is essentially bubble sort.

### 1.3.3 Multilinear Functions

A function $f : V^k \to \mathbb{R}$ is *k-linear* if it is linear in each of its $k$ arguments. It is customary to write bilinar and trilinear instead of 2-linear and 3-linear. A $k$-linear function on $V$ is also called a *k-tensor* on $V$. We denote the vector space of all $k$-tensors on $V$ by $L_k(V)$. If $f$ is a $k$-tensor on $V$, we say that the *degree* of $f$ is $k$.

**Example 1.3.3**
The dot product on $\mathbb{R}^n$ is bilinear.

**Example 1.3.4**
If we view the determinant as a function of the $n$ column vectors of a matrix, then it is $n$-linear in $\mathbb{R}^n$.

**Definition 1.3.1 (Symmetric $k$-Linear Function)**
A $k$-linear function $f : V^k \to \mathbb{R}$ is *symmetric* if

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = f(v_1, \ldots, v_k)$$

for all permutations $\sigma \in S_k$.

**Definition 1.3.2 (Alternating $k$-Linear Function)**
A $k$-linear function $f : V^k \to \mathbb{R}$ is *alternating* if

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) f(v_1, \ldots, v_k)$$

for all permutations $\sigma \in S_k$.

**Example 1.3.5**
(i) The dot product on $\mathbb{R}^n$ is symmetric
(ii) The determinant on $\mathbb{R}^n$ is alternating
(iii) The cross product $v \times w$ on $\mathbb{R}^3$ is alternating

**Example 1.3.6**
For any two linear functionals $f, g \in V^\vee$, the function

$$(f \wedge g)(u, v) := f(u)g(v) - f(v)g(u)$$

is alternating. This is a special case of the wedge product which we will see soon.

We are especially interested in the space $A_k(V)$ of all alternating $k$-linear functions on a vector space $V$ for $k > 0$. These are also known as *alternating $k$-tensors*, *$k$-covectors*, or *multicovectors of degree $k$* on $V$. For $k = 0$, we define a *0-covector* to be a constant so that $A_0(V) = \mathbb{R}$ by convention. A 1-covector is simply a covector.

### 1.3.4 The Permutation Action on Multilinear Functions

If $f$ is a $k$-linear function on some vector space $V$ and $\sigma$ is a permutation in $S_k$, we define a new $k$-linear function $\sigma f$ by

$$(\sigma f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Thus $f$ is symmetric if and only if $\sigma f = f$ for all $\sigma \in S_k$ and $f$ is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$.

**Lemma 1.3.7**
If $\sigma, \tau \in S_k$ and $f \in L_k(V)$, then $\tau(\sigma f) = (\tau\sigma) f$.

Thus the map $S_k \times L_k \to L_k$ described above is a *left action*.

**Definition 1.3.3 (Left Action)**
If $G$ is a group and $X$ an arbitrary set, a map $G \times X \to X$ written as

$$(\sigma, x) \mapsto \sigma \cdot x$$

is a *left action* of $G$ on $X$ if

(i) $e \cdot x = x$ for every $x \in X$, where $e$ is the identity element in $G$

(ii) $\tau \cdot (\sigma \cdot x) = (\tau\sigma) \cdot x$ for all $\tau, \sigma \in G$ and $x \in X$.

A right action can be similarly defined.

Recall too the following definition.

**Definition 1.3.4 (Orbit)**
The *orbit* of an element $x \in X$ with respect to a group action is defined to be the set

$$Gx := \{\sigma \cdot x \in X : \sigma \in G\}.$$

Note that each permutation $S_k$ itself acts as a linear operator $L_k(V) \to L_k(V)$, since $\sigma f$ is $\mathbb{R}$-linear in $f$.

### 1.3.5 The Symmetrizing and Alternating Operators

Given any $k$-linear function $f$ on a vector space $V$, there is a standard trick to make a symmetric or alternating $k$-linear function from $f$.

$$\begin{aligned}
(Sf)(v_1, \ldots, v_k) &= \sum_{\sigma \in S_k} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \\
&=: \sum_{\sigma \in S_k} \sigma f \\
(Af)(v_1, \ldots, v_k) &= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \sigma f.
\end{aligned}$$

**Remark 1.3.8** This trick turns up in different places when taking an average over permutations can "simply" the problem. For example, we can reduce some instances of semidefinite programs from algebraic graphs to linear programs by taking an average over permutations from its automorphism group.

**Proposition 1.3.9**
If $f \in L_k(V)$,

(i) $Sf \in L_k(V)$ is symmetric

(ii) $Af \in L_k(V)$ is alternating

**Proof**
We omit the proof of (i) since it follows the same flow as (ii). For $\tau \in S_k$,

$$\begin{aligned}\tau(Af) &:= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\tau(\sigma f) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)(\tau\sigma f) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\tau)^2(\operatorname{sgn}\sigma)(\tau\sigma f) \\ &= (\operatorname{sgn}\tau)\sum_{\sigma \in S_k} (\operatorname{sgn}\tau\sigma)(\tau\sigma f) \\ &= (\operatorname{sgn}\tau)Af.\end{aligned}$$

**Lemma 1.3.10**
If $f \in L_k(V)$ is alternating, then

$$Af = (k!)f.$$

**Proof**
By computation,

$$\begin{aligned}Af &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)\sigma f \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)^2 f \\ &= (k!)f.\end{aligned}$$

### 1.3.6 The Tensor Product

Let $f \in L_k(V)$ and $g \in L_\ell(V)$. Their *tensor product* is the $(k+\ell)$-linear function $f \otimes g$ defined by

$$(f \times g)(v_1, \dots, v_{k+\ell}) := f(v_1, \dots, v_k)g(v_{k+1}, \dots, v_{k+\ell}).$$

**Example 1.3.11 (Bilinear Maps)**
Let $\{e_i : i \in [n]\}$ be a basis for a vector space $V$ $\{\alpha^i\}$ the dual basis for $V^\vee$, and $\langle , \rangle : V \times V \to \mathbb{R}$ a bilinear map on $V$.

Set $g_{ij} := \langle e_i, e_j \rangle \in \mathbb{R}$. If $v = \sum_i v^i e_i$ and $w = \sum w^i e_i$, we can express $\langle , \rangle$ in terms of the tensor product

$$\begin{aligned}
\langle v, w \rangle &= \sum_{i,j} v^i w^j \langle e_i, e_j \rangle \\
&= \sum_{i,j} \alpha^i(v)\alpha^j(w) g_{ij} \\
&= \sum_{i,j} g_{ij}(\alpha^i \otimes \alpha^j)(v, w).
\end{aligned}$$

**Proposition 1.3.12**
The tensor product of multi-linear functions is associative since multiplication is associative in $\mathbb{R}$.

### 1.3.7 The Wedge Product

If two multilinear functions $f, g \in L_k(V)$ are alternating, we would like to have a product that is alternating as well. This motivates the definition of the *wedge product*, also called the *exterior product*: for $f \in A_k(V), g \in A_\ell(V)$,

$$\begin{aligned}
(f \wedge g)(v_1, \dots, v_{k+\ell}) &:= \frac{1}{k!\ell!} A(f \otimes g)(v_1, \dots, v_{k+\ell}) \\
&= \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).
\end{aligned}$$

**Remark 1.3.13** This construction is also alternating for $f \in L_k(V), g \in A_\ell(V)$. However, the terms in the denominator only make sense for alternating functions which we see next.

When $k = 0$, the element $f \in A_0(V)$ is just some constant $c$. Then the wedge product $c \wedge g$ is simply scalar multiplication

$$c \wedge g = \frac{1}{\ell!} \sum_{\sigma \in S_\ell} (\operatorname{sgn} \sigma) c g(v_{\sigma(1)}, \dots, v_{\sigma(\ell)}) = cg.$$

The coefficient $1/k!\ell!$ compensates for repetitions in the sum. For every permutation $\sigma \in S_{k+\ell}$, there are $k!$ permutations $\tau \in S_k$ that permute the first $k$ arguments and leave the arguments of $g$ alone. For every such $\tau \in S_k$, the composition $\sigma\tau \in S_{k+\ell}$ contributes the same term to the sum, since $f$ is alternating. This we divide by $k!$ to get rid of the $k!$ repeating terms in the sum and similarly for $\ell!$.

Another way to avoid redundancies in the definition of $f \wedge g$ is to stipulate that in the sum, $\sigma(1), \ldots, \sigma(k)$ is in ascending order and also $\sigma(k+1), \ldots, \sigma(k+\ell)$.

**Definition 1.3.5 (Shuffle)**
A permutation $\sigma \in S_{k+\ell}$ is a $(k, \ell)$-*shuffle* if

$$\sigma(1) < \cdots < \sigma(k)$$

and

$$\sigma(k+1) < \cdots < \sigma(k+\ell).$$

We can alternatively define the wedge product by summing over all $(k, \ell)$-shuffles. Thus this is a sum over $(k+\ell)!/k!\ell! = \binom{k+\ell}{k}$ terms. rather than $(k+\ell)!$ terms.

**Example 1.3.14**
The wedge product of two covectors $f, g \in A_1(V)$ is given by

$$(f \wedge g)(v, w) = f(v)g(w) - f(w)g(v).$$

### 1.3.8 Anticommutativity of the Wedge Product

By the definition of the wedge product, $f \wedge g$ is bilinear in $f$ and $g$ since it is a sum of bilinear functions.

**Proposition 1.3.15**
The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_\ell(V)$,

$$f \wedge g = (-1)^{k\ell} g \wedge f.$$

**Proof**
Define $\tau \in S_k$ as

$$\tau(i) := \begin{cases} i + k, & i \leq \ell \\ i - \ell \equiv i + k - (k + \ell), & i > \ell \end{cases}$$

Thus $\tau$ is the $k$-th product of the $k+\ell$ cycle and

$$\begin{aligned}
\sigma(1) &= \sigma\tau(\ell+1) \\
&\dots \\
\sigma(k) &= \sigma\tau(\ell+k) \\
\sigma(k+1) &= \sigma\tau(1) \\
&\dots \\
\sigma(k+\ell) &= \sigma\tau(\ell).
\end{aligned}$$

For any $v_1, \dots, v_{k+\ell} \in V$,

$$\begin{aligned}
A(f \otimes g)(v_1, \dots, v_{k+\ell}) &= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}) \\
&= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f(v_{\sigma\tau(\ell+1)}, \dots, v_{\sigma\tau(\ell+k)}) g(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(\ell)}) \\
&= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma\tau) g(v_{\sigma\tau(1)}, \dots, v_{\sigma\tau(\ell)}) f(v_{\sigma\tau(\ell+1)}, \dots, v_{\sigma\tau(\ell+k)}) \\
&= (\operatorname{sgn} \tau) A(g \otimes g)(v_1, \dots, v_{k+\ell}).
\end{aligned}$$

The statement can then be proven by dividing by $k!\ell!$ and verifying that $\operatorname{sgn} \tau = (-1)^{k\ell}$. Indeed, the $(k+\ell)$-cycle has sign $(-1)^{k+\ell-1}$. Taking the composition of $k$ of them yields

$$(-1)^{k\ell + k(k-1)} = (-1)^{k\ell}.$$

**Corollary 1.3.15.1**
If $f$ is a multicovector of odd degree on $V$, then $f \wedge f = 0$.

**Proof**
Let $k$ be the degree of $f$. By anticommutativity,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f.$$

This is only possible if $f \wedge f = 0$.

### 1.3.9 Associativity of the Wedge Product

The wedge product of a $k$-covector $f$ and an $\ell$-covector $g$ on a vector space $V$ is by definition the $(k+\ell)$-covector

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g).$$

Our goal in this section is to show that this product is associative.

**Lemma 1.3.16**
Suppose $f \in L_k(V)$ and $g \in L_\ell(V)$.
(i) $A(A(f) \otimes g) = k!A(f \otimes g)$
(ii) $A(f \otimes A(g)) = \ell!A(f \otimes g)$

**Proof**
We prove (i) and omit the proof of (ii) as it follows the exact train of thought. By definition,

$$A(A(f) \otimes g) = \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\sigma \left( \sum_{\tau \in S_k} (\operatorname{sgn} \tau)(\tau f) \otimes g \right).$$

We can view $\tau \in S_k$ as a permutation in $S_{k+\ell}$. Hence

$$A(A(f) \otimes g) = \sum_{\sigma \in S_{k+\ell}, \tau \in S_k} (\operatorname{sgn} \sigma)(\operatorname{sgn} \tau)(\sigma\tau)(f \otimes g).$$

For each $\mu \in S_{k+\ell}$ and $\tau \in S_k$, $\sigma = \mu\tau^{-1}$ is the unique element such that $\mu = \sigma\tau$. Hence each $\mu \in S_{k+\ell}$ appears once in the double sum for each $\tau \in S_k$, and hence $k!$ times in total. Thus

$$A(A(f) \otimes g) = k! \sum_{\mu \in S_{k+\ell}} (\operatorname{sgn} \mu)\mu(f \otimes g) = k!A(f \otimes g).$$

**Proposition 1.3.17 (Associativity of the Wedge Product)**
Let $V$ be a real vector space and $f \in A_k(V), g \in A_\ell(V), h \in A_m(V)$. Then

$$(f \wedge g) \wedge h = f \wedge (g \wedge h).$$

**Proof**
By the definition of the wedge product,

$$\begin{aligned}
(f \wedge g) \wedge h &= \frac{1}{(k+\ell)!m!} A((f \wedge g) \otimes h) \\
&= \frac{1}{(k+\ell)!m!} \frac{1}{k!\ell!} A(A(f \otimes g) \otimes h) \\
&= \frac{(k+\ell)!}{(k+\ell)!m!k!\ell!} A((f \otimes g) \otimes h) && \text{lemma} \\
&= \frac{1}{k!\ell!m!} A((f \otimes g) \otimes h).
\end{aligned}$$

We can also show that that $f \wedge (g \wedge h)$ is also equal to the last term, concluding the proof by the associativity of the tensor product.

Since associativity holds, it is customary to omit the parenthesis in multiple wedge products.

**Corollary 1.3.17.1**
If $f_i \in A_{d_i}(V)$, then

$$f_1 \wedge \cdots \wedge f_r = \frac{1}{(d_1)! \dots (d_r)!} A(f_1 \otimes \cdots \otimes f_r).$$

Let $[b^i_j]$ denote the matrix whose $(i, j)$-th entry is $b^i_j$. We have the following proposition.

**Proposition 1.3.18 (Wedge Product of 1-Covectors)**
If $\alpha^1, \dots, \alpha^k \in L_1(V)$ and $v_1, \dots, v_k \in V$,

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) = \det\left[\alpha(v_j)^i\right].$$

**Proof**
We have

$$\begin{aligned}
(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) &= A(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \dots, v_k) \\
&= \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha^1(v_{\sigma(1)}) \dots \alpha^k(v_{\sigma(k)}) \\
&= \det\left[\alpha^i(v_j)\right].
\end{aligned}$$

Recall the notation

$$A = \bigoplus_{k=0}^{\infty} A^k$$

means that each nonzero element of $A$ is uniquely a finite sum

$$a = a_{i_1} + \cdots + a_{i_m}$$

where each $0 \neq a_{ij} \in A^{i_j}$.

**Definition 1.3.6 (Graded Algebra)**
An algebra $A$ over a field $\mathbb{K}$ is *graded* if it can be written as a direct sum $A = \oplus_{k=0}^{\infty} A^k$ of vector spaces over $\mathbb{K}$ such that the multiplication map from $A$ sends $A^k \times A^\ell$ to $A^{k+\ell}$.

**Definition 1.3.7 (Graded Commutative)**
A graded algebra $A = \oplus_{k=0}^{\infty} A^k$ is said to be *graded commutative* or *anticommutative* if for all $a \in A^k$ and $b \in A^\ell$,

$$ab = (-1)^{k\ell} ba.$$

A *homomorphism of graded algebras* is an algebra homomorphism that preserves the degree.

**Example 1.3.19**
The polynomial algebra $A = \mathbb{R}[x, y]$ is the graded by degree; $A^k$ consists of all homogeneous polynomials of total degree $k$ in the variables $x$ and $y$.

For a vector space $V$ of finite dimension $n$, define

$$A_*(V) := \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^{n} A_k(V).$$

Note the second equality is due to linear dependence. With the wedge product of multicovector as multiplication, $A_*(V)$ becomes an anticommutative graded algebra, known as the *exterior algebra* or the *Grassman algebra* of multicovectors on the vector space $V$.

### 1.3.10 A Basis for $k$-Covectors

Let $e_1, \ldots, e_n$ be a basis for a real vector space $V$ and $\alpha^1, \ldots, \alpha^n$ the dual basis for $V^\vee$. We introduce the multi-index notation

$$I = (i_1, \ldots, i_k)$$

to write $e_I := (e_{i_1}, \ldots, e_{i_k})$ and $\alpha^I$ for $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$.

A $k$-linear function $f \in L_k(V)$ is completely determined by its values on all $k$-tuples $e^I$. If $f$ is alternating, it is completely determined by its values on $e_I$ with $1 \leq i_1 < \cdots < i_k \leq n$. This it suffices to consider $e_I$ with $I$ in strictly ascending order.

**Lemma 1.3.20**
Let $e_1, \ldots, e_n$ be a basis for a vector space $V$ and $\alpha^1, \ldots, \alpha^n$ its dual basis in $V^\vee$. If $I, J$ are strictly ascending multi-indices of length $k$, then

$$\alpha^I(e_J) = \delta^I_J = \begin{cases} 1, I = J \\ 0, I \neq J \end{cases}$$

**Proof**

We have previously shown that

$$\alpha^I(e_J) = \det\big[\alpha^i(e_j)\big]_{i\in I, j\in J}.$$

If $I = J$, then the matrix is the identity matrix and its determinant is 1.

If $I \neq J$, there is some minimal sub-index $\ell$ such that $i_\ell \neq j_\ell$. Without loss of generality, assume that $i_\ell < j_\ell$. Then $i_\ell$ is not equal to any of $j_1, \dots, j_\ell$. But then the $\ell$-th row of the matrix $[\alpha^i(e_j)]$ is all zero and the determinant evaluates to 0.

**Proposition 1.3.21**

The alternating $k$-linear functions $\alpha^I$ for all strictly ascending multi-indices $I$, form a basis for the space $A_k(V)$.

**Proof**

Linear Independence. Suppose $\sum_I c_I \alpha^I = 0$. Applying both sides to an arbitrary $e_J$, we get that

$$0 = \sum_I c_I \alpha^I(e_J) = \sum_I c_I \delta^I_J = c_J.$$

Spanning. Let $f \in A_k(V)$. We claim that

$$f = \sum_I f(e_I)\alpha^I.$$

Indeed, by $k$-linearity and the alternating property if two $k$-covectors agree on all $e_J$, then they are equal. We have

$$\sum_I f(e_I)\alpha^I(e_J) = \sum_I f(e_I)\delta^I_J = f(e_J).$$

**Corollary 1.3.21.1**

If $\dim V = n$, then $A_k(V)$ has dimension $\binom{n}{k}$.

**Proof**

The number of strictly ascending multi-indices is the number of size $k$ subsets of $[n]$.

**Corollary 1.3.21.2**

If $k > \dim V$, then $A_k(V) = 0$.

**Proof**
In the multi-index $I$, at least two will be the same, say $i = j$. But then $\alpha^i \wedge \alpha^j = 0$ since $\alpha^i, \alpha^j$ are equal covectors of odd degree 1 and so $\alpha^I = 0$ as well.

### 1.3.11 Useful Results

**Proposition 1.3.22 (Characterizations of Alternating Tensors)**
Let $f \in L_k(V)$. The following are equivalent.

(i) $f$ is alternating

(ii) $f$ changes signs when two arguments are interchanged

(iii) $f$ changes signs when two successive arguments are interchanged

(iv) $f(v_1, \dots, v_k) = 0$ whenever two of its arguments are equal

**Theorem 1.3.23 (Transformation Rule for a Wedge Product of Covectors)**
Suppose two sets of covectors $\beta^1, \dots, \beta^k, \gamma^1, \dots, \gamma^k \in L_1(V)$ satisfy

$$\beta^i = \sum_{j=1}^k a^i_j \gamma^j$$

for each $i \in [k]$. Then

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k$$

where $A = [a^i_j]$.

**Proof**
Using the fact that $\alpha \wedge \alpha = 0$ for all $\alpha \in L_1(V)$,

$$\begin{aligned}
\beta^1 \wedge \cdots \wedge \beta^k &= \left(\sum_{j_1=1}^k a^1_{j_1}\gamma^{j_1}\right) \wedge \cdots \wedge \left(\sum_{j_k=1}^k a^k_{j_k}\gamma^{j_k}\right) \\
&= \sum_{\sigma \in S_k} \prod_{i=1}^k a^i_{\sigma(i)} \gamma^{\sigma(1)} \wedge \cdots \wedge \gamma^{\sigma(k)} \\
&= \sum_{\sigma \in S_k} \prod_{i=1}^k a^i_{\sigma(i)} (\operatorname{sgn}\sigma)\gamma^1 \wedge \cdots \wedge \gamma^k \\
&= (\det A)\gamma^1 \wedge \cdots \wedge \gamma^k.
\end{aligned}$$

**Theorem 1.3.24 (Transformation Rule for Multi-Covectors)**
Let $f \in A_k(V)$ and suppose vectors $u_1, \dots, u_k, v_1, \dots, v_k \in V$ satisfy

$$u_j = \sum_{i=1}^{k} a_j^i v_i$$

for each $j \in [k]$. Then

$$f(u_1, \dots, u_k) = (\det A) f(v_1, \dots, v_k).$$

**Lemma 1.3.25**
Let $\alpha^1, \dots, \alpha^k \in A_1(V)$. Then $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$ if and only if $\alpha^1, \dots, \alpha^k$ are linearly independent within the dual space $V^\vee$.

The lemma can be shown using the transformation rule.

**Proposition 1.3.26 (Exterior Multiplication)**
Let $0 \neq \alpha \in A_1(V)$ and $\gamma \in A_k(V)$. Then $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some $(k-1)$-covector $\beta \in L_{k-1}(V)$.

**Proof (Sketch)**
If the latter holds, then clearly $\alpha \wedge \gamma = 0$.

Suppose $\alpha \wedge \gamma = 0$ and write $\gamma$ as a linear combination of wedge products formed by any basis of $V^\vee$ that includes $\alpha$. By linear independence, we see that we can factor out a $\alpha$ term in each non-zero term in the linear combination.

## 1.4 Differential Forms on $\mathbb{R}^n$

### 1.4.1 Differential 1-Forms and the Differential of a Function

**Definition 1.4.1 (Cotangent Space)**
The *cotangent space* to $\mathbb{R}^n$ at $p$, denoted by $T_p^*(\mathbb{R}^n)$ or $T_p^*\mathbb{R}^n$ is defined to be the dual space $(T_p\mathbb{R}^n)^\vee$ of the tangent space $T_p(\mathbb{R}^n)$.

Thus an element of the cotangent space $T_p^*(\mathbb{R}^n)$ is a covector on the tangent space $T_p(\mathbb{R}^n)$.

The following definition is the analogue to the vector field.

**Definition 1.4.2 (Differential 1-Form)**
A *covector field* or a *differential 1-form* on an open subset $U \subseteq \mathbb{R}^n$ is a function $\omega$ that assigns to each point $p \in U$ a covector $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \to \bigsqcup_{p \in U} T_p^*(\mathbb{R}^n)$$
$$p \mapsto \omega_p \in T_p^*(\mathbb{R}^n)$$

We call a differential 1-form a 1-*form* for short.

From any $f \in C^\infty(U, \mathbb{R})$, we can construct a 1-form $df$ called the *differential* of $f$ as follows. For $p \in U$ and $X_p \in T_pU$, $df$ maps $p$ to the covector $(df)_p : T_p(\mathbb{R}^n) \to \mathbb{R}$ given by

$$(df)_p(X_p) = X_p f.$$

The directional derivative of a function in the direction of a tangent vector at a point $p$ sets up a bilinear function

$$T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) \to \mathbb{R}$$
$$(X_p, f) \mapsto X_p f = \langle X_p, f \rangle.$$

We can think of a tangent vector as a function on the second argument of this pairing: $\langle X_p, \cdot \rangle$. The differential $(df)_p$ at $p$ is then a function on the first argument of the pairing: $\langle \cdot, f \rangle$. The value of the differential $df$ at $p$ is also written $df|_p$.

Recall the partial derivatives form a basis to the tangent space $T_p(\mathbb{R}^n)$.

**Proposition 1.4.1**
Let $x^1, \ldots, x^n$ denote the standard basis on $\mathbb{R}^n$. At every point $p \in \mathbb{R}^n$, $(dx^1)_p, \ldots, (dx^n)_p$ is the dual basis for the cotangent space $T_p^*(\mathbb{R}^n)$ with respect to the basis $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ for the tangent space $T_p(\mathbb{R}^n)$.

Thus each $(dx^i)_p$ is an evaluation functional at point $p$ on the tangent space.

**Proof**
By definition,

$$(dx^i)_p \left( \left. \frac{\partial}{\partial x^j} \right|_p \right) = \left. \frac{\partial}{\partial x^j} \right|_p x^i = \delta^i_j.$$

If $\omega$ is a 1-form on an open subset $U \subseteq \mathbb{R}^n$, the proposition above shows that at every $p \in U$, we can write

$$\omega_p = \sum_i a_i(p)(dx^i)_p$$

for some $a_i(p) \in \mathbb{R}$. As $p$ varies over $U$, the coefficients $a_i$ become functions on $U$ and we can write $\omega = \sum_i a_i dx^i$. The covector field $\omega$ is said to be $C^\infty$ on $U$ if the coefficient functions $a_i$ are all $C^\infty$ on $U$.

**Proposition 1.4.2**
If $f : U \to \mathbb{R}$ is a $C^\infty$ function on an open subset $U \subseteq \mathbb{R}^n$, then

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

**Proof**
We know that

$$(df)_p = \sum_i a_i(p)(dx^i)_p$$

at every $p \in U$ for some $a_i(p)$. Conclude by evaluating this at the coordinate vector field $\partial/\partial x^j$:

$$\begin{aligned} \frac{\partial f}{\partial x^j} &= df\left(\frac{\partial}{\partial x^j}\right) \\ &= \sum_i a_i dx^i \left(\frac{\partial}{\partial x^j}\right) \\ &= \sum_i a_i \delta^i_j \\ &= a_j. \end{aligned}$$

The proposition above shows that if $f \in C^\infty$, then the 1-form $df$ is also $C^\infty$. Moreover, we see that $dx^i$ is nothing more than the coordinate function with respect to the basis $\partial/\partial x^j$.

### 1.4.2 Differential $k$-Forms

**Definition 1.4.3 (Differential $k$-Form)**
A *differential form $\omega$ of degree $k$* or *$k$-form* on an open subset $U \subseteq \mathbb{R}^n$ is a function that assigns to each point $p \in U$ an alternating $k$-linear function on the tangent space $T_p(\mathbb{R}^n)$, ie $\omega_p \in A_k(T_p\mathbb{R}^n)$.

Since $A_1(T_p\mathbb{R}^n) = T_p^*(\mathbb{R}^n)$, the definition of a $k$-form generalizes that of a 1-form we have just seen.

We have shown that one possible basis for $A_k(V)$ where $\dim V < \infty$, is the wedge product of $k$-coordinate functions with strictly ascending indices. Thus a basis for $A_k(T_p\mathbb{R}^n)$ is

$$dx_p^I = dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k}$$

for $1 \le i_1 \le \cdots \le i_k \le n$. It follows taht at each $p \in U$, $\omega_p$ is a linear combination

$$\omega_p = \sum_I a_I(p) dx_p^I$$

and a $k$-form is a linear combination

$$\omega = \sum_I a_I dx^I$$

with function coefficients $a_I : U \to \mathbb{R}$. We say that a $k$-form is $C^\infty$ on $U$ if all the coefficients $a_I$ are $C^\infty$ functions on $U$.

Denote by $\Omega^k(U)$ the vector space of $C^\infty$ $k$-forms on $U$. A 0-form on $U$ assigns to each point $p \in U$ an element of $A_0(T_p\mathbb{R}^n) = \mathbb{R}$. Thus $\Omega^0(U) = C^\infty(U)$.

The *wedge product* of a $k$-form $\omega$ and an $\ell$-form $\tau$ on some open $U \subseteq \mathbb{R}^n$ is defined pointwise:

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p$$

for each $p \in U$. If $\omega = \sum_I a_I dx^I$ and $\tau = \sum_J b_J dx^J$, then

$$\omega \wedge \tau = \sum_{I, J \text{ disjoint}} (a_I b_J) dx^I \wedge dx^J,$$

which shows that the wedge product of two $C^\infty$ forms is $C^\infty$. Thus the wedge product is a bilinear map

$$\wedge : \Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U).$$

Recall that the wedge product is anticommutative and associative.

For the case $k = 0$, the wedge product is just the pointwise multiplication of a $C^\infty$ function and a $C^\infty$ $\ell$-form

$$(f \wedge \omega)_p = f(p) \wedge \omega_p = f(p)\omega_p.$$

Thus if $f \in C^\infty(U)$ and $\omega \in \Omega^\ell(U)$, then $f \wedge \omega = f\omega$.

**Example 1.4.3**
Let $x, y, z$ be the coordinates in $\mathbb{R}^3$. The $C^\infty$ $k$-forms on $\mathbb{R}^3$ are

$$\begin{aligned} f dx + g dy + h dz & \qquad k = 1 \\ f dy \wedge dz + g dx \wedge dz + h dx \wedge dy & \qquad k = 2 \\ f dx \wedge dy \wedge dz & \qquad k = 3 \end{aligned}$$

**Example 1.4.4**
Let $x^1, \ldots, x^4$ be the coordinates in $\mathbb{R}^4$ and $p \in \mathbb{R}^4$. A possible basis for the vector space $A_3(T_p\mathbb{R}^4)$ is given by

$$(dx^1 \wedge dx^2 \wedge dx^3)_p, (dx^1 \wedge dx^2 \wedge dx^4)_p, (dx^1 \wedge dx^3 \wedge dx^4)_p, (dx^2 \wedge dx^3 \wedge dx^4)_p.$$

With the wedge product as multiplication and the degree of a form as the grading, the direction sum

$$\Omega^*(U) = \bigoplus_{k=0}^{n} \Omega^k(U)$$

becomes an anticommutative graded algebra over $\mathbb{R}$. Since we can multiply $C^\infty$ $k$-forms by $C^\infty$ functions, the set $\Omega^k(U)$ is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$. Thus the direct sum $\Omega^*(U)$ is also a module over the ring $C^\infty(U)$.

### 1.4.3 Differential Forms as Multilinear Functions on Vector Fields

If $\omega$ is a $C^\infty$ 1-form and $X$ is a $C^\infty$ vector field on an open subset $U \subseteq \mathbb{R}^n$, we define a function $\omega(X)$ on $U$ by the formula

$$\omega(X)_p = \omega_p(X_p)$$

for each $p \in U$. In coordinates,

$$\begin{aligned} \omega &= \sum_i a_i dx^i & a_i &\in C^\infty(U) \\ X &= \sum_j b^j \frac{\partial}{\partial x^j} & b^j &\in C^\infty(U) \\ \omega(X) &= \sum_i a_i b^i. \end{aligned}$$

Recall that we write $\mathfrak{X}(U)$ to denote the set of all $C^\infty$ vector fields on $U$. Thus a $C^\infty$ 1-form gives rise to a map $\mathfrak{X}(U) \to C^\infty(U)$.

This function is linear over the ring $C^\infty(U)$. To see this, consider some $f \in C^\infty(U)$.

$$\begin{aligned} (\omega(fX))_p &:= \omega_p(f(p)X_p) \\ &= f(p)\omega_p(X_p) && \omega_p \text{ is linear} \\ &=: (f\omega(X))_p. \end{aligned}$$

In the notation $\mathfrak{F}(U) = C^\infty(U)$, a 1-form gives rise to a $\mathfrak{F}(U)$-linear map $\mathfrak{X}(U) \to \mathfrak{F}(U)$. Similarly, a $k$-form $\omega$ on $U$ gives rise to a $k$-linear map over $\mathfrak{F}(U)$,

$$\begin{aligned} \mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U) &\to \mathfrak{F}(U) \\ (X_1, \ldots X_k) &\mapsto \omega(X_1, \ldots, X_k). \end{aligned}$$

**Example 1.4.5**
Let $\omega$ be a 2-form and $\tau$ a 1-form on $\mathbb{R}^3$. If $X, Y, Z$ are vector fields on $\mathbb{R}^3$,

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X).$$

### 1.4.4 The Exterior Derivative

In order to define the *exterior derivative* of a $C^\infty$ $k$-form on an open subset $U \subseteq \mathbb{R}^n$, we first define it on 0-forms. Recall $\Omega^k(U)$ denotes the $C^\infty$ $k$-forms on $U$. The exterior derivative of a function $f \in C^\infty(U)$ is defined to be its differential $df \in \Omega^1(U)$. Recall we showed the coordinate expansion

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

**Definition 1.4.4 (Exterior Derivative)**
For $k \geq 1$, if $\omega = \sum_I a_I dx^I \in \Omega^k(U)$ for some $a_I \in C^\infty(U)$, define

$$\begin{aligned} d\omega &:= \sum_I da_I \wedge dx^I \\ &= \sum_I \left( \sum_j \frac{\partial a_I}{\partial x^j} dx^j \right) \wedge dx^I \\ &\in \Omega^{k+1}(U). \end{aligned}$$

**Example 1.4.6**
Let $\omega = f dx + g dy \in \Omega^2(\mathbb{R}^2)$ be a 2-form where $f, g \in C^\infty(\mathbb{R}^2)$. We write $f_x = \partial/\partial x$ and $f_y = \partial f/\partial y$ for simplicity. Then

$$\begin{aligned} d\omega &= df \wedge dx + dg \wedge dy \\ &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy \\ &= (g_x - f_y) dx \wedge dy. \end{aligned}$$

**Definition 1.4.5 (Antiderivation)**
Let $A = \oplus_{k=0}^\infty A^k$ be a graded algebra over a field $\mathbb{K}$. An *antiderivation* of the graded algebra $A$ is a $\mathbb{K}$-linear map $D : A \to A$ such that for $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^k a(Db).$$

Furthermore, if there is some $m$ such that the antiderivation $D$ sends $A^k \to A^{k+m}$ for all $k$, then we say that it is an antiderivation of degree $m$.

By defining $A_k := 0$ for $k < \infty$, we can extend the grading of a graded algebra $A$ to negative integers. With this extension, the degree $m$ of an antiderivation can be negative.

**Proposition 1.4.7**

(i) The exterior differentiation $d : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1:

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.$$

(ii) $d^2 = 0$

(iii) If $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

**Proof**

(i) Since both sides of the claimed equality is linear in $\omega, \tau$, it suffices to check the equality for $\omega = f dx^I$ and $\tau = g dx^J$. Then

$$\begin{aligned} d(\omega \wedge \tau) &= d(fg dx^I \wedge dx^J) \\ &= \sum_i \frac{\partial (fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum_i \frac{\partial f}{\partial x^i} g dx^i \wedge dx^I \wedge dx^J + \sum_i f \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J. \end{aligned}$$

Moving the 1-form $(\partial g / \partial x^i) dx^i$ across the $k$-form $dx^I$ results in the sign $(-1)^k$ as desired.

(ii) Again, it suffices to check for $\omega = f dx^I$. We have

$$\begin{aligned} d^2(f dx^I) &= d\left( \sum_i \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \right) \\ &= \sum_{j,i} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I. \end{aligned}$$

In the sum, if $i = j$, then $dx^j \wedge dx^i = 0$. If $i \neq j$, then $\partial^2 f / \partial x^i \partial x^j$ is symmetric in $i, j$, but $dx^j \wedge dx^i$ is alternating in $i, j$, hence the terms with $i \neq j$ pair up and cancel.

(iii) This is by definition.

The three properties of the proposition above uniquely characterize exterior differentiation on an open set $U \subseteq \mathbb{R}^n$.

**Proposition 1.4.8 (Characterization of the Exterior Derivative)**
If $D : \Omega^*(U) \to \Omega^*(U)$

(i) is an antiderivation of degree 1

(ii) satisfies $D^2 = 0$

(iii) satisfies $(Df)(X) = X$ for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$

then $D = d$.

**Proof**
By linearity, it suffices to check that $D = d$ on a basic $k$-form $f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

(iii) states that $Df = df$ on $C^\infty$ functions $f$. It follows that $Ddx^i = DDx^i = 0$ by (ii). But then by the antiderivation property and induction, $D(dx^I) = D(dx^{i_1} \wedge (dx^{i_2} \wedge \cdots \wedge dx^{i_k})) = 0$.

Finally, for every $k$-form $f dx^I$,

$$\begin{aligned} D(f dx^I) &= (Df) \wedge dx^I + f D(dx^I) && \text{(i)} \\ &= (df) \wedge dx^I && D(dx^I) = 0 \\ &=: d(f dx^I). \end{aligned}$$

### 1.4.5 Closed & Exact Forms

**Definition 1.4.6 (Closed $k$-Form)**
A $k$-form $\omega$ on $U$ is *closed* of $d\omega = 0$.

**Definition 1.4.7 (Exact $k$-Form)**
A $k$-form $\omega$ on $U$ is *exact* if there is a $(k-1)$-form $\tau$ such that $\omega = d\tau$.

Note that since $d(d\tau) = 0$, every exact form is closed.

**Example 1.4.9**
Define a 1-form on $\mathbb{R}^2 - \{0\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-ydx + xdy).$$

Then $\omega$ is closed since

$$\begin{aligned}
d\omega &= d\left(\frac{x}{x^2+y^2}\right) \wedge dy - d\left(\frac{y}{x^2+y^2}\right) \wedge dx \\
&= db \wedge dy - da \wedge dx \\
&= \partial_x b(dx \wedge dy) - \partial_y a(dy \wedge dx) \\
&= \frac{y^2-x^2}{(x^2+y^2)^2}(dx \wedge dy) + \frac{x^2-y^2}{(x^2+y^2)^2}(dx \wedge dy) \\
&= 0.
\end{aligned}$$

**Definition 1.4.8 (Differential Complex/Cochain Complex)**
A collection of vector spaces $\{V^k\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a *differential complex/cochain complex*.

For any open subset $U \subseteq \mathbb{R}^n$, the exterior derivative $d$ makes the vector space $\Omega^*(U)$ of $C^\infty$ forms on $U$ into a cochain complex, called the *de Rham complex* of $U$:

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \dots.$$

The closed forms are precisely the elements of the kernel of $d$ and the exact forms are the elements of the image of $d$.

### 1.4.6 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus. By a *vector-valued function* on an open subset $U \subseteq \mathbb{R}^3$, we mean some $F = (P, Q, R) : U \to \mathbb{R}^3$. Thus this is precisely a vector field on $U$.

Three operators *gradient*, *curl*, and *divergence* are defined below:

$$\begin{aligned}
\operatorname{grad} f &= \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} f = \begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix}\\
\operatorname{curl}\begin{bmatrix}P\\ Q\\ R\end{bmatrix} &= \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \times \begin{bmatrix}P\\ Q\\ R\end{bmatrix} = \begin{bmatrix}R_y - Q_z\\ -(R_x - P_z)\\ Q_x - P_y\end{bmatrix}\\
\operatorname{div}\begin{bmatrix}P\\ Q\\ R\end{bmatrix} &= \begin{bmatrix}\partial_x\\ \partial_y\\ \partial_z\end{bmatrix} \cdot \begin{bmatrix}P\\ Q\\ R\end{bmatrix} = P_x + Q_y + R_z.
\end{aligned}$$

Now, every 1-form on $U$ is a linear combination with function coefficients of $dx, dy, dz$. Thus we can identify 1-forms with vector fields on $U$ via

$$P dx + Q dy + R dz \leftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

2-forms can be identified with vector fields on $U$ via

$$P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy \leftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

Finally, 3-forms on $U$ can be identified with functions on $U$:

$$f dx \wedge dy \wedge dz \leftrightarrow f.$$

Following these identifications, the exterior derivative of a 0-form $f$ is

$$df = f_x dx + f_y dy + f_z dz \leftrightarrow \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \operatorname{grad} f.$$

The exterior derivative of a 1-form is

$$\begin{aligned} & d(P dx + Q dy + R dz) \\ & = (R_y - Q_z) dy \wedge dz - (R_x - P_z) dz \wedge dx + (Q_x - P_y) dx \wedge dy \\ & \updownarrow \\ & \begin{bmatrix} R_y - Q_z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix} = \operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}. \end{aligned}$$

Lastly, the exterior derivative of a 2-form is

$$\begin{aligned} & d(P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy) \\ & = (P_x + Q_y + R_z) dx \wedge dy \wedge dz \\ & \updownarrow \\ & P_x + Q_y + R_z = \operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}. \end{aligned}$$

In summary, the exterior derivative on 0-forms, 1-forms, and 2-forms are simply the operators grad, curl, and div, under the appropriate identifications. Under these identifications, a vector field $\langle P, Q, R \rangle$ on $\mathbb{R}^3$ is the gradient of a $C^\infty$ function $f$ if and only if the corresponding 1-form $P dx + Q dy + R dz = df$.

We now state three basic facts concerning grad, curl, and div.

**Proposition 1.4.10**
The following hold:

(a) $\operatorname{curl}(\operatorname{grad} f) = \langle 0,0,0\rangle$

(b) $\operatorname{div}(\operatorname{curl}\langle P,Q,R\rangle) = 0$

This proposition expresses the property $d^2 = 0$ on open subsets of $\mathbb{R}^3$.

**Proposition 1.4.11**
On $\mathbb{R}^3$, a vector field $F$ is the gradient of some scalar function $f$ if and only if $\operatorname{curl} F = 0$.

This proposition expresses the fact that a 1-form on $\mathbb{R}^3$ is exact if and only if it is closed. This need not hold on a region other than $\mathbb{R}^3$, as the following examplle shows.

**Example 1.4.12**
Define $U := \mathbb{R}^3 - \{(0,0,z) : z \in \mathbb{R}\}$ and consider the familiar vector field

$$F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$$

on $\mathbb{R}^3$. Then $\operatorname{curl} F = 0$ as we computed, but $F$ is not the gradient of any $C^\infty$ function on $U$. This can be proven by contradition using the fundamental theorem for line integrals.

In the language of differential forms, the corresponding 1-form $\omega \leftrightarrow F$ is closed but not exact.

It turns out whether the previous proposition is true for a region $U$ depends only on the topology of $U$. One measure of failure of a closed $k$-form to be exact is the quotient vector space

$$H^k(U) := \frac{\{\text{closed } k\text{-forms on } U\}}{\{\text{exact } k\text{-forms on } U\}}$$

called the *$k$-th de Rham cohomology of $U$*. The generalization of the previous proposition to any differential form on $\mathbb{R}^n$ is called the *Poincaré lemma*: for $k \geq 1$, every closed $k$-form on $\mathbb{R}^n$ is exact. This is equivalent to proving the vanishing of the $k$-th deRham cohomology $H^k(\mathbb{R}^n)$ for $k \geq 1$.

The theory of differential forms allows us to generalize vector calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$ and then manifolds of any dimension. The general Stokes theorem subsumes and unifies the fundamental theorem for line integrals, Green's theorem in the plane, the classical Stokes theorem for a surface in $\mathbb{R}^3$, and the divergence theorem.

### 1.4.7 Convention on Subscripts and Superscripts

In differential geometry it is customary to index vector fields with subscripts $e_1, \ldots, e_n$ and differential forms with superscripts $\omega^1, \ldots, \omega^n$. Being 0-forms, coordinate functions take superscripts $x^1, \ldots, x^n$. Their differentials, beign 1-forms, should also have superscripts $dx^1, \ldots, dx^n$. Coordinate vector fields $\partial/\partial_i$ are considered to have subscripts since the $i$ appears in the lower half of the fraction.

Coefficient functions may have superscripts or subscripts depending on whether they are the coefficients of a vector field or a differential forms. For a vector field $X = \sum_i a^i e_i$, the coefficients have superscripts since mismatch "cancels out". For the same reason, the coefficients in a differential form $\omega = \sum_j b_j dx^j$ have subscripts.

This convention conveniently leads to a "conservation of indices". For example, if $X = \sum_i a_i \partial/\partial x^i$, then

$$a^i = (dx^i)(X)$$

so both sides of the equality have a net superscript $i$. If $\omega = \sum_j b_j dx^j$,

$$\omega(X) = \left(\sum_j b_j dx^j\right)\left(\sum_i a^i \frac{\partial}{\partial x^i}\right) = \sum_i b_i a^i.$$

After cancellation of superscripts and subscripts, both sides of the quality sign have zero net index. This convention is a useful mnemonic aid in some of the transformation formulas in differential geometry.

### 1.4.8 Miscellaneous Results

**Definition 1.4.9 (Superderivation)**
Let $A = \oplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field $\mathbb{K}$ with $A^k = 0$ for $k < 0$ and $m \in \mathbb{Z}$.
A *superderivation of $A$ of degree $m$* is a $\mathbb{K}$-linear map $D : A \to A$ such that for all $k$, $D(A^k) \subseteq A^{k+m}$ and for all $a \in A^k, b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{km} a(Db).$$

**Proposition 1.4.13**
If $D_1, D_2$ are two superderivations of $A$ of respective degrees $m_1, m_2$, their *commutator*

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1$$

is a superderivation of degree $m_1 + m_2$.

This proposition can be verified by checking the definitions.

A super derivation is said to be *even* or *odd* depending on the parity of its degree. An even superderivation is a derivation is a derivation and an odd superderivation is an antiderivation.

# Chapter 2

# Manifolds

Intuitively, a manifold is a generalization of curves and surfaces to higher dimension. It is locally Euclidean in the sense that every point has a neighborhood, called a chart, that is homeomorphic to an open subset of $\mathbb{R}^n$. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, so may concepts from $\mathbb{R}^n$, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold.

Our goal is to define and explore basic properties of a smooth manifold and smooth maps between manifolds. Initially, the only way to verify that a space is a manifold is to exhibit a collection of $C^\infty$ compatible charts covering the space. We eventually see a set of sufficient conditions under which a quotient topological space becomes a manifold.

## 2.1 Manifolds

### 2.1.1 Topological Manifolds

We recall a few definitions from point-set topology. A topological space is *second countable* if it has a countable basis. It is said to be *Hausdorff* if any two distinct points are repectively contained in disjoint open neighborhoods.

**Definition 2.1.1 (Locally Euclidean)**
A topological space $M$ is *locally Euclidean of dimension $n$* if every point $p \in M$ has a neighborhood $U$ such that there is a homeomorphism $\phi : U \to \mathbb{R}^n$ from $U$ onto an open subset of $\mathbb{R}^n$.

We call the pair $(U, \phi)$ a *chart*, $U$ a *coordinate neighborhood* or a *coordinate open set*, and $\phi$ a *coordinate map* or a *coordinate system* on $U$. Moreover, we say a chart $(U, \phi)$ is *centered* *at* $p \in U$ if $\phi(p) = 0$.

**Definition 2.1.2 (Topological Manifold)**
A *topological manifold* is a Hausdorff, second countable, locally Euclidean space.

**Remark 2.1.1** We require manifolds to be Hausdorff and second countable to ensure that manifolds behave as expected from our experience with Euclidean spaces. For instance, finite subsets are closed and limits of sequences are unique in Hausdorff spaces. The motivation for second-countability is based on the existence of the so called *partitions of unity*.

**Remark 2.1.2** We can restrict our definition of manifolds by forcing each coordinate map $\phi : U \to \mathbb{R}^n$ to be homeomorphic to an open ball in $\mathbb{R}^n$ or to $\mathbb{R}^n$ itself.

We say a manifold is of *dimension* $n$ or is an $n$*-manifold* if it is locally Euclidean of dimension $n$. For the dimension of a manifold to be well-defined, we need to know that for $n \neq m$, an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This is known as the *invariance of dimension* but is not easy to prove directly. Since we are interested in smooth manifolds, the analogous result is much easier to prove.

**Example 2.1.3**
The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$ and is hence an $n$-manifold. Similarly, every open subset of $\mathbb{R}^n$ is also an $n$-manifold.

Let $(X, \tau)$ be a topological space and $A \subseteq X$. Recall that $A$ endowed with the subspace topology $\{U \cap A : U \in \tau\}$ is called a *subspace* of $X$. The Hausdorff condition and second countability are inherited by subspaces. Thus any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

**Example 2.1.4**
The graph of a contiuous function $f : \mathbb{R} \to \mathbb{R}$ is a 1-manifold since it is a subspace of $\mathbb{R}^2$ and is locally Euclidean through the homeomorphism $(x, f(x)) \mapsto x$.

**Example 2.1.5**
The union of the $x$ and $y$-axis in $\mathbb{R}^2$ is not a topological manifold.

Suppose towards a contradition that there is a neighborhood of the intersection $p$ that is homeomorphic to an open ball $B$ of $\mathbb{R}^n$ with $p$ mapping to 0. The homeomorphism $U \to B$ restricts to a homeomorphism $U - \{p\} \to B - \{0\}$. Now, $B - \{0\}$ is either connected for $n \geq 2$ or disconnected if $n = 1$. Since $U - \{p\}$ has four connected components, this is a contradiction.

### 2.1.2 Compatible Charts

Suppose $(U, \phi), (V, \psi)$ are two charts of an $n$-manifold. Since $U \cap V$ is open in $U$ and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto a open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

**Definition 2.1.3 (Smoothly Compatible)**
Two charts $(U, \phi), (V, \psi)$ of a topological manifold are $C^\infty$-compatible if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$
$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are $C^\infty$ as functions from $\mathbb{R}^n \to \mathbb{R}^n$.

These compositions are called *transition functions* between the charts. We often omit "smoothly" and simply speak of compatible charts.

If $U \cap V$ is empty, then the two charts are automatically smoothly compatible. For simplicity of notation, we will sometimes write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and similarly $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$.

**Definition 2.1.4 (Smooth Atlas)**
A $C^\infty$ *atlas* or simply an *atlas* on a local Euclidean space $M$ is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise $C^\infty$-compatible charts that cover $M$, ie $M = \cup_\alpha U_\alpha$.

**Example 2.1.6**
The unit circle $S^1$ in the complex plane $\mathbb{C}$ can be described as the set of points

$$\{e^{it} \in \mathbb{C} : t \in [0, 2\pi]\}.$$

Let $U_1, U_2$ be the two open subsets of $S^1$

$$U_1 = \{e^{it} : t \in (-\pi, \pi)\}$$
$$U_2 = \{e^{it} : t \in (0, 2\pi)\}$$

and define $\phi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha = 1, 2$ by

$$\phi_1(e^{it}) = t \qquad t \in (-\pi, \pi)$$
$$\phi_2(e^{it}) = t \qquad t \in (0, 2\pi)$$

Then $\{(U_1, \phi_1), (U_2, \phi_2)\}$ form a $C^\infty$ atlas on $S^1$.

Although the smooth compatibility of charts is reflexive and symmetric, it is not transitive.

Indeed, Suppose $(U_1, \phi_1)$ is compatible with $(U_2, \phi_2)$ and $(U_2, \phi_2)$ is compatible with $(U_3, \phi_3)$. Note that the three coordinate functions are simultaneously defined on the intersection $U_{123}$. Thus the composite

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$

is smooth but only on $\phi_1(U_{123})$ and not necessarily on $\phi_1(U_{13})$. Here the equality means on the restriction to their common domains.

**Definition 2.1.5**
A chart $(V, \psi)$ is *compatible with an atlas* $\mathfrak{U}$ if it is compatible with all the charts of the atlas.

**Lemma 2.1.7**
Let $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts $(V, \psi), (W, \sigma)$ are both compatible with the atlas, then they are compatible with each other.

**Proof**
Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is $C^\infty$ at $\psi(p)$. Since $\mathfrak{U}$ is an atlas for $M$, we can choose $U_\alpha \ni p$ for some $\alpha$. Then $p$ is in the intersection $V \cap W \cap U_\alpha$.

From the remark above,

$$\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$$

is $C^\infty$ on $\psi(V \cap W \cap U_\alpha)$, and hence at $\psi(p)$. But $p \in V \cap W$ was arbitrary, hence $\sigma \circ \psi^{-1}$ is $C^\infty$ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is $C^\infty$ on $\sigma(V \cap W)$.

### 2.1.3 Smooth Manifolds

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be maximal if it is not contained in a larger atlas. In other words, if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.

**Definition 2.1.6 (Smooth Manifold)**
A smooth manifold is a topological manifold $M$ together with a maximal atlas.

the maximal atlas is also called a differentiable structure on $M$. A manifold is said to have dimension $n$ if all of its connected components have dimension $n$. A 1-dimensional manifold is also called a *curve*, a 2-dimensional manifold a *surface*, and an $n$-dimensional manifold an $n$-manifold as before.

In practice, we need not check that a topological manifold $M$ has a maximal atlas. The existence of any atlas on $M$ will do, due to the following proposition.

**Proposition 2.1.8**
Any atlas $\mathfrak{U}$ on a locally Euclidean space is contained in a unique maximal atlas.

**Proof**
Adjoin to the atlas $\mathfrak{U}$ all charts that are compatible with $\mathfrak{U}$. But then add the added charts are compartible with each other by the previous proposition. Thus the enlarged collection of charts is still an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{U}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Uniqueness follows by adding all charts from another maximal atlas and noting we have yet another atlas.

In summary, to show that a topological space $M$ is a smooth manifold, it suffices to check

(i) $M$ is Hausdorff and second countable

(ii) $M$ has a (not necessarily smooth) atlas.

From hereonforth, "manifold" will mean a smooth manifold. In the context of manifolds, we denote the standard coordinates on $\mathbb{R}^n$ by $r^1, \dots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of some manifold, we let $x^i := r^i \circ \phi$ be the $i$-th component of $\phi$ and write $\phi = (x^1, \dots, x^n)$ and $(U, \phi) = (U, x^1, \dots, x^n)$. Thus for $p \in U$, $(x^1(p), \dots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions are called *(local) coordinates on* $U$. By abuse of notation, we sometimes omit the $p$ so the notation $(x^1, \dots, x^n)$ standard for local coordinates on the open set $U$ and for a point in $\mathbb{R}^n$. By a *chart* $(U, \phi)$ *about* $p$ in a manifold $M$, we will mean a chart in the differentiable structure of $M$ such that $p \in U$.

### 2.1.4 Examples of Smooth Manifolds

**Example 2.1.9 (Euclidean Space)**
$\mathbb{R}^n$ is an $n$-manifold with a single chart $(\mathbb{R}^n, r^1, \dots, r^n)$ where the $r^i$'s are the standard coordinates on $\mathbb{R}^n$.

**Example 2.1.10 (Open Submanifolds)**
Any open subset $V$ of a manifold $M$ is also a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for $M$, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})\}$ is an atlas for $V$.

**Example 2.1.11 (0-Manifolds)**
In a 0-manifold, every point $p$ has a neighborhood that is homoemorphic to $\mathbb{R}^0$, meaning the neighborhood consists only $p$. Thus a 0-manifold is a discrete set. By second

countability, this discrete set must be countable.

**Example 2.1.12 (Graphs of Smooth Functions)**
For $A \subseteq \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^m$, the *graph* of $f$ is defined as

$$\Gamma(f) := \{(x, f(x)) \in A \times \mathbb{R}^m\}.$$

If $U$ is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is $C^\infty$, the two maps

$$\begin{aligned} &\phi : \Gamma(f) \to U && (x, f(x)) \mapsto x \\ &(1, f) : U \to \Gamma(f) && x \mapsto (x, f(x)) \end{aligned}$$

are continuous and inverse to each other and hence homeomorphisms.

The graph $\Gamma(f)$ of a smooth function $f$ has a single chart atlas $(\Gamma(f), \phi)$ and is therefore a smooth manifold.

This shows that many of the familiar surfaces of calculus, for example an illiptic paraboloid or a hyperbolic paraboloid, are manifolds.

**Example 2.1.13 (General Linear Groups)**
For any two positive integers $m$ and $n$, let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is by definition

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\} = \overset{-1}{\det}(\mathbb{R} - \{0\}).$$

Since the determinant is a polynomial of the entries of the matrix, $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ and is therefore a manifold.

By the same reasoning, $\mathrm{GL}(n, \mathbb{C})$ is a manifold of dimension $2n^2$.

**Example 2.1.14 (Unit Circle in the Plane)**
We know that $S^1$ is a manifold since it has a two chart atlas. Let us consider $S^1$ as a subset of $\mathbb{R}^2$ with defining equation $x^2 + y^2 = 1$.

We can cover $S^1$ with the upper, lower semicircles $U_1, U_2$ and right, left semicircles $U_3, U_4$. We can take $\phi_1, \phi_2$ to be the coordinate function $x$ and $\phi_3, \phi_4$ to be the coordinate function $y$. It can be checked that $\{(U_i, \phi_i)\}_{i=1}^4$ is a smooth atlas on $S^1$.

**Example 2.1.15 (Product Manifold)**
If $M, N$ are smooth manifolds, then $M \times N$ with its product topology is Hausdorff and second countable. To see that $M \times N$ is a manifold, we need to exhibit an atlas on it.

**Proposition 2.1.16**
Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ be smooth atlases for the manifolds $M, N$ of dimensions $m, n$ respectively. Then the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

of charts is a smooth atlas on $M \times N$. Thus $M \times N$ is a smooth $(m+n)$-manifold.

**Example 2.1.17**
The infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds. Moreover, the $n$-dimensional torus $\times_{i=1}^n S^1$ is a manifold.

**Remark 2.1.18** Let $S^n$ denote the unit sphere

$$(x^1)^2 + \dots (x^{n+1})^2 = 1$$

in $\mathbb{R}^{n+1}$. It is not hard to write down a smooth atlas of size $2n$ on $S^n$ by considering overlapping semi-spheres. The manifold $S^n$ with this differentiable structure is called the *standard n-sphere*.

One of the most surprising achievements in topology was John Milnor's discovery of exotic 7-spheres, smooth manifolds homeomorphic but not diffeomorphic to the standard 7-sphere. In 1963, Michel Kervaire and John Milnor determined that there are exactly 28 nondiffeomorphic differentiable structures on $S^7$.

It is known that in dimensions $< 4$, every topological manifold has a unique differentiable structure and in dimensions $> 4$, every compact topological manifold has a finite number of differentiable structures. Dimension 4 is a mystery and the statement that $S^4$ has a unique differentiable structure is called the *smooth Poincaré conjecture*.

Michel Kervaire was the first to construct an example of topological manifolds with no differentiable structure.

## 2.2 Smooth Maps on a Manifold

Having defined smooth manifolds, we now consider maps between them. Through coordinate charts, we can transfer the notion of smooth maps from Euclidean spaces to manifolds. It turns out that smooth compatibility of charts in an atlas means the smoothness of a map is independent of the choice of charts and is therefore well-defined. We give various criteria for the smoothness of a map and provide examples of such maps.

Then we transfer the notion of partial derivatives from Euclidean space to a coordinate chart on a manifold. This enables us to generalize the inverse function theorem to manifolds. Using this result, we formulate a criterion for a set of smooth functions to serve as local coordinates near a point.

### 2.2.1 Smooth Functions on a Manifold

**Definition 2.2.1 (Smooth at a Point)**
Let $M$ be a smooth $n$-manifold. A function $f : M \to \mathbb{R}$ is $C^\infty$/*smooth a a point* $p \in M$ if there is a chart $(U, \phi)$ about $p$ such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is $C^\infty$ at $\phi(p)$.

The function $f$ is said to be $C^\infty$ *on* $M$ if it is smooth at every point of $M$.

**Remark 2.2.1** The definition of smoothness is independent of the chosen chart. Indeed, if $(U, \phi), (V, \psi)$ are charts about $p \in M$ and $f \circ \phi^{-1}$ is smooth at $\phi(p)$, then by the smooth compatibility of the atlas,

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1})$$

is smooth at $\psi(p)$. Note we may need to restrict to a smaller neighborhood about $p$ but this does not change the result.

**Remark 2.2.2** In the definition above, $f : M \to \mathbb{R}$ is not assumed to be continuous. However, if $f$ is smooth at $p \in M$, then

$$f = (f \circ \phi^{-1}) \circ \phi$$

is a composition of continuous functions at $p$ and is therefore continuous at $p$.

Since we are only interested in smooth functions on an open set, there is no loss of generality in assuming at the outset that $f$ is continuous.

**Proposition 2.2.3 (Smoothness of Real-Valued Function)**
Let $M$ be a manifold of dimension $n$ and $f : M \to \mathbb{R}$ a real-valued function on $M$. The following are equivalent:

(i) $f$ is $C^\infty$

(ii) The manifold $m$ has an atlas such that for every chart $(U, \phi)$ in the atlas, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is $C^\infty$

(iii) For every chart $(V, \psi)$ on $M$, the function $f \circ \psi^{-1} : \psi(V) \to \mathbb{R}$ is $C^\infty$

The idea of this proposition will be a recurrent motif: to prove the smoothness of an object, it suffices to show a smoothness criterion holds on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on every chart of the manifold.

**Definition 2.2.2 (Pullback)**
Let $F : N \to M$ be a map and $h$ a function on $M$. The *pullback of $h$ by $F$*, denoted $F^*h$, is the composition $h \circ F$.

Thus a function $f : M \to \mathbb{R}$ is smooth on a chart $(U, \phi)$ if and only if its pullback $(\phi^{-1})^* f$ by $\phi^{-1}$ is smooth on $\phi(U)$.

### 2.2.2 Smooth Maps between Manifolds

**Definition 2.2.3 (Smooth at a Point)**
Let $M$ be a smooth $m$-manifold and $N$ a smooth $n$-manifold. A continuous map $F : N \to M$ is $C^\infty$ *at a point* $p \in N$ if there are charts $(V, \psi)$ about $F(p) \in M$ and $(U, \phi)$ about $p \in N$ such that the composition

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \mathbb{R}^m$$

is $C^\infty$ at $\phi(p)$.

The continuous map $F : N \to M$ is said to be $C^\infty$ if it is $C^\infty$ at every point of $N$.

**Remark 2.2.4** In the definition above, we assume $F : N \to M$ is continuous so $F^{-1}(V)$ is open in $N$. Thus smooth maps between manifolds are by definition continuous.

Note that if we take $M = \mathbb{R}$ with the single chart $(\mathbb{R}, \mathrm{Id})$, then we recover the definition of smooth maps $f : M \to \mathbb{R}$.

**Proposition 2.2.5**
Suppose $F : N \to M$ is smooth at $p \in N$. If $(U, \phi)$ is any chart about $p$ in $N$ and $(V, \psi)$ is any chart about $F(p) \in M$, then $\psi \circ F \circ \phi^{-1}$ is smooth at $\phi(p)$.

**Proof**
Since $F$ is smooth at $p \in N$, there are charts $(U_\alpha, \phi_\alpha)$ about $p \in N$ and $(V_\beta, \psi_\beta)$ about $F(p) \in M$ such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is smooth at $\phi_\alpha(p)$.

By the smooth compatility of charts in an atlas, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are smooth on open subsets of Euclidean space. Hence the composition

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1})$$

is $C^\infty$ at $\phi(p)$.

Next, we present a way to check smoothness of a map without specifying points in the domain.

**Proposition 2.2.6**
Let $N, M$ be smooth manifolds and $F : N \to M$ be continuous. The following are equivalent:

(i) $F$ is $C^\infty$

(ii) There are atlases $\mathfrak{U}$ for $N$ and $\mathfrak{B}$ for $M$ such that for every chart $(U, \phi)$ in $\mathfrak{U}$ and $(V, \psi)$ in $\mathfrak{B}$, the following map is $C^\infty$

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

(iii) For every chart $(U, \phi)$ on $N$ and $(V, \psi)$ on $M$, the following map is $C^\infty$

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

**Proposition 2.2.7 (Composition of Smooth Maps)**
If $F : N \to M$ and $G : M \to P$ are smooth maps of manifolds, then the composition $G \circ F : N \to P$ is smooth.

**Proof**
Let $(U, \phi), (V, \psi)$ and $(W, \sigma)$ be charts on $N, M, P$ respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1})$$

under a suitable restriction. By a previous proposition, this is then a composition of smooth maps on Euclidean space and is therefore smooth. The same proposition now ensures $G \circ F$ is smooth. Note we need to let $\psi$ vary over all charts for $M$ to check the composition is smooth over its entire domain.

### 2.2.3 Diffeomorphisms

**Definition 2.2.4 (Diffeomorphism of Manifolds)**
A *diffeomorphism* of manifolds is a bijective $C^\infty$ map $F : N \to M$ whose inverse $F^{-1}$ is also smooth.

The next two propositions hsow that coordinate maps are diffeomorphisms and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

**Proposition 2.2.8**
If $(U, \phi)$ is a chart on an $n$-manifold $M$, the coordinate map $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism.

**Proof**
$\phi$ is a homeomorphism by definition so it suffices to check that $\phi, \phi^{-1}$ are both smooth.

We use a previous proposition to check that $\phi : U \to \phi(U)$ is a smooth map between the open submanifolds $U, \phi(U)$ of $M, \mathbb{R}^n$ respectively. It suffices to show that $\phi$ is smooth with respect to particular atlases for $U, \phi(U)$. Indeed, consider the single chart atlases $\{(U, \phi)\}$ and $\{(\phi(U), \mathrm{Id})\}$. We see that

$$\mathrm{Id} \circ \phi \circ \phi^{-1} = \mathrm{Id}$$

is trivially $C^\infty$. The same atlases can be used to show that $\phi^{-1}$ is also $C^\infty$.

**Proposition 2.2.9**
Let $U \subseteq M$ be an open subset of the $n$-manifold $M$. If $F : U \to F(U) \subseteq \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then $(U, F)$ is a chart in the differentiable structure of $M$.

**Proof**
For any chart $(U_\alpha, \phi_\alpha)$ in the maximal atlas of $M$, both $\phi_\alpha, \phi_\alpha^{-1}$ are $C^\infty$ by the previous proposition. As compositions of smooth maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are $C^\infty$. Hence $(U, F)$ is compatible with the maximal atlas so that $(U, F)$ must belong in the atlas by maximality.

### 2.2.4 Smoothness in Terms of Components

We now derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

**Proposition 2.2.10 (Smoothness of Vector-Valued Functions)**
Let $N$ be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. The following are equivalent:

(i) $F$ is $C^\infty$

(ii) $N$ has an atlas such that for every chart $(U, \phi)$ in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is smooth

(iii) For every chart $(U, \phi)$ on $N$, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is $C^\infty$

**Proposition 2.2.11 (Smoothness in terms of Components)**
Let $N$ be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is $C^\infty$ if and only if its component functions $F^1, \ldots, F^m : N \to \mathbb{R}$ are all $C^\infty$.

**Example 2.2.12**
The map $F : \mathbb{R} \to S^1$ given by

$$F(t) = (\cos t, \sin t)$$

is $C^\infty$.

**Proposition 2.2.13 (Smoothness in term of Vector-Valued Functions)**
Let $F : N \to M$ be a continuous maps between an $n$-manifold and an $m$-manifold. The following are equivalent:

(i) $F$ is $C^\infty$

(ii) $M$ has an atlas such that for every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ in the atlas, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is smooth

(iii) For every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ on $M$, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is $C^\infty$

This smoothness criterion then translates into a smoothness criterion in terms of the components of the map.

**Proposition 2.2.14 (Smoothness in terms of Components)**
Let $F : N \to M$ be a continuous map from an $n$-manifold to an $m$-manifold. The following are equivalent.

(i) $F$ is smooth

(ii) $M$ has an atlas such that for every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ in the atlas, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of $F$ relative to the chart are all $C^\infty$

(iii) For every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ on $M$, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of $F$ relative to the chart are all smooth

### 2.2.5 Examples of Smooth Maps

**Proposition 2.2.15 (Smoothness of Projection Map)**
Let $M, N$ be manifolds and $\pi : M \times N \to M$ given by

$$\pi(p, q) = p$$

the projection to the first factor. Then $\pi$ is a smooth map.

**Proof**
Fix $(p,q) \in M \times N$ and let $(U,\phi) = (U, x^1, \ldots, x^m)$ and $(V,\psi) = (V, y^1, \ldots, y^m)$ be coordinate neighborhoods of $p, q$ respectively. We know that $(U \times V, \phi \times \psi) = (U \times V, x^1, \ldots, x^m, y^1, \ldots, y^n)$ is a coordinate neighborhood of $(p,q)$. But then

$$\phi \circ \pi \circ (\phi \times \psi)^{-1}(a^1, \ldots, a^m, b^1, \ldots, b^n) = (a^1, \ldots, a^m)$$

is a smooth map from $(\phi \times \psi)(U \times V) \subseteq \mathbb{R}^{m+n}$ to $\phi(U) \subseteq \mathbb{R}^m$. Thus $\pi$ is smooth at $(p,q)$ and by the arbitrary choice of $(p,q)$, $\pi$ is $C^\infty$ on $M \times N$.

**Proposition 2.2.16**
Let $M_1, M_2, N$ be manifolds of dimension $m_1, m_2, n$ respectively. A map $(f_1, f_2) : N \to M_1 \times M_2$ is smooth if and only if $f_1, f_2$ are both smooth.

**Proof**
If $(f_1, f_2)$ is smooth, then the projection is smooth by the previous proposition.

Conversely, suppose $f_1, f_2$ are smooth. Then the components of $f_1, f_2$ on any local coordinate neighborhoods of $M_1, M_2$ respectively are smooth. But there is an atlas $\mathfrak{U}$ of $M_1 \times M_2$ which is just the cross product of charts from $M_1, M_2$ and the components of $(f_1, f_2)$ on the local coordinate neighborhoods of $\mathfrak{U}$ are smooth by assumption. This suffices to show that $(f_1, f_2)$ is indeed smooth.

**Proposition 2.2.17**
A smooth function $f(x,y)$ on $\mathbb{R}^2$ restricts to a smooth function on $S^1$.

**Proof (Sketch)**
We denote a point on $S^1$ as $p = (a,b)$ and use $x, y$ to mean the standard coordinate functions on $\mathbb{R}^2$. If we show that $x, y$ restricts to $C^\infty$ functions on $S^1$, then the inclusion map $i(p) = (x(p), y(p))$ is then smooth on $S^1$ and the composition $f|_{S^1} = f \circ i$ will therefore be smooth.

We can check that $x, y$ are smooth on a particular atlas.

The definition of a smooth map between manifolds allows us to define a Lie group.

**Definition 2.2.5 (Lie Group)**
A *Lie group* is a smooth manifold $G$ with a group structure such that the multiplication map

$$\mu : G \times G \to G$$

and inverse map

$$\iota : G \to G$$

are both smooth.

We can similarly define a *topological group* which is a topological space with a group structure wher the multiplication and inverse maps are both continuous. Note a topological group is not required to the a topological manifold.

**Example 2.2.18**

(i) $\mathbb{R}^n$ is a Lie group under addition

(ii) $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is a Lie group under multiplication

(iii) The unit circle $S^1 \subseteq \mathbb{C}^\times$ is a Lie group under multiplication

(iv) The Cartesian product $G_1 \times G_2$ of two Lie groups $(G_1, \mu_1)$ and $(G_2, \mu_2)$ is a Lie group under coordinatewise multiplicative $\mu_1 \times \mu_2$

**Example 2.2.19 (General Linear Group)**
The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a open submanifold of $\mathbb{R}^{n\times m}$. Matrix multiplication is a polynomial in the coordinates of the input and is hence a smooth map. By Cramer's rule, the entries in the inverse of a matrix is a rational function of the input entries. Thus the inverse map is also smooth and $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

The notation for matrices is difficult. $A \in \mathbb{R}^{n\times n}$ can represent a linear transformation $y = Ax$ so $y^i = \sum_j a^i_j x^j$ and we write $A = [a^i_j]$. But $A$ can also represent a bilinear form $\langle x, y\rangle = x^T Ay$. But then $\langle x, y\rangle = \sum_{i,j} x^i a_{ij} y^j$ and $A = [a_{ij}]$.

In the absence of context, we write $A = [a_{ij}]$.

### 2.2.6 Partial Derivatives

**Definition 2.2.6**
Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on an $n$-manifold $M$, $r^1, \ldots, r^n$ the standard coordinates on $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ a smooth function.
The *partial derivative* $\partial f / \partial x^i$ of $f$ with respect to $x^i$ at $p$ is given by

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p f &:= \frac{\partial f}{\partial x^i}(p) \\ &:= \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) \\ &:= \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}). \end{aligned}$$

The partial derivative $\partial f / \partial x^i$ is $C^\infty$ on $U$ because its pullback $(\partial f / \partial x^i) \circ \phi^{-1}$ is $C^\infty$ on $\phi(U)$.

The next proposition states that partial derivatives on a manifold satisfy the same duality property $\partial r^i / \partial r^j = \delta^i_j$ as the coordinate functions on $\mathbb{R}^n$.

**Proposition 2.2.20**
Suppose $(U, x^1, \ldots, x^n)$ is a chart on a manifold. Then $\partial x^i / \partial x^j = \delta^i_j$.

**Proof**
At a point $p \in U$, by the definition of $\partial / \partial x^j|_p$,

$$\begin{aligned} \frac{\partial x^i}{\partial x^j}(p) &= \frac{\partial (x^i \circ \phi^{-1})}{\partial r^j}(\phi(p)) \\ &= \frac{\partial (r^i \circ \phi \circ \phi^{-1})}{\partial r^j}(\phi(p)) \\ &= \frac{\partial r^i}{\partial r^j}(\phi(p)) \\ &= \delta^i_j. \end{aligned}$$

**Definition 2.2.7 (Jacobian)**
Let $F : N \to M$ be a smooth map $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on $N$, and $(V, \psi) = (V, y^1, \ldots, y^m)$ be a chart on $M$ such that $F(U) \subseteq V$.
Let $F^i = y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$ denote the $i$-th component of $F$ in the chart $(V, \psi)$. Then the matrix $[\partial F^i / \partial x^j]$ is the *Jacobian matrix of* $F$ *relative to the charts* $(U, \phi), (V, \psi)$.

If $M, N$ have the same dimension, then $\det[\partial F^i/\partial x^j]$ is known as the *Jacobian determinant of $F$ relative to the two charts.* The Jacobian determinant is also written as $\partial(F^1,\ldots,F^n)/\partial(x^1,\ldots,x^n)$.

**Example 2.2.21 (Jacobian Matrix of a Transition Map)**
Let $(U,\phi) = (U, x^1,\ldots,x^n)$ and $(V,\psi) = (V, y^1,\ldots,y^n)$ be overlapping charts on a manifold $M$. The transition map $\psi\circ\phi^{-1} : \phi(U\cap V)\to\psi(U\cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Its Jacobian matrix $J(\psi\circ\phi^{-1})$ at $\phi(p)$ is the matrix $[\partial y^i/\partial x^j]$ of partial derivatives at $p$.

Indeed,

$$\begin{aligned}\frac{\partial(\psi\circ\phi^{-1})^i}{\partial r^j}(\phi(p)) &= \frac{\partial(r^i\circ\psi\circ\phi^{-1})}{\partial r^j}(\phi(p))\\ &= \frac{\partial(y^i\circ\phi^{-1})}{\partial r^j}(\phi(p))\\ &= \frac{\partial y^i}{\partial x^j}(p).\end{aligned}$$

### 2.2.7 Inverse Function Theorem

Recall that any diffeomorphism $F : U \to F(U) \subseteq \mathbb{R}^n$ of an open subset $U$ of a manifold can be interpreted as a chart on $U$.

**Definition 2.2.8 (Locally Invertible/Local Diffeomorphism)**
We say that a $C^\infty$ map $F : N \to M$ is *locally invertible* or a *local diffeomorphism* at $p\in N$ if $p$ has a neighborhood $U$ on which $F|_U : U \to F(U)$ is a diffeomorphism.

Given $n$ smooth functions $F^1,\ldots,F^n$ in a neighborhood of a point $p \in N$ of an $n$-manifold, we would like to know whether they form a coordiante system, possibly on a smaller neighborhood of $p$. The inverse function theorem provdies an answer.

**Theorem 2.2.22 (Inverse Function Theorem for $\mathbb{R}^n$)**
Let $F : W \to \mathbb{R}^n$ be a smooth map defined on an open subset $W \subseteq \mathbb{R}^n$. For any $p \in W$, the map $F$ is locally invertible at $p$ if and only if the Jacobian determinant $\det[\partial F^i/\partial r^j(p)]$ is non-zero.

This theorem is typically proved in a standard course on multivariate calculus/real analysis. Since the statement is a local result, it easily translates to manifolds.

**Theorem 2.2.23 (Inverse Function Theorem for Manifolds)**
Let $F : N \to M$ be a smooth map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = (U, x^1, \dots, x^n)$ about $p \in N$ and $(V, \psi) = (V, y^1, \dots, y^n)$ about $F(p) \in M$ we have $F(U) \subseteq V$. Set $F^i := y^i \circ F$. Then $F$ is locally invertible at $p$ if and only if its Jacobian determinant $\det[\partial F^i / \partial x^j(p)]$ is non-zero.

**Proof**
Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of $F$ relative to the charts $(U, \phi)$ and $(V, \psi)$ is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial (r^i \circ \psi \circ F)}{\partial x^j}(p)\right] = \left[\frac{\partial (r^i \circ \psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right],$$

which is precisely the Jacobian matrix at $\phi(p)$ of the map

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

between two open subsets of $\mathbb{R}^n$.

By the inverse function theorem for $\mathbb{R}^n$,

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[\frac{\partial r^i \circ (\psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right] \neq 0$$

if and only if $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. Since $\psi, \phi$ are diffeomorphisms, this last statement is equivalent to the local invertibility of $F$ at $p$.

We typically apply the inverse function theorem in the following form.

**Corollary 2.2.23.1**
Let $N$ be an $n$-manifold. A set of $n$ smooth functions $F^1, \dots, F^n$ defined on a coordinate neighborhood $(U, x^1, \dots, x^n)$ of a point $p \in N$ forms a coordinate system aout $p$ if and only if the Jacobian determinant $\det[\partial F^i / \partial x^j(p)]$ is non-zero.

**Example 2.2.24**
Consider the functions $x^2 + y^2 - 1, y$ on $\mathbb{R}^2$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$(x, y) \mapsto (x^2 + y^2 - 1, y).$$

$F$ can serve as a coordinate map in a neighborhood of $p$ if and only if it is a local diffeomorphism at $p$. The inverse function theorem states that this is equal to the condition

$$0 \neq \frac{\partial (F^1, F^2)}{\partial (x, y)} = \det\begin{bmatrix} 2x & 2y \\ 0 & 1 \end{bmatrix} = 2x.$$

Thus $F$ can serve as a coordinate system at any point not on the $y$-axis.

## 2.3 Quotients

Recall that given an equivalence relation on a topological space, we can always imbue the quotient space with a topology such that the natural projection map is continuous. However, even if the original space is a mnaifold, a quotient space is often not a manifold. We study conditions under which a quotient space remains second countable and Hausdorff.

### 2.3.1 The Quotient Topology

Recall that an equivalence relation $\sim$ on a set $S$ is reflexive, symmetric, and transitive relation. The *equivalence class* $[x]$ of $x \in S$ is the set of all elements in $S$ equivalent to $x$. We denote the set of equivalence classes by $S/\sim$ and refer to this set as the *quotient* of $S$ by $\sim$. The *natural projection map* $\pi : S \to S/\sim$ is given by $x \mapsto [x]$.

If $S$ is a topological space, we can define a topology on $S/\sim$ by declaring a set $U \subseteq S/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in $S$. Note the projection map is automatically continuous by definition under this topology. We call this the *quotient topology* on $S/\sim$ and under this topology, $S/\sim$ is called the *quotient space*.

### 2.3.2 Continuity of a Map on a Quotient

Let $\sim$ be an equivalence relation on the topological space $S$ and give $S/\sim$ the quotient topology. Suppose $f : S \to Y$ takes values in another topology space $Y$ is constant on each equivalence class. Then it induces a map $\bar{f} : S/\sim \to Y$ given by

$$[p] \mapsto f(p).$$

**Proposition 2.3.1**
The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

This proposition gives a useful criterion for verifying continuous of $\bar{f}$: lift the function to $\bar{f} = f \circ \pi$ on $S$ and check the continuous of the lifted map $f$. The proof of the proposition follows by checking the definitions.

### 2.3.3 Identification of Subset to a Point

If $A$ is a subspace of a topological space $S$, we can define a relation $\sim$ on $S$ by declaring $x \sim y$ for all $x, y \in A$. This is an equivalence relation and we say the quotient space $S/\sim$ is obtained from $S$ by *identifying $A$ to a point.*

**Example 2.3.2**
Let $I = [0,1]$ be the closed unit interval and $I/\sim$ the quotient space obtained from $I$ by identifying $\{0,1\}$. Denote by $S^1$ the unit circle in the complex plane. The function $f : I \to S^1$ given by

$$x \mapsto \exp(2\pi i x)$$

assumes the same value at 0 and 1 and so induces a function $\bar{f} : I/\sim \to S^1$.

**Proposition 2.3.3**
The function $\bar{f} : I/\sim \to S^1$ is a homeomorphism.

**Proof**
Since $f$ is continuous, $\bar{f}$ is also continuous. Clearly, $\bar{f}$ is a bijection. As the continuous image of the compact set $I$, the quotient $I/\sim$ is compact. Thus $\bar{f}$ is a continuous bijection from a compact space to a Hausdorff space.

From elementary point-set topology, the continuous image of compact (and thus closed) spaces are again compact (and thus closed) so $\bar{f}$ is a closed mapping. But then the inverse is continuous since the pre-image of closed sets are closed. Hence $\bar{f}$ is a homeomorphism.

### 2.3.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not in general preserve the Hausdorff or second countability properties. We can derive a necessary condition through the following observation: Every singleton in a Hausdorff space is closed. Thus if $\pi : S \to S/\sim$ is the projection and the quotient is Hausdorff, then for any $p \in S$, its image $\{\pi(p)\}$ is closed in $S/\sim$. By the continuity of $\pi$, the inverse image $\pi^{-1}(\pi(p)) = [p]$ is closed in $S$.

**Proposition 2.3.4**
If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point $p \in S$ is necessarily closed in $S$.

**Example 2.3.5**
Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $(0, \infty)$ to a point. Then the quotient $\mathbb{R}/\sim$ is not Hausdorff since the $(0, \infty)$ is not closed in $\mathbb{R}$.

### 2.3.5 Open Equivalence Relations

We derive conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f : X \to Y$ of topological spaces is an *open mapping* if the image of open sets under $f$ is open.

**Definition 2.3.1 (Open Equivalence Relation)**
An equivalence relation $\sim$ on a topological space $S$ is said to be *open* if the projection map $\pi : S \to S/\sim$ is open.

In other words, $\sim$ on $S$ is open if and only if for every open set $U \subseteq S$, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalence to some point of $U$ is open.

**Example 2.3.6**
Consider $\mathbb{R}$ with $\sim$ the equivalence relation that identifies $1, -1$. Then

$$\pi^{-1}(\pi(-2,0)) = (-2,0) \cup \{1\}$$

is not open in $\mathbb{R}$ so $\sim$ cannot be open.

Given an equivalence relation $\sim$ on $S$, let $R \subseteq S \times S$ be the subset that defines the relation

$$R := \{(x,y) \in S \times S : x \sim y\}.$$

We say that $R$ is the *graph* of $\sim$.

**Theorem 2.3.7**
Let $\sim$ be an open equivalence relation on a topological space $S$. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph $R$ of $\sim$ is closed in $S \times S$.

**Proof**
By definition, $R$ is closed in $S \times S$ if and only if $(S \times S) - R$ is open in $S \times S$. In other words, for every $(x,y) \in S \times S - R$, there is a basic open set $U \times V$ containing $(x,y)$ such that $U \times V \subseteq S \times S - R$. This can be restated as for every $x \nsim y \in S$, there are neighborhoods $U \ni x, V \ni y$ in $S$ such that no element of $U$ is equivalent to an element of $V$. This is equivalent to the statement that for any two points $[x] \neq [y] \in S/\sim$, there are neighborhoods $U \ni x, V \ni y$ in $S$ such that $\pi(U) \cap \pi(V) = \varnothing$ in $S/\sim$.

($\implies$) Suppose $R$ is closed in $S \times S$ and consider the last equivalent statement. By definition, $S/\sim$ is Hausdorff.

($\impliedby$) Conversely, suppose that $S/\sim$ is Hausdorff. Choose $[x] \neq [y] \in S/\sim$. There are disjoint open sets $A, B \subseteq S/\sim$ such that $[x] \in A, [y] \in B$. By the surjectivity of $\pi$, we have $A = \pi(\pi^{-1}A)$ and $\pi(\pi^{-1}B)$. Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U, y \in V$ and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $S/\sim$.

This is the last equivalent condition to $R$ being closed in $S \times S$.

**Corollary 2.3.7.1**
A topological space $S$ is Hausdorff if and only if the diagonal $\Delta$ in $S \times S$ is closed.

**Theorem 2.3.8**
Let $\sim$ be an open equivalence relation on a topological space $S$ with projection $\pi : S \to S/\sim$. If $\mathcal{B} := \{B_\alpha\}$ is a basis of $S$, then the image $\{\pi(B_\alpha)\}$ under $\pi$ is a basis for $S/\sim$.

**Proof**
Let $W \subseteq S/\sim$ be open and pick $[x] \in W$ for some $x \in S$. Then $\pi^{-1}(W) \ni x$ is open in $S$ and there is some $B_\alpha \ni x$ contained in $\pi^{-1}(W)$. Then $[x] = \pi(x) \in \pi(B_\alpha) \subseteq W$ as desired.

**Corollary 2.3.8.1**
If $\sim$ is an open equivalence relation on a second-countable space $S$, then the quotient space $S/\sim$ is second countable.

### 2.3.6 The Real Projective Space

**Definition 2.3.2 (Real Projective Space $\mathbb{R}P^n$)**
Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by

$$x \sim y \iff \exists t \geq 0, y = tx.$$

The *real projective space* $\mathbb{R}P^n$ is defined as the quotient space

$$(\mathbb{R}^{n+1} - \{0\})/\sim.$$

We denote the equivalence class of a point $(a^0, \ldots, a^n) \in \mathbb{R}^{n+1} - \{0\}$ by $[a^0, \ldots, a^n]$ and let $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection. We call $[a^0, \ldots, a^n]$ *homogeneous coordinates* on $\mathbb{R}P^n$.

Geometrically speaking, two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. Such a line uniquely determines a pair of antipodal points on $S^n$. This suggests that we define an equivalence relation on $S^n$ by identifying antipodal points:

$$x \sim y \iff x = \pm y.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$.

**Proposition 2.3.9 (Real Projective Space as a Quotient of a Sphere)**
For $x = (x^1, \dots, x^n) \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean norm of $x$. The map $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$x \mapsto \frac{x}{\|x\|}$$

induces a homeomorphism $\bar{f} : \mathbb{R}P^n \to S^n/\sim$.

**Proof**
Since $f$ is continuous on $\mathbb{R}^{n+1} - \{0\}$, $\pi_\sim \circ f : \mathbb{R}^{n+1} - \{0\} \to S^n/\sim$ is a composition of continuous functions and is therefore continuous. By a previous proposition, $\pi_\sim \circ f$ is constant on the equivalence classes of $\mathbb{R}P^n$ and the natural quotient function $\bar{f}$ is therefore also continuous.

Consider the identity map $\mathrm{Id} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. This is certainly continuous. But then $g : S^n \to \mathbb{R}^{n+1}/\sim$ given by $g = \pi_\sim \circ \mathrm{Id} \circ \iota$ where $\iota$ is the the inclusion map is continuous. Since $g$ is constant over the equivalence classes of $\sim$, the natural quotient function $\bar{g}$ is also continuous.

By observation,

$$\bar{f}[x^0, \dots, x^n] = \left[\frac{x}{\|x\|}\right]$$
$$\bar{g}[x] = [x^0, \dots, x^n]$$

are inverses and we conclude the proof.

**Example 2.3.10 (The Real Projective Line $\mathbb{R}P^1$)**
$\mathbb{R}P^1$ is homeomorphic to the quotient $S^1/\sim$, which in turn homeomorphic to the closed upper semicircle with two identified endpoints. This is in turn homeomorphic to $S^1$ and so $\mathbb{R}P^1 \cong S^1$.

**Example 2.3.11 (The Real Projective Plane $\mathbb{R}P^2$)**
$\mathbb{R}P^2$ is homeomorphic to $S^2/\sim$. We can interpret this as the closed upper hemisphere $H^2 \subseteq \mathbb{R}^3$ in which each pair of antipodal points on the equator is identified, denoted $H^2/\sim$. Let $D^2 \subseteq \mathbb{R}^2$ denote the closed unit disk and remark the map $\varphi : H^2 \to D^2$ given by

$$(x, y, z) \mapsto (x, y)$$

is a homeomorphism. Let $D^2/\sim$ denote the closed unit disk with antipodal points identified. $\varphi$ induces a homeomorphism from $H^2/\sim \to D^2\sim$. Thus $\mathbb{R}P^2 \cong D^2/\sim$.

$\mathbb{R}P^2$ cannot be embedded as a submanifold into $\mathbb{R}^3$. However, if we allow self-intersection, we can map $\mathbb{R}P^2 \to \mathbb{R}^3$ as a *cross-cap*. This map is not injective.

**Proposition 2.3.12**
The equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ in the definition of $\mathbb{R}P^n$ is an open equivalence relation.

**Proof**
For an open subset $U \subseteq \mathbb{R}^{n+1} - \{0\}$, the image $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} - \{0\}$. But $\pi^{-1}(\pi(U))$ consists of all nonzero scalar multiples of points of $U$. That is,

$$\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}^\times} tU = \bigcup \{tp : p \in U\}.$$

But multiplication by $t \in \mathbb{R}^\times$ is a homeomorphism of $\mathbb{R}^{n+1} - \{0\}$, the set $tU$ is open for any $t$ and so the union of all such sets remains open.

**Corollary 2.3.12.1**
The real projective space $\mathbb{R}P^n$ is second countable.

**Proposition 2.3.13**
The real projective space $\mathbb{R}P^n$ is Hausdorff.

**Proof**
Let $S = \mathbb{R}^{n+1} - \{0\}$ and consider the set

$$R = \{(x, y) \in S \times S : \exists t \in \mathbb{R}^\times, y = tx\}.$$

If we write $x, y$ as column vectors, then $\begin{bmatrix} x & y \end{bmatrix}$ is an $(n+1) \times 2$ matrix. $R$ can then be characterized as the set of matrices $\begin{bmatrix} x & y \end{bmatrix} \in S \times S$ of rank at most 1. By a standard fact from linear algebra, this is equivalent to all $2 \times 2$ minors of $\begin{bmatrix} x & y \end{bmatrix}$ being zero. Thus as the zero set of finitely many polynomials, $\mathbb{R}$ is a closed subset of $S \times S$. Since $\sim$ is an open equivalence relation on $S$, and $R$ is closed in $S \times S$, a previous theorem informs us that $S/\!\sim \,\cong \mathbb{R}P^n$ is Hausdorff.

### 2.3.7 The Standard Smooth Atlas on a Real Projective Space

Let $[a^0, \ldots, a^n]$ be homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although $a^0$ is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \neq 0$ is indepedent of the choice of representative. Hence the condition $a^0 \neq 0$ makes sense on $\mathbb{R}P^n$ and we can define

$$U_i := \{[a^0, \ldots, a^n] : a^i \neq 0\}$$

for each $i \in 0, 1, \ldots, n$. Define $\phi_i : U_i \to \mathbb{R}^n$ given by

$$[a^0, \ldots, a^n] \mapsto \left( \frac{a^0}{a^i}, \ldots, \frac{\widehat{a^i}}{a^i}, \ldots, \frac{a^n}{a^i} \right)$$

where the carat sign indicates the entry is to be omitted. This map has a continuous inverse

$$(b^1, \ldots, b^n) \mapsto [b^1, \ldots, \underbrace{1}_{i\text{-th entry}}, \ldots, b^n].$$

It follows that $\mathbb{R}P^n$ is locally Euclidean with $(U_i, \phi_i)$ as charts.

On the intersection $U_0 \cap U_1$, we have $a^0 \neq 0$ and $a^1 \neq 0$, and there are two coordinates systems. We refer to the coordinate functions on $U_0$ as $x^1, \ldots, x^n$

$$x^i = \frac{a^i}{a^0}, i \in [n].$$

and the coordinate functions on $U_1$ as $y^1, \ldots, y^n$

$$y^1 = \frac{a^0}{a^1}, y^i = \frac{a^i}{a^1}, i \in \{2, \ldots, n\}.$$

So on $U_0 \cap U_1$,

$$(\phi_1 \circ \phi_0^{-1})(x) = \left( \frac{1}{x^1}, \frac{x^2}{x^1}, \ldots, \frac{x^n}{x^1} \right).$$

This is a smooth function since $x^1 \neq 0$ on $\phi_0(U_0 \cap U_1)$. A similar formula holds on any other $U_i \cap U_j$. Thus the collection $\{(U_i, \phi_i)\}_{i=0,\ldots,n}$ is a smooth atlas for $\mathbb{R}P^n$, called the *standard atlas*. We conclude that $\mathbb{R}P^n$ is a smooth manifold as desired.

### 2.3.8 The Grassmannian Manifold

**Proposition 2.3.14 (Orbit Space of a Continuous Group Action)**
Suppose a right action of a topological group $G$ on a topolgical space $S$ is continuous. Define two points $x, y$ to be equivalent if they are in the same orbit, so if there is some $g \in G$ such that $y = xg$.
Define $S/G$ to be the quotient space, also known as the *orbit space* of the action. Then the projection map $\pi : S \to S/G$ is an open map.

**Proof**
It suffices to show that $\pi^{-1}(\pi(U))$ is open for every $U \subseteq S$ open.

Note that right multiplication by some $g \in G$ is a homeomorphism $S \to S$. We can see this by considering $\{g\}$ as a topological space under the subspace topology and using the continuity of the inclusion map.

We can express
$$\pi^{-1}(\pi(U)) = \bigcup_{g\in G} Ug$$
as a union of open sets which is therefore open as desired.

The Grassmannian $G(k,n)$ is the set of all $k$-planes in $\mathbb{R}^n$, or in other words, the set of all $k$-dimensional subspaces of $\mathbb{R}^n$. Such a subspace is completely determined by a full-rank matrix $A \in \mathbb{R}^{n\times k}$, known as a *matrix representative* of the $k$-plane. Two matrices $A, B$ determine the same $k$-plane if there is a change of basis matrix $g \in \mathrm{GL}(k,\mathbb{R})$ such that $B = Ag$.

Let $F(k,n)$ be the set of all $n\times k$ matrices of rank $k$, topologized as a subspace of $\mathbb{R}^{n\times k}$. We define the equivalence relation
$$A \times B \iff \exists g \in \mathrm{GL}(k,\mathbb{R}), B = Ag.$$

Then there is a bijection between $G(k,n)$ and $F(k,n)/\sim$ and we give $G(k,n)$ the subspace topology $F(k,n)/\sim$.

**Proposition 2.3.15**
$F(k,n)/\sim$ is Hausdorff and second-countable.

**Proof**
We can view $F/\sim$ as the orbit space of the action
$$\begin{aligned} F(n,k)\times \mathrm{GL}(k,\mathbb{R}) &\to F(n,k) \\ (A,g) &\mapsto Ag \end{aligned}$$
which is certainly a continuous group action. But then $\sim$ is an open relation by the previous proposition. It follows immediately that $F(n,k)/\sim$ is second countable.

In order to show that $F(n,k)/\sim$ is Hausdorff, it suffices to show the graph $R \subseteq F(n,k)\times F(n,k)$ of the relation $\sim$ is closed. We have
$$\begin{aligned} R &= \{(A,B) : \exists g, B = Ag\} \\ &= \{(A,B) : \operatorname{rank} \begin{bmatrix} A & B \end{bmatrix} \leq k\} \\ &= \{(A,B) : \text{all } (k+1)\times(k+1) \text{ minors have determinant } 0\}. \end{aligned}$$

The last set is a finite intersection of zero-sets of polynomials, which is certainly closed.

We now construct a smooth atlas for $G(n,k)$.

Let
$$\mathcal{I} := \{(i_1,\dots,i_k) : 1 \leq i_1 < \cdots < i_k \leq n\}$$

be the set of increasing row indices. Write $A_I$ to denote the submatrix of $A$ made of the rows of $A$ indexed by $I$. For $I \in \mathcal{I}$, define

$$V_I := \{A \in F(k,n) : \det A_I \neq 0\}$$

and remark that this is an open set. Define $\tilde{\phi}_I : V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$A \mapsto (AA_I^{-1})_{I^c} = A_{I^c}A_I^{-1}$$

where $(\cdot)_{I^c}$ indicates taking the rows not indexed by $I$. This is certainly a continuous map.

Note that if $A \in V_I$, then $Ag \in V_I$ for any $g \in \mathrm{GL}(k,\mathbb{R})$ since

$$(Ag)_I = A_I g \in \mathrm{GL}(k,\mathbb{R}).$$

**Proposition 2.3.16**
Let $\phi_I : (U_I := F(k,n)/\sim) \to \mathbb{R}^{(n-k)\times k}$ denote the map induced by $\tilde{\phi}_I$. Then $\phi_I$ is well-defined and

$$\{(U_I, \phi_I)\}_{I\in\mathcal{I}}$$

is a smooth structure on $G(n,k)$.

**Proof**
We first check that $\tilde{\phi}_I$ is constant on the orbits of $\mathrm{GL}(k,n)$. Indeed,

$$\begin{aligned}(Ag)_{I^c}(Ag)_I^{-1} &= A_{I^c}gg^{-1}A_I^{-1} \\ &= A_{I^c}A_I^{-1}.\end{aligned}$$

Then we note that $\tilde{\phi}_I : V_I \to \mathbb{R}^{(n-k)\times k}$ is continuous since matrix inversion and multiplication are smooth mappings. But then $\phi_I : U_I \to \mathbb{R}^{(n-k)\times k}$ is continuous as desired.

To see that it has a continuous inverse, define $B^{(I)}$ as the matrix obtained from $B$ by inserting the identity matrix in the $I$-th rows and consider the mapping

$$\begin{aligned}\mathbb{R}^{(n-k)\times k} &\to U_I \\ B &\mapsto [B^{(I)}].\end{aligned}$$

This is certainly continuous since it is the composition of a continuous map with the projection map. To see that this is the inverse of $\phi_I$, observe that every element $[A] \in U_I$ has a canonical representative

$$AA_I^{-1}$$

whose $I$-th rows form the identity matrix. $\tilde{\phi}$ selects the $I^c$ rows from the canonical representation and the map above sends the $I^c$ rows back to the equivalence class of the canonical representation.

Finally, we need to show the smooth compatibility of charts. Pick $I, J \in \mathcal{I}$ and consider

$$\begin{aligned}\phi_I \circ \phi_J^{-1} : \phi_J(U_I \cap U_J) &\to \phi_I(U_I \cap U_J) \\ B &\mapsto (B^{(J)})_{I^c}(B^{(J)})_I^{-1}.\end{aligned}$$

This is smooth since matrix inversion and multiplication are both smooth mappings.

Thus we see that $G(n, k)$ is a smooth $(n-k) \times k$-manifold.

# Chapter 3

# The Tangent Space

By definition, the tangent space to a manifold at a point is the vector space of derivations at the point. A smooth map of manifolds induces a linear map between tangent spaces at corresponding points, called its *differential*. In local coordinates, the differential is represented by the Jacobian of partial derivatives of the map.

A basic principle in manifold theory is the linearization principle, meaning a manifold can be approximated near a point by its tangent space at the point, and a smooth map can be approximated by the differential of the map. One example of this is the inverse function theorem.

## 3.1 The Tangent Space

One can define a tangent vector as an arrow in the image of the chart. However, this approach is complicated since a different chart would give rise to a different set of tangent vectors. The cleaner approach is to consider point-derivations.

### 3.1.1 The Tangent Space at a Point

We define germs similarly to $\mathbb{R}^n$.

**Definition 3.1.1 (Germ)**
A *germ* of a $C^\infty$ function $M \to N$ is an equivalence class of $C^\infty$ functions defined in a neighborhood of some $p \in M$. Two such functions are equivalent if they agree on some (possibly smaller) neighborhood of $p$.

The set of germs of $C^\infty$ real-valued functions at $p \in M$ is denoted $C_p^\infty(M)$. Addition and multiplication of functions make $C_p^\infty(M)$ into a ring. Scalar multiplication by reals make $C_p^\infty(M)$ and algebra over $\mathbb{R}$.

We similarly generalize the definition of a point-derivation from $\mathbb{R}^n$.

**Definition 3.1.2 (Point-Derivation)**
A *derivation at a point* $p \in M$/*point-derivation* of $C_p^\infty(M)$ is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

**Definition 3.1.3 (Tangent Vector)**
A *tangent vector* at $p \in M$ is a derivation at $p$.

Just as for $\mathbb{R}^n$, the tangent vectors at $p$ form a vector space $T_p(M) = T_pM$.

**Remark 3.1.1** If $U$ is an open set containing $p$ in $M$, then the algebra $C_p^\infty(U)$ is the same as $C_p^\infty(M)$. Hence $T_pU = T_pM$.

Given a coordinate neighborhood $(U, \phi) = (U, x^1, \dots, x^n)$ about some $p \in M$, recall the definition of the partial derivative

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}) \in \mathbb{R}$$

where $r^i$ is the $i$-th standard coordinate on $\mathbb{R}^n$. It can be checked that $\partial/\partial x^i$ satisfies the derivation property and so is a tangent vector at $p$.

When $M$ is one-dimensional and $t$ is a local coordinate, it is customary to write $d/dt|_p$ for $\partial/\partial t|_p$. We often omit $|_p$ if it is understood at which point the tangent vector is located.

### 3.1.2 The Differential of a Map

**Definition 3.1.4 (Differential)**
Let $F : N \to M$ be a smooth map between manifolds. At each point $p \in N$, the map induces a linear map of tangent spaces $F_* : T_pN \to T_{F(p)}M$ which sends $X_p \in C_p^\infty$ to $F_*(X_p)$ defined by

$$F_*(X_p)f := X_p(f \circ F) \in \mathbb{R}$$

for $f \in C^\infty_{F(p)}M$.

Note in the definition above, $f$ is a representative of a germ at $F(p)$.

**Proposition 3.1.2**
$F_*(X_p)$ is a derivation at $F(p)$ and $F_* : T_pN \to T_{F(p)}M$ is a linear map.

**Proof**
Let $f, g \in C_p^\infty$.

$$\begin{aligned} F_*(X_p)(fg) &:= X_p(fg \circ F) \\ &= X_p((f \circ F)(g \circ F)) \\ &= X_p(f \circ F)(g \circ F(p)) + (f \circ F(p))X_p(g \circ F(p)) \\ &= F_*(X_p)fg(F(p)) + f(F(p))F_*(X_p)g \end{aligned}$$

as desired.

To see linearity, let $\alpha \in \mathbb{R}$ and $X_p, Y_p \in T_pN$. For any $f \in C^\infty_{F(p)}$,

$$\begin{aligned} F_*(\alpha X_p + Y_p) &:= (\alpha X_p + Y_p)(f \circ F) \\ &= \alpha X_p(f \circ F) + Y_p(f \circ F) \\ &= \alpha F_*(X_p)f + F_*(Y_p)f. \end{aligned}$$

We sometimes write $F_{*,p} = F_*$ to make the dependence on $p$ explicit.

**Example 3.1.3 (Differential of a Map between Euclidean Spaces)**
Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $p \in \mathbb{R}^n$. The tangent vectors $\partial/\partial x^j|_p, j \in [n]$ and $\partial/\partial y^i|_p, i \in [m]$ form bases for the tangent spaces $T_p\mathbb{R}^n$ and $T_{F(p)}\mathbb{R}^m$, respectively. The linear map $F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is described by a matrix $[a^i_j]$ relative to the two bases.

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_k a^k_j a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}.$$

Let $F^i := y^i \circ F$. We can find $a^i_j$ by evaluating both sides at $y^i$.

$$\begin{aligned} \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i &= \sum_k a^k_j \delta^i_k = a^i_j \\ F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) &= \left.\frac{\partial}{\partial x^j}\right|_p (y^i \circ F) = \frac{\partial F^i}{\partial x^j}(p). \end{aligned}$$

Thus the matrix of $F_*$ is precisely the Jacobian matrix for derivative of $F$ at $p$.

### 3.1.3 The Chain Rule

**Theorem 3.1.4 (Chain Rule)**
If $F : N \to M$ and $G : M \to P$ are smooth maps between manifolds. Then for $p \in N$,

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

**Proof**
Let $X_p \in T_pN$ and $f$ a smooth function at $G(F(p)) \in P$.

$$\begin{aligned}((G \circ F)_*X_p)f &= X_p(f \circ G \circ F)\\ &=: (F_*X_p)(f \circ G)\\ &=: (G_*(F_*X_p))f\\ &= ((G_* \circ F_*)X_p)f.\end{aligned}$$

**Remark 3.1.5** The differential of the identity map $M \to M$ is the identity map $T_pM \to T_pM$.

**Corollary 3.1.5.1**
If $F : M \to M$ is a diffeomorphism of manifolds, then $F_* : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces for any $p \in N$.

**Proof**
By assumption, $F$ has a smooth inverse $G : M \to N$. By the chain rule,

$$\begin{aligned}F_* \circ G_* &= (F \circ G)_*\\ &= (\mathrm{Id}_M)_*\\ &= \mathrm{Id}_{T_pM}\end{aligned}$$

and similarly for $G_* \circ F_*$.

**Corollary 3.1.5.2 (Invariance of Dimension)**
If an open subset $U \subseteq \mathbb{R}^n$ is diffeomorphic to an open subset $V \subseteq \mathbb{R}^m$, then $n = m$.

**Proof**
Let $F : U \to V$ be any diffeomorphism and $p \in U$. The previous corollary informs us that $F_{*,p} : T_pU \to T_{F(p)}$ is an isomorphism of vector spaces. Since there are vector space isomorphisms $T_pU \cong \mathbb{R}^n$ and $T_{F(p)} \cong \mathbb{R}^m$, we must have $n = m$.

### 3.1.4 Bases of Tangent Space at a Point

As usual, we denote by $\{r^i\}$ the standard coordinates on $\mathbb{R}^n$ and if $(U, \phi)$ is a chart about a point $p$ in a an $n$-manifold $M$, we set $x^i = r^i \circ \phi$. Since $\phi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image, a previous corollary tells us that the differential

$$\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

is a vector space isomorphism. In particular, $\dim T_pM = n = \dim M$.

**Proposition 3.1.6**
Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart about $p \in M$. Then

$$\phi_* \left( \left. \frac{\partial}{\partial x^i} \right|_p \right) = \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)}.$$

**Proof**
Let $f \in C^\infty_{\phi(p)}\mathbb{R}^n$.

$$\begin{aligned} \phi_* \left( \left. \frac{\partial}{\partial x^i} \right|_p \right) f &= \left. \frac{\partial}{\partial x^i} \right|_p (f \circ \phi) \\ &= \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)} (f \circ \phi \circ \phi^{-1}) \\ &= \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)} f. \end{aligned}$$

**Proposition 3.1.7**
Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart containig $p$, then the tangent space $T_pM$ has basis

$$\left. \frac{\partial}{\partial x^1} \right|_p, \ldots, \left. \frac{\partial}{\partial x^n} \right|_p.$$

**Proof**
An isomorphism of vector spaces carries a basis to a basis. The image of the tangent vectors above map to the partial derivatives at $\phi(p)$, which forms a basis of $T_{\phi(p)}\mathbb{R}^n$.

**Proposition 3.1.8 (Transition Matrix for Coordiante Vectors)**
Suppose $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two coordinate charts on a manifold $M$. Then on $U \cap V$,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}.$$

**Proof**
At each $p \in U \cap V$, there are two bases for the tangent space $T_pM$ and so there is a matrix $[a^i_j(p)]$ such that

$$\frac{\partial}{\partial x^j} = \sum_k a^k_j \frac{\partial}{\partial y^k}.$$

Evaluating both sides at $y^i$, we get

$$\begin{aligned} \frac{\partial y^i}{\partial x^j} &= \sum_k a^k_j \frac{\partial y^i}{\partial y^k} \\ &= \sum_k a^k_j \delta^i_k \\ &= a^i_j. \end{aligned}$$

### 3.1.5 A Local Expression for the Differential

Give a smooth map $F : N \to M$ of manifolds and $p \in N$, let $(U, x^1, \ldots, x^n)$ be a chart about $p \in N$ and $(V, y^1, \ldots, y^m)$ be a chart about $F(p) \in M$. We find a local expression for the differential relative to the two charts.

The partial derivatives at $p, F(p)$ form a basis for $T_pN, T_{F(p)}M$ respectively. Thus the differential $F_* = F_{*,p}$ is completely determined by the numbers $a^i_j$

$$F_*\left( \left.\frac{\partial}{\partial x^j}\right|_p \right) = \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}.$$

Applying both sides to $y^i$, we find that

$$\begin{aligned} a_j^i &= \left( \sum_k a_j^k \frac{\partial}{\partial y^k}\bigg|_{F(p)} \right) y^i \\ &= F_* \left( \frac{\partial}{\partial x^j}\bigg|_p \right) y^i \\ &= \frac{\partial}{\partial x^j}\bigg|_p (y^i \circ F) \\ &= \frac{\partial F^i}{\partial x^j}(p). \end{aligned}$$

**Proposition 3.1.9**
Given a smooth map $F : N \to M$ of manifolds and a point $p \in N$, let $(U, (x^j))$ and $(V, (y^i))$ be coordinate charts about $p \in N$ and $F(p) \in M$ respectively. Relative to the bases $\{\partial/\partial x^j|_p\}$ for $T_pN$ and $\{\partial/\partial y^i|_{F(p)}\}$ for $T_{F(p)}M$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is represented by the matrix

$$\left[ \frac{\partial F^i}{\partial x^j}(p) \right].$$

**Remark 3.1.10** The inverse function theorem for manifolds has a coordinate-free description: A smooth map $F : N \to M$ between two manifolds of the same dimension is locally invertible at $p \in N$ if and only if its differential at $p$ $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism.

### 3.1.6 Curves in a Manifold

**Definition 3.1.5 (Smooth Curve)**
A *smooth curve* in a manifold $M$ is a smooth map $c : (a, b) \to M$.

Usually we assume $0 \in (a, b)$ and say that $c$ is a *curve starting at* $p$ if $c(0) = p$.

**Definition 3.1.6 (Velocity Vector)**
The velocity vector $c'(t_0)$ of the curve $c$ at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) := c_* \left( \frac{d}{dt}\bigg|_{t_0} \right) \in T_{c(t_0)}M.$$

Thus $c'(t_0)$ is simply the differential of the curve at the point $t = t_0$. We say that $c'(t_0)$ is the velocity of $c$ at the point $c(t_0)$.

If we need to distinguish between the standard calculus notation $c'(t)$and the tangent vector $c'(t_0)$, we will write $\dot{c}(t)$ to denote the calculus derivative.

**Proposition 3.1.11 (Velocity Vector & Calculus Derivative)**
Let $c : (a, b) \to \mathbb{R}$ be a curve with co-domain $\mathbb{R}$. Then

$$c'(t_0) = \dot{c}(t_0)\, \frac{d}{dx}\bigg|_{c(t_0)}.$$

**Proof**
Pick $f \in C^\infty_{c(t_0)}$. By the chain rule from calculus,

$$\begin{aligned} c'(t_0)f &:= c_*\left(\frac{d}{dt}\bigg|_{t_0}\right) f \\ &:= \frac{d}{dt}\bigg|_{t_0} (f \circ c) \\ &= \dot{c}(t_0)\, \frac{d}{dx}\bigg|_{c(t_0)} f. \end{aligned}$$

This concludes the proof.

**Example 3.1.12**
Define $c : \mathbb{R} \to \mathbb{R}^2$ by

$$c(t) := (t^2, t^3).$$

Then $c'(t)$ is a linear combination of $\partial/\partial x$ and $\partial/\partial y$ at $c(t)$:

$$c_*\left(\frac{d}{dt}\right) = c'(t) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}.$$

We can evaluate both sides at $x, y$ respectively to extract the coefficients $a = 2t, b = 3t^2$.

More generally, to compute the velocity vector of a smooth curve $c \in \mathbb{R}^n$, one can simply differentiate the components of $c$.

**Proposition 3.1.13 (Velocity of a Curve in Local Coordinates)**
Let $c : (a, b) \to M$ be a smooth curve, and $(U, x^1, \ldots, x^n)$ a coordinate chart about $c(t)$. Write $c^i = x^i \circ c$ for the $i$-th component of $c$ in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^n \dot{c}^i(t)\, \frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

**Proof**
We already know that $c'(t)$ is a linear combination of the partial derivatives at $c(t)$, say with coefficients $a^i$. Evaluating the tangent vector at $x^i$ yields

$$\begin{aligned} a^i &= c'(t)x^i \\ &:= c_*\left(\frac{\partial}{\partial t}\right)x^i \\ &= \frac{\partial}{\partial t}(x^i \circ c) \\ &= \frac{\partial}{\partial t}c^i \\ &= \dot{c}^i(t). \end{aligned}$$

Every smooth curve $c$ at a point $p \in M$ gives rise to a tangent vector $c'(0) \in T_pM$. Conversely, we can show that every tangent vector $X_p \in T_pM$ is precisely the velocity vector of some curve at $p$.

**Proposition 3.1.14 (Existence of a Curve with a given Initial Vector)**
For any point $p \in M$ of a manifold and any tangent vector $X_p \in T_pM$, there are $\varepsilon > 0$ and a smooth curve $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

**Proof**
Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart centered at $p$, ie $\phi(p) = 0 \in \mathbb{R}^n$. Suppose $X_p = \sum_i a^i \partial/\partial x^i|_p$ at $p$. Let $\{r^i\}$ be the standard coordinates on $\mathbb{R}^n$ and write $x^i = r^i \circ \phi$.

We wish to find a curve $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $\alpha(0) = 0$ and $\alpha'(0) = \sum_i a^i \partial/\partial r^i|_0$. The previous proposition states that one possible choice of $\alpha$ is given by

$$\alpha(t) = (a^1 t, \dots, a^n t)$$

where we must choose the domain $(-\varepsilon, \varepsilon)$ to be sufficiently small such that $\alpha(t)$ still lies in $\phi(U)$.

Next, we map $\alpha$ to $M$ via $\phi^{-1}$. Define $c := \phi^{-1} \circ \alpha : (-\varepsilon, \varepsilon) \to M$. Then

$$c(0) = \phi^{-1}(\alpha(0)) = \phi^{-1}(0) = p.$$

By the chain rule and the fact that charts are tangent space isomorphisms,

$$\begin{aligned} c'(0) &= (\phi^{-1})_* \alpha_* \left( \left. \frac{d}{dt} \right|_0 \right) && \text{chain rule} \\ &= (\phi^{-1})_* \left( \sum_i a^i \left. \frac{\partial}{\partial r^i} \right|_0 \right) && \text{previous proposition} \\ &= \sum_i a^i \left. \frac{\partial}{\partial x^i} \right|_p && \text{tangent space isomorphism} \\ &= X_p. \end{aligned}$$

we can now interpret the abstract definition of a tangent vector geometrically as a directional derivative using curves.

**Proposition 3.1.15**
Suppose $X_p$ is a tangent vector at a point $p \in M$ of a manifold and $f \in C_p^\infty(M)$. If $c : (-\varepsilon, \varepsilon) \to M$ is a smooth curve starting at $p$ with $c'(0) = X_p$, then

$$X_p f = \left. \frac{d}{dt} \right|_0 (f \circ c).$$

**Proof**
By the definitions of $c'(0)$ and $c_*$,

$$\begin{aligned} X_p f &= c'(0) f \\ &= c_* \left( \left. \frac{d}{dt} \right|_0 \right) f \\ &= \left. \frac{d}{dt} \right|_0 (f \circ c). \end{aligned}$$

### 3.1.7 Computing the Differential Using Curves

We have two ways of computing the differential of a smooth map: the function definition in terms of point derivations and a matrix representation in terms of local coordinates. We now give another way of computing the differential using curves.

**Proposition 3.1.16**
Let $F : N \to M$ be a smooth map between manifolds, $p \in N$, and $X_p \in T_pN$. If $c$ is a smooth curve starting at $p \in N$ with velocity $X_p$ at $p$, then

$$F_{*,p}(X_p) = (F \circ c)'(0).$$

Thus $F_{*,p}(X_p)$ is the velocity vector of the image curve $F \circ c$ at $F(p)$.

**Proof**
By assumption, $c(0) = p$ and $c'(0) = X_p$. Thus

$$\begin{aligned} F_{*,p}(X_p) &= F_{*,p}(c'(0)) \\ &= (F_{*,p} \circ c_{*,0}) \left( \left. \frac{d}{dt} \right|_0 \right) \\ &= (F \circ c)_{*,0} \left( \left. \frac{d}{dt} \right|_0 \right) \\ &= (F \circ c)'(0). \end{aligned}$$

**Example 3.1.17 (Differential of Left Matrix Multiplication)**
Let $g \in \mathrm{GL}(n, \mathbb{R})$ and $\ell_g : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ denote left multiplication by $g$. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of the vector space $\mathbb{R}^{n \times n}$, the tangent space $T_g(\mathrm{GL}(n, \mathbb{R}))$ can be identified with $\mathbb{R}^{n \times n}$ by mapping partial derivatives to canonical basis vectors. Let $\Phi : T_g(\mathrm{GL}(n, \mathbb{R})) \to \mathbb{R}^{n \times n}$ denote this identification.

Let $X \in T_{\mathrm{Id}}(\mathrm{GL}(n, \mathbb{R})) = \mathbb{R}^{n \times n}$. To compute $(\ell_g)_{*,\mathrm{Id}}(X)$, we can choose a curve $c(t) \in \mathrm{GL}(n, \mathbb{R})$ such that $c(0) = \mathrm{Id}$ and $c'(0) = X$. From a previous proposition, this means that $c'(0)f = (f \dot{\circ} c)(0)$ for any $f \in C^\infty_{\mathrm{Id}}(\mathrm{GL}(n, \mathbb{R}))$. We have shown that such a curve always exists.

Then $\ell_g(c(t)) = gc(t)$ is simply matrix multiplication and

$$\begin{aligned} (\ell_g)_{*,\mathrm{Id}}(X) &= (\ell_g \circ c)'(0) && \text{previous proposition} \\ \Phi((\ell_g)_{*,\mathrm{Id}}(X)) &= \Phi((\ell_g \circ c)'(0)) \\ &= (\ell_g \dot{\circ} c)(0) && \text{previous proposition} \\ &= g\dot{c}(0) \\ &= g\Phi(X) \end{aligned}$$

Thus with the identification above, the differential

$$(\ell_g)_{*,\mathrm{Id}} : T_{\mathrm{Id}}(\mathrm{GL}(n, \mathbb{R})) \to T_g(\mathrm{GL}(n, \mathbb{R}))$$

is also left multiplication by $g$.

### 3.1.8 Immersions and Submersions

Two important cases of differential maps are immersions and submersions.

**Definition 3.1.7 (Immersion)**
A smooth map $F : N \to M$ between manifolds is said to be an *immersion* at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is injective. If this holds at all $p \in N$, we say $F$ is an *immersion.*

**Definition 3.1.8 (Submersion)**
A smooth map $F : N \to M$ between manifolds is said to be an *submersion* at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective. If this holds at all $p \in N$, we say $F$ is an *submersion.*

**Remark 3.1.18** Recall that if $N, M$ have dimensions $n, m$ respectively, then $\dim T_pN$ and $\dim T_{F(p)}M = m$. The injectivity of the differential $F_{*,p}$ implies immediately that $n \leq m$. Similarly, surjectively implies $n \geq m$.

**Example 3.1.19**
The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher-dimensional $\mathbb{R}^m$:

$$\iota(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

The prototypical submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi(x^1, \dots, x^m, x^{m+1}, \dots, x^n) = (x^1, \dots, x^m).$$

**Example 3.1.20**
if $U \subseteq M$ is an open subset, then the inclusion $\iota : U \to M$ is both an immersion and a submersion. Thus a submersion need not be onto.

### 3.1.9 Rank, Critical & Regular Points

Recall the *rank* of a linear transformation between finite-dimensional vector spaces is the dimension of the image.

**Definition 3.1.9 (Rank of Smooth Map)**
The *rank* of a smooth map $F : N \to M$ between manifolds is defined as the rank of the differential $F_{*,p} : T_pN \to T_{F(p)}M$.

Relative to local coordinates $(U, x^1, \dots, x^n)$ at $p \in N$ and $(V, y^1, \dots, y^m)$ at $F(p) \in M$, the differential is represented by the Jacobian $JF_p = [\partial F^i/\partial x^j(p)]$. Hence

$$\operatorname{rank} F(p) = \operatorname{rank} \left[\frac{\partial F^i}{\partial x^j}(p)\right].$$

Note that the differential is independent of coordinate charts, and so is the rank of the Jacobian matrix.

**Definition 3.1.10 (Critical/Regular Point)**
A point $p \in N$ is a *critical point* of $F : N \to M$ if the differential $F_{*,p}$ is not surjective. Otherwise, it is a *regular point*.

Thus $p$ is a regular point of $F$ if and only if $F$ is a submersion at $p$.

**Definition 3.1.11 (Critical/Regular Value)**
Let $F : N \to M$ be a smooth map between manifolds. A point in $M$ is a *critical value* if it is the image of a critical point. Otherwise, it is a *regular value*.

Note we do not define regular values as the image of some regular point. We require all points in the pre-image to be regular. On the other hand, $c \in M$ is critical if there is a single critical point in the pre-image.

**Proposition 3.1.21**
Let $f : M \to \mathbb{R}$ be smooth. A point $p \in M$ is critical if and only if relative to some chart $(U, x^1, \dots, x^n)$ containing $p$, all the partial derivatives satisfy

$$\frac{\partial f}{\partial x^j}(p) = 0$$

for $j \in [n]$.

**Proof**
The differential $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} \cong \mathbb{R}$ is represented by the Jacobian matrix. Since the image is a linear subspace of $\mathbb{R}$, it is either the zero map or a surjective map. Thus $f_{*,p}$ fails to be surjective if and only if all partial derivatives are zero.

### 3.1.10 Useful Results

**Proposition 3.1.22**
Let $M, N$ be manifolds and $\pi_M : M \times N \to M, \pi_N : M \times N \to N$ be two projections. For $(p, q) \in M \times N$,

$$(\pi_{M*}, \pi_{N*}) : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

**Proof**
Recall that if $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^m)$ are charts about $p \in M, q \in N$ respectively, $(U \times V, \pi_M x^1, \dots, \pi_N y^1, \dots)$ is a chart about $(p,q) \in M \times N$. Write $\bar{x}^i := \pi_M x^i$ and $\bar{y}^i = \pi_N y^i$. We have

$$\begin{aligned}\pi_{M*}\left(\left.\frac{\partial}{\partial \bar{x}^j}\right|_{(p,q)}\right) x^i &= \frac{\partial}{\partial \bar{x}^j}(x^i \circ \pi_M) \\ &= \frac{\partial \bar{x}^i}{\partial \bar{x}^j} \\ &= \delta_{ij}.\end{aligned}$$

Hence

$$\pi_{M*}\left(\left.\frac{\partial}{\partial \bar{x}^j}\right|_{(p,q)}\right) = \left.\frac{\partial}{\partial x^j}\right|_p.$$

By checking how $\pi_{M*}, \pi_{N*}$ acts on the canonical basis of $T_{(p,q)(M\times N)}$, we see that it is a linear map between basis elements and is therefore an isomorphism.

**Proposition 3.1.23**
Let $G$ be a Lie group equipped with its multiplication map $\mu : G \times G \to G$ and inverse map $\iota : G \to G$. The differential of $\mu$ at the identity $e \in G$ is addition:

$$\begin{aligned}\mu_{*,(e,e)} : T_eG \times T_eG &\to T_eG \\ (X_e, Y_e) &\mapsto X_e + Y_e\end{aligned}$$

while the differential of $\iota$ at the identity $e \in G$ is negation

$$\begin{aligned}\iota_{*,e} : T_eG &\to T_eG \\ X_e &\mapsto -X_e.\end{aligned}$$

**Proof**
For the first claim, it suffices to show that $\mu*, (e,e)(X_e, 0) = X_e$ and $\mu_{*,(e,e)}(0, Y_e) = Y_e$. The result then follows by linearity.

To compute $\mu_{*,(e,e)}(X_e, 0)$, We construct a curve $\alpha(t)$ in $G \times G$ at which $\alpha(0) = (e,e)$ and $\alpha'(0) = (X_e, 0)$ as follows. Take any curve $c(t)$ in $G$ such that $c(0) = e$ and $c'(0) = X_e$. Then define $\alpha(t) := (c(t), e)$. We have

$$\begin{aligned}\mu_{*,(e,e)}(X_e, 0) &= (\mu \circ \alpha)'(0) \\ &= c'(0) \\ &= X_p\end{aligned}$$

The computation for $(0, Y_e) \mapsto Y_e$ is identical.

For the second claim, we need only show that $X_e - \iota_{*,e}(X_e) = 0$. Indeed, take the same curve $c(t)$ in $G$ as above but define $\alpha(t) := (c(t), (\iota \circ c)(t))$. Note that $\alpha'(0) = (c'(0), (\iota \circ c)'(0))$ by construction, which can be checked by evaluating both sides at the coordinate functions. Then

$$\begin{aligned} e &= (\mu \circ \alpha)(t) \\ 0 &= (\mu \circ \alpha)'(t) \\ &= \mu_{*,(e,e)}(X_e, (\iota \circ c)'(0)) \\ &= X_e + (\iota \circ c)'(0) \\ &= X_e + \iota_{*,e}(X_e). \end{aligned}$$

**Proposition 3.1.24 (Transforming Vectors to Coordinate Vectors)**
Let $X_1, \ldots, X_n$ be vector fields on an open subset $U$ of an $n$-manifold. Suppose at some $p \in U$, the tangent vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Then there is a chart $(V, x^1, \ldots, x^n)$ about $p$ such that

$$(X_i)_p = \left.\frac{\partial}{\partial x^i}\right|_p .$$

**Proof**
Let $(U, y^1, \ldots, y^n)$ be any chart about $p$. Write

$$(X_j)_p = \sum_i a^i_j \left.\frac{\partial}{\partial y^i}\right|_p .$$

Since the $(X_j)_p$'s are linearly independent, the matrix given by $A = [a^i_j]$ is non-singular, and we can define a new coordinate system $x^1, \ldots, x^n$ by

$$y^i = \sum_j a^i_j x^j.$$

In other words, $x^j = \sum_i (A^{-1})^j_i y^i$.

By the change of basis formula, for which we recall can be verified by evaluating both sides at $y^k$,

$$\begin{aligned} \frac{\partial}{\partial x^j} &= \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} \\ &= \sum_i a^i_j \frac{\partial}{\partial y^i}. \end{aligned}$$

By construction, this evaluates to $(X_j)_p$ at the point $p$.

**Definition 3.1.12 (Local Maxima)**
A real-valued function $f : M \to \mathbb{R}$ on a manifold is said to have a *local maximum* at $p \in M$ if there is a neighborhood $U \ni p$ such that $f(p) \geq f(q)$ for all $q \in U$.

Recall that if a differentiable function $f : I \to \mathbb{R}$ on an interval has a local maximum at some $p \in I$, then $f'(p) = 0 \in T_{f(p)}\mathbb{R}$. This can be shown by the definition of the calculus derivative in terms of Newton quotients.

**Proposition 3.1.25**
A local maximum of a smooth function $f : M \to \mathbb{R}$ is a critical point of $f$.

**Proof**
We need to show that $f_{*,p}$ is not surjective, meaning it is the zero map, or sends any tangent vector to 0.

Fix $X_p \in T_pM$. Let $c(t)$ be a curve starting at $p$ with initial velocity $X_p$. then $f \circ c$ is a differentiable function on an interval with a local maximum at $t = 0$ so that $(f \circ c)'(0) = 0$. But then

$$f_{*,p}(X_p) = (f \circ c)'(0) = 0$$

for any $X_p \in T_pM$, which concludes the proof.

## 3.2 Submanifolds

Currently, we can check that a given topological space is a manifold either by definition or by exhibiting it as an appropriate quotient space. We now derive another way: exhibiting the topological space as a *(regular) submanifold* of another manifold.

### 3.2.1 Submanifolds

The $xy$-plane in $\mathbb{R}^3$ is the prototype of a *regular manifold* of a manifold. It is defined by the vanishing of the coordinate function $z$.

**Definition 3.2.1 (Regular Submanifold)**
A subset $S$ of an $n$-manifold is a *regular submanifold* of dimension $k$ if for every $p \in S$, there is a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ of $p$ in the maximal atlas of $N$ such that $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. By renumbering the coordinates, we may assume that these $n-k$ coordinate functions are $x^{k+1}, \ldots, x^n$.

On $U \cap S$, $\phi = (x^1, \ldots, x^k, 0, \ldots, 0)$. Let $\phi_S : U \cap S \to \mathbb{R}^k$ be the restriction of the first $k$ components of $\phi$ to $U \cap S$, so $\phi_S = (x^1, \ldots, x^k)$. We call such a chart $(U, \phi)$ in $N$ an *adapted chart* relative to $S$. Note that $(U \cap S, \phi_S)$ is a chart for $S$ in the subspace topology.

**Definition 3.2.2 (Codimension)**
If $S$ is a regular submanifold of dimension $k$ in an $n$-manifold $N$, then we say the *codimension* of $S$ in $N$ is $n - k$.

**Remark 3.2.1** We remark that as a topological space, a regular submanifold of $N$ is required to have the subspace topology.

**Example 3.2.2 (Open Submanifolds are Regular Submanifolds)**
The dimension $k$ of the submanifold may be equal to $n$, the dimension of the manifold. In this case, the charts of the submanifold have domain $U \cap S = U$. Thus an open subset of a manifold is a regular submanifold of the same dimension.

**Example 3.2.3**
The interval $S = (-1, 1)$ on the $x$-axis is a regualr submanifold of the $xy$-plane. We can take the open square $(-1, 1) \times (-1, 1)$ as an adapted chart. Then $U \cap S$ is precisely the zero set of $y$ on $U$.

**Example 3.2.4 (Topologist's Sine Curve)**
Let $\Gamma$ e the graph of the function $f(x) = \sin(1/x)$ over the interval $(0, 1)$ and $S$ the union of $\Gamma$ and the open interval

$$I = \{(0, y) \in \mathbb{R}^2 : y \in (-1, 1)\}.$$

$S \subseteq \mathbb{R}^2$ cannot be a regular submanifold. Indeed, for any $p \in I$, there is no adapted chart containing $p$, since any sufficiently small neighborhood $U \ni p$ in $\mathbb{R}^2$ intersects $S$ in infinitely many components.

Recall that the closure of $\Gamma$ in $\mathbb{R}^2$ is called the *topologist's since curve*. It differs from $S$ in including the endpoints $(1, \sin 1), (0, 1), (0, -1)$.

**Proposition 3.2.5 (Regular Submanifolds are Manifolds)**
Let $S$ be a regular submanifold of $N$ and $\mathfrak{U} = \{(U, \phi)\}$ a collection of compatible adapted charts of $N$ covering $S$. Then $\{(U \cap S, \phi_S)\}$ is at atlas for $S$. Thus a reglar submanifold is itself a manifold.
Moreover, if $N$ has dimension $n$ and $S$ is locally defined by the vanishing of $n - k$ coordinates, then $\dim S = k$.

**Proof**
Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \Psi) = (V, y^1, \dots, y^n)$ be two adapted charts in the given collection. Assume that they intersect. As we remarked in the definition of a regular submanifold, we can renumber of coordinates of any adapted chart relative to a submanifold $S$ so that the last $n-k$ coordinates vanish on points of $S$. Then for $p \in U \cap V \cap S$,

$$(\Psi_S \circ \phi_S^{-1})(x^1, \dots, x^k) = (y^1, \dots, y^k).$$

Since $\Psi \circ \phi^{-1}$ is smooth, the $y^i$'s are smooth functions of the $x^j$'s and $\Psi_S \circ \phi_S^{-1}$ is therefore smooth as well. Similarly, $\phi_S \circ \Psi_S^{-1}$ is also smooth.

This shows that any two charts in $\{(U \cap S), \phi_S\}$ are smoothly compatible. The collection $\{(U \cap S, \phi_S)\}$ is thus a smooth structure on $S$ since the $U \cap S$'s cover $S$ by assumption.

### 3.2.2 Level Sets of a Function

**Definition 3.2.3 (Level Set)**
The *level set* of a map $F : N \to M$ is a subset

$$F^{-1}(\{c\}) = F^{-1}(c) = \{p \in N : F(p) = c\}$$

for some $c \in M$.

The value $c \in M$ is called the *level* of $F^{-1}(c)$. In the special case of $F : N \to \mathbb{R}^m$, we say $Z(F) := F^{-1}(0)$ is the *zero set* of $F$.

Recall that $c$ is a regular value of $F$ if and only if either $c$ is not in the image of $F$ or at every $p \in F^{-1}(c)$, the differential $F_{*,p} : T_pN \to T_pM$ is surjective. The inverse image of a regular value $c$ is called a *regular level set*. If the zero set of some $F : N \to \mathbb{R}^m$ is regular, it is called a *regular zero set*.

Remark that if a regular level set is nonempty, then the map $F : N \to M$ is a submersion at $p$. In particular, $\dim N \geq \dim M$.

**Example 3.2.6 (The 2-Sphere in $\mathbb{R}^3$)**
The unit 2-sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 - 1 = 0\}$$

is the zero set of the function

$$f(x, y, z) = x^1 + y^2 + z^2 - 1.$$

Now, the only critical point of $f$ is 0, which does not lie on the sphere. Thus all points on the sphere are regular points of $f$ and 0 is a regular value of $f$.

Let $p$ be a point of $S^2$ at which $(\partial f/\partial x)(p) = 2x(p) \neq 0$. Then the Jacobian matrix of the map $(f, y, z) : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the Jacobian determinant is nonzero. Hence the inverse function applies and we can find a neighborhood $U_p$ of $p \in \mathbb{R}^3$ such that $(U_p, f, y, z)$ is a chart in the atlas of $\mathbb{R}^3$. In this chart, the set $U_p \cap S^2$ is defined by the vanishing of the first coordinate $f$. Thus $(U_p, f, y, z)$ is an adapted chart relative to $S^2$ and $(U_p \cap S^2, y, z)$ is a chart for $S^2$.

Similarly, if either $(\partial f/\partial y)(p) \neq 0$ or $(\partial f/\partial z)(p) \neq 0$, then we can find an adapted chart $(V_p, x, f, z)$ or $(V_p, x, y, f)$ containing $p$ in which the set $V_p \cap S^2$ is the zero set of the second or third coordinate $f$. But at least one of the partial derivatives must be nonzero, so as $p$ varies over all points of the sphere, we obtain a collection of adapted charts of $\mathbb{R}^3$ covering $S^2$. Hence $S^2$ is a regular submanifold of $\mathbb{R}^3$ with dimension 2.

This is an important example since we can nearly translate it verbatim for the regular zero set of a function $F : N \to \mathbb{R}$. First, we note that any regualr level set $g^{-1}(c)$ of a smooth function $g$ on a manifold can be expressed as a regular zero set. Indeed, consider $f = g - c$ so that

$$g(p) = c \iff f(p) = 0.$$

Moreover, the differentials of $f, g$ are point-wise equal and so $f, g$ have the exact same critical points. So if $g^{-1}(c)$ is a regular level set, then $f^{-1}(0)$ is a regular zero set.

**Theorem 3.2.7**
Let $g : N \to \mathbb{R}$ be a smooth function on the manifold $N$. Then a non-empty regular level set $S = g^{-1}(c)$ is a regular submanifold of $N$ with codimension 1.

**Proof**
Let $f = g - c$ so that $S = f^{-1}(0)$ is the regular zero set of $f$. Let $p \in S$. Since $p$ is a regular point of $f$, relative to any chart $(U, x^1, \dots, x^n)$ about $p$, $(\partial f/\partial x^i)(p) \neq 0$ for some $i$. By relabelling, we can assume without loss of generality that $i = 1$. The Jacobian of the smooth map $(f, x^2, \dots, x^n) : U \to \mathbb{R}^n$ is given by

$$\begin{bmatrix} \frac{\partial f}{\partial x^1} & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

This is nonzero at the point $p$ by construction. Thus the inverse function theorem applies and there is a neighborhood $U_p$ of $p$ on which $f, x^1, \dots, x^n$ forms a coordinate system.

Relative to the chart $(U_p, f, x^2, \dots, x^n)$, the level set $U_p \cap S$ is defined by setting the first coordiante $f$ to 0, hence the chart is adapted relative to $S$. But $p \in N$ was arbitrary and so $S$ is a regular submanifold of codimension 1 in $N$.

### 3.2.3 The Regular Level Set Theorem

Our next step is to extend the previous hteorem to a regular level set of a map between smooth manifolds. This useful theorem is known under various names such as the implicit function theorem, preimage theorem, and the regular level set theorem. We will follow the last.

**Theorem 3.2.8 (Regular Level Set Theorem)**
Let $F : N \to M$ is a smooth map between manifolds with $\dim N = n, \dim M = m$. Then a nonempty regular level set $F^{-1}(c)$ for some $c \in M$ is a regular submanifold of $N$ with codimension $m$.

**Proof**
Choose a chart $(V, \Psi) = (V, y^1, \dots, y^m)$ of $M$ centered at $c$ such that $\Psi(c) = 0 \in \mathbb{R}^m$. Then $F^{-1}(V)$ is an open set in $N$ containing $F^{-1}(c)$. Moreover, in $F^{-1}(V)$, $F^{-1}(c) = (\Psi \circ F)^{-1}(0)$ and the level set $F^{-1}(c)$ is the zero set of $\Psi \circ F$. If $F^i := y^i \circ F = r^i \circ (\Psi \circ F)$, then $F^{-1}(c)$ is also the common zero set of the functions $F^1, \dots, F^m$ on $F^{-1}(V)$.

Since we assumed the regular level set to be nonempty, we must have $n \geq m$. Fix a point $p \in F^{-1}(c)$ and let $(U, x^1, \dots, x^n)$ be a coordinate neighborhood of $p \in N$ contained in $F^{-1}(V)$. Since $F^{-1}(c)$ is a regular level set, $p \in F^{-1}(c)$ is by definition a regular point of $F$. Thus the $m \times n$ Jacobian matrix of $F$ has rank $m$. By relabelling if necessary, we may assume without loss of generality that the first $m \times m$ block is nonsingular.

Replace the first $m$ coordinates $x^1, \dots, x^m$ of the chart $(U, \phi)$ by $F^1, \dots, F^m$. We claim that there is a neighborhood $U_p$ of $p$ such that $(U_p, F^1, \dots, F^m, x^{m+1}, \dots, x^n)$ is a chart in the atlas of $N$. It suffices to compute the Jacobian matrix of the chart function at $p$. But the first $m \times m$ block is nonsingular by construction and hence the the inverse function theorem applies.

In the chart $(U_p, F^1, \dots, F^m, x^{m+1}, \dots x^n)$, the set $S := F^{-1}(c)$ is obtained by setting the first $m$ coordinates to 0. Since this is true for every point $p \in S$, $S$ is by definition a regular submanifold of $N$ with codimension $m$.

The proof of the regular level set theorem yields the following lemma.

**Lemma 3.2.9**
Let $F : N \to \mathbb{R}^m$ be a smooth map on a manifold $N$ of dimension $n$ and let $S$ be the level set $F^{-1}(0)$. Suppose relative to some coordinate chart $(U, x^1, \ldots, x^n)$ about $p \in S$, the determinant of the Jacobian matrix with respect to $x^{j_1}, \ldots, x^{j_m}$ is nonzero. Then in some neighborhood of $p$, we can replace $x^{j_1}, \ldots, x^{j_m}$ by $F_1, \ldots, F_m$ to obtain an adapted chart for $N$ relative to $S$.

**Remark 3.2.10** The regular level set theorem gives a sufficient but not necessary condition for a level set to be a regular manifold. For example, if $f : \mathbb{R}^2 \to \mathbb{R}$ is the map $f(x, y) = y^2$, then the zero set $Z(f) = Z(y^2)$ is the $x$-axis, a regular submanifold of $\mathbb{R}^2$. But $\partial f/\partial x = \partial f/\partial y = 0$ on the $x$-axis, and every point in $Z(f)$ is a critical point of $f$. Thus, although $Z(f)$ is a regular submanifold of $\mathbb{R}^2$, it is not a regular level set of $f$.

### 3.2.4 Examples of Regular Manifolds

**Example 3.2.11 (Hypersurface)**
The solution set $S$ pf $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$ is a 2-manifold by the regular level set theorem.

**Example 3.2.12**
The subset $S \subseteq \mathbb{R}^3$ satisfying

$$\begin{aligned} x^3 + y^3 + z^3 &= 1 \\ x + y + z &= 0 \end{aligned}$$

is a 1-manifold.

This can be checked by consider the function

$$F(x, y, z) = (x^3 + y^3 + z^3, x + y + z)$$

and checking that its Jacobian has rank 2 for every point in $S$. This implies $S$ is a regular level set of $F$ and is thus a manifold.

**Example 3.2.13 (Special Linear Group)**
The recall the special linear group

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) : \det A = 1\}.$$

Then $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(1)$ is a level set of the determinant map.

We wish to check that no critical points of det live in $\mathrm{SL}(n, \mathbb{R})$. Let $m_{ij}$ denote the

$(i, j)$-minor of $A$. Recall the cofactor expansion formula

$$\det A = (-1)^{i+1} a_{i1} m_{i1} + \cdots + (-1)^{i+n} a_{in} m_{in}.$$

Since $m_{ij}$ is not a function of $a_{ij}$,

$$\frac{\partial f}{\partial a_{ij}} = (-1)^{i+j} m_{ij}.$$

Hence a matrix $A \in \mathrm{GL}(n, \mathbb{R})$ is a critical point of $f$ if and only if all the $(n-1) \times (n-1)$ minors $m_{ij}$ of $A$ are zero. But then such a matrix necessarily has determinant 0 by cofactor expansion. Thus we can apply the regular level set theorem to conclude that $\mathrm{SL}(n, \mathbb{R})$ is a regular $(n^2 - 1)$-submanifold of $\mathrm{GL}(n, \mathbb{R})$.

### 3.2.5 Transversality

**Definition 3.2.4 (Transversal)**
A $C^\infty$ map $f : N \to M$ is said to be *transversal* to a submanifold $S \subseteq M$ if for every $p \in f^{-1}(S)$,

$$f_*(T_p N) + T_{f(p)} S = T_{f(p)} M$$

where the addition is taken to be the Minkowski sum.

We write $f \pitchfork S$ to denote this condition.

**Theorem 3.2.14 (Tranversality)**
If a smooth map $f : N \to M$ is transversal to a regular submanifold $S$ of codimension $k$ in $M$, then $f^{-1}(S)$ is a regular submanifold of codimension $k$ in $N$.

Remark that when $S = \{c\}$ is a single point, transversality of $f$ to $S$ simply means that

$$f_*(T_p N) = T_{f(p)} M$$

for every $p \in f^{-1}(c)$, ie $f_{*,p}$ is surjective at every $p \in f^{-1}(c)$, ie $f^{-1}(c)$ is a regular level set. Thus the transversality theorem is a generalization of the regular level set theorem. It is useful in giving conditions under which the intersection of two submanifolds is a submanifold.

**Proof**
Let $p \in f^{-1}(S)$ and $(U, x^1, \ldots, x^m)$ be an adapted chart centered at $f(p)$ for $M$ relative to $S$ such that $U \cap S = Z(x^{m-k+1}, \ldots, x^m)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $g : U \to \mathbb{R}^k$ to be the map

$$g = (x^{m-k+1}, \ldots, x^m).$$

Consider $g \circ f : f^{-1}(U) \to \mathbb{R}^k$. By construction,

$$\begin{aligned} f^{-1}(U) \cap f^{-1}(S) &= f^{-1}(U \cap S) \\ &= \{p \in f^{-1}(U) : g(f(p)) = 0\} \\ &= (g \circ f)^{-1}(0). \end{aligned}$$

We claim that $f^{-1}(U \cap S)$ is a regular level set of the function $g \circ f$ above. It suffices to show that $(g \circ f)_{*,p}$ is surjective at every $p \in f^{-1}(U \cap S)$. Fix $Z \in T_0\mathbb{R}^k$. $g_*$ is by construction surjective at every point in $U$ as $g$ is a subset of coordinate functions. Hence there is some $Y \in T_{f(p)}M$ such that $g_*(Y) = Z$. the transversality of $f$ implies that there is some $X \in T_pN$ and $Y' \in T_{f(p)}S$ such that $f_*(X) + Y' = Y$. Since $g$ is constant on $S$ and $f(p) \in U \cap S$, we must have $g_*(Y') = 0$ so that

$$\begin{aligned} Z &= g_*(Y) \\ & g_*(f_*(X) + Y') \\ &= (g \circ f)_*(X) + 0 \\ &= (g \circ f)_*(X). \end{aligned}$$

By the arbitrary choice of $p, Z$, we conclude that $f^{-1}(U \cap S) = (g \circ f)^{-1}(0)$ is a regular level set of the function $g \circ f$.

It follows that for every $p \in f^{-1}(S)$, there is a neighborhood $V \ni p$, where we can replace the local coordinates $(V, y^1, \dots, y^n)$ with $(V, g^1, \dots, g^k, y^{m+1}, \dots, y^n)$ to obtain an adapted chart of $N$ relative to $f^{-1}(S)$. By definition, $f^{-1}(S)$ is a regular submanifold of $N$.

**Remark 3.2.15** As part of the proof above, we showed that

$$(g \circ f)_*^{-1}(T_0\mathbb{R}^k) = T_pN.$$

But

$$\begin{aligned} (g \circ f)_*^{-1}(T_0\mathbb{R}^k) &= f_*^{-1}(g_*^{-1}(T_0\mathbb{R}^k)) \\ &= f_*^{-1}(T_{f(p)}S) \end{aligned}$$

Hence

$$f_*^{-1}(T_{f(p)}S) = T_pN.$$

We say two submanifolds $S, S'$ *intersect transversely*, denoted $S \pitchfork S'$, if for each $p \in S \cap S'$,

$$T_pS + T_pS' = T_pM.$$

**Corollary 3.2.15.1**
If $S' \subseteq M$ is a regular submanifold that intersects the regular submanifold $S$ transversely, then $S \cap S'$ is a regular submanifold of $M$ whose codimension is equal to the sum of the codimensions of $S, S'$.

**Proof**
Apply the transversality theorem with $f = \iota_{S'}$ as the inclusion map $S' \to M$. Then $f^{-1}(S) = S \cap S'$ is a regular submanifold of $S'$. The codimension can be seen from the definition of a regular submanifold.

**Theorem 3.2.16 (Parametric Transversality)**
Let $F : M \times S \to N$ be a smooth map between manifolds and assume that $F \pitchfork Q$ for some regular submanifold $Q \subseteq N$. Then for almost every $s \in S$, the map $F_s : M \to N$ given y $F_s(x) = F(x, s)$ is transverse to $Q$ as well.

The proof of this theorem requires an application of Sard's theorem, which we will see after.

**Proof**
Since $F \pitchfork Q$, $W := F^{-1}(Q)$ is a regular submanifold of $M \times S$ by the transversality theorem. Consider the projection on the second factor, $\pi : M \times S \to S$ given by $\pi(x, s) = s$. By Sard's theorem, it suffices to show that whenever $s \in S$ is a regular value of $\pi|_W : W \to S$, then $F_S \pitchfork Q$.

Fix a regular value $s \in S$ of $\pi|_W$ and consider any $y \in F_s^{-1}(Q) \subseteq M$. Write $q := F_s(y) \in Q$. Since $(y, s) \in (\pi|_W)^{-1}(Q)$ by construction, $(\pi|_W)_{*,(y,s)} : T_{(y,s)}W \to T_sS$ is surjective by the definition of a regular value. Moreover, $F_*(T_{(y,s)}W) = T_{F_s(y)}Q$ by the remark above since $F \pitchfork Q$.

Fix any $Z_q \in T_{F_s(y)}N$. We need to find some $Y_q \in T_{F_s(y)}Q, X_y \in T_yM$ such that

$$Z_q = Y_q + (F_s)_*(X_y).$$

By assumption, $F \pitchfork Q$ so there are $Y'_q \in T_{F_s(y)}Q, X'_y \in T_yM, X_s \in T_sS$ such that

$$Z_q = Y'_q + F_*(X'_y, X'_s).$$

But $(\pi|_W)_*(T_{(y,s)}W) = T_sS$ so we can find some $(X''_y, X_s) \in T_{(y,s)}W$ such that

$$(\pi|_W)_*(X''_y, X_s) = X_s.$$

Note that this is the same $X_s$ since $\pi$ is a projection. By linearity,

$$Z_q = Y'_q + F_*(X''_y, X_s) + F_*(X'_y - X''_y, 0).$$

But $F_*(X''_y, X_s) \in T_qQ$ as $(X''_y, X_s) \in T_{(y,s)}W$ and $F_*(X'_y - X''_y, 0) = F_s(X'_y - X''_y)$, concluding the proof.

### 3.2.6 Useful Results

**Proposition 3.2.17 (Regular Submanifolds)**
Suppose $S \subseteq \mathbb{R}^2$ has the property that locall on $S$, one of the coordinates is a smooth function of the other coordinate. Then $S$ is a regular submanifold of $\mathbb{R}^2$.

**Proof**
Let $p \in S$ and a neighborhood $U \ni p$ of $S$ such that on $U \cap S$, there is a smooth function $f : A \to B$ such that $V := A \times B \subseteq U$ and $y = f(x)$ for all $(x, y) \in V \cap S$. Consider the function $F : V \to \mathbb{R}^2$ given by

$$F(x, y) = (x, y - f(x)).$$

The Jacobian is given by

$$JF(x, y) = \begin{bmatrix} 1 & 0 \\ \frac{\partial f}{\partial x} & 1 \end{bmatrix}$$

Thus $F$ is a diffeomorphism onto its image and can be used as a coordinate map. But in the chart $(V, x, y - f(x))$, $V \cap S$ is defined by the vanishing of the second coordinate $y - f(x)$. $S$ is by definition a regular submanifold of $\mathbb{R}^2$.

**Proposition 3.2.18**
The graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$

$$\Gamma(f) := \{(x, y, f(x, y)) \in \mathbb{R}^3\}$$

is a submanifold of $\mathbb{R}^3$.

**Proof**
Let $p \in \Gamma(f)$. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(x, y, z) := (x, y, z - f(x, y)).$$

Its Jacobian is given by

$$JF(x, y, z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial f}{\partial x} & -\frac{\partial f}{\partial y} & 1 \end{bmatrix}.$$

Thus $F$ is a local diffeomorphism and can be used as a coordinate map. Moreover, $S$ is globally defined by the vanishing of the third coordinate $z - f(x, y)$ and so is a regular submanifold of $\mathbb{R}^3$.

Recall a *homogeneous polynomial of degree* $k$ $F(x_0, \dots, x_n) \in \mathbb{R}[x_0, \dots, x_n]$ is a linear combination of monomials $x_0^{i_0} \dots x_n^{i_n}$ of degree $\sum_j i_j = k$.

**Lemma 3.2.19 (Euler's Formula)**
Let $F(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree $k$. For any $t \in \mathbb{R}$,

$$F(tx_0, \ldots, tx_n) = t^k F(x_0, \ldots, x_n)$$
$$\sum_i x_i \frac{\partial F}{\partial x_i} = kF.$$

**Proof**
Differentiate with respect to $t$.

On a projective space $\mathbb{R}P^n$, a homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree $k$ is not a function, since its value at a point is not necessarily unique. However, the zero set in $\mathbb{R}P^n$ of a homogeneous polynomial $F(x_0, \ldots, x_n)$ is well defined, since $F(a_0, \ldots, a_n) = 0$ if and only if

$$F(ta_0, \ldots, ta_n) = t^k F(a_0, \ldots, a_n) = 0$$

for all $t \in \mathbb{R}^\times$.

**Definition 3.2.5 (Real Projective Variety)**
The zero set of finitely many homogeneous polynomials in $\mathbb{R}P^n$ is called a *real projective variety.*

A projective variety defined by a single homogeneous polynomial of degree $k$ is called a *hypersurface* of degree $k$.

**Proposition 3.2.20**
The hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smooth if the partial derivatives are not simultaneously zero on $Z(F)$.

**Proof**
Let $p \in Z(F)$. We claim that at least one of $\partial F/\partial x_1, \partial F/\partial x_2$ is non-zero at $p$. Suppose otherwise, But then

$$0 = kF(p) = \sum_i x_i \frac{\partial F}{\partial x_i} = x_0 \frac{\partial F}{\partial x_0}.$$

But $x_0 \neq 0$ on $U_0$ so all partial derivatives vanish, a contradiction.

Recall the standard coordinates on $U_0 \simeq \mathbb{R}^2$ are $x = x_1/x_0, y = x_2/x_0$. In $U_0$,

$$F(x_0, x_1, x_2) = x_0^k F(1, x_1/x_0, x_2/x_0) = x_0^k F(1, x, y).$$

Define $f(x, y) := F(1, x, y)$ so that $f, F$ have the same zero set in $U_0$. Now, the Jacobian of $f$ is given by

$$Jf(x,y) = \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{bmatrix}.$$

But at least one of this non-zero. We can similarly show this for $U_1, U_2$.

All in all, $Z(F)$ is a regular level set of $F$ and is hence a regular submanifold of $\mathbb{R}P^n$.

**Proposition 3.2.21 (Product of Regular Submanifolds)**
If $S_i$ is a regular submanifold of the manifold $M_i, i = 1, 2$, then $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$.

**Proof**
Fix some $(p, q) \in S_1 \times S_2$ as well as adapted charts $(U, x^1, \dots, x^n), (V, y^1, \dots, y^m)$ relative to $S_1, S_2$ and about $p, q$, respectively so that $S_1 \cap U, S_2 \cap V$ are defined by the vanishing of the last $k, \ell$ coordinates, respectively. Then $(U \times V, x^1, \dots, x^n, y^1, \dots, y^m)$ is a chart about $(p, q)$ in the product manifold $M_1 \times M_2$. By construction, $(S_1 \times S_2) \cap (U \cap V)$ is defined by the vanishing the same $k + \ell$ coordinates. It follows by definition that $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$ with codimension $k + \ell$.

Recall that the *complex special linear group* $\mathrm{SL}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of complex matrices with determinant 1.

**Proposition 3.2.22**
$\mathrm{SL}(n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$.

**Proof**
It suffices to show that $\mathrm{SL}(n, \mathbb{C})$ is a regular level set of det. That it is a level set is clear. Since det is complex-valued, it suffices to show that the differential at every $A \in \mathrm{SL}(n, \mathbb{C})$ is not identically zero.

Let $A \in \mathrm{SL}(n, \mathbb{C})$ and consider the curve

$$A(t) := (1 + t)A$$

which starts at $A(0) = A$ with initial velocity $A \in T_A \mathbb{C}^{n \times n}$ (under the appropriate identification) and additionally satisfies

$$\det A(t) = (1 + t)^n \det A = (1 + t)^n.$$

But then $(\det A(t))'(0) = n \neq 0$, concluding the proof.

## 3.3 Categories and Functors

Many problems in mathematics share common features. In topology, one is interested in knowing if two topological spaces are homeomorphic and in groups theory we wish to know if two groups are isomorphic. This has given rise to the theories of categories and functors.

A category is essentially a collection of objects and arrow (*morphisms*) between objects. These arrows satisfy the abstract properties of maps and are often structure preserving. For instance, smooth manifolds and smooth maps form a category and so do vector spaces and linear maps. A functor from one category to another preserves the identity morphism and the composition of morphisms. If the target category is simpler than the domain category, this provides a way to simply problems in the original category. The tangent space construction with the differential of a smooth map is a functor from the category of smooth manifolds with a distinguished point to the category of vector spaces. The existence of the tangent space functor shows that if two manifolds are diffeomorphic, then their tangent spaces at corresponding points must be isomorphic, thereby proving the smooth invariance of dimension. Invariance of dimension in the continuous category of topological spaces and continuous maps is more difficult to show since there is no tangent space functor in the continuous category.

For a functor to be useful, it should be simple enough to be computable, yet complex enough to preserve essential features of the original category. For smooth manifolds, this delicate balance is achieved in the de Rham cohomology functor. In the rest of the book, we introduce various functors of smooth manifolds, such as the tangent bundle and differential forms, culminating in the de Rham cohomology.

As an concrete step, we first study the dual construction on vector spaces as a nontrivial example of a functor.

### 3.3.1 Categories

**Definition 3.3.1 (Category)**
A *category* is a collection of elements, called *objects*, and for any two objects $A, B$, a set $\operatorname{Mor}(A, B)$ of elements, called *morphisms* from $A$ to $B$ such that given any $f \in \operatorname{Mor}(A, B), g \in \operatorname{Mor}(B, C)$, the *composite* $g \circ f \in \operatorname{Mor}(A, C)$ is defined. Furthermore, the composition of morphisms satisfies two properties:

(i) (Identity Axiom) for each object $A$, there is an identity morphism $\operatorname{Id}_A \in \operatorname{Mor}(A, A)$ such that for any $f \in \operatorname{Mor}(A, B)$ and $\operatorname{Mor}(B, A)$,

$$f \circ \operatorname{Id}_A = f, \operatorname{Id}_A \circ g = g.$$

(ii) (Associative Axiom) for $f \in \operatorname{Mor}(A, B), g \in \operatorname{Mor}(B, C)$ and $h \in \operatorname{Mor}(C, D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

If $f \in \operatorname{Mor}(A, B)$, we often write $f : A \to B$.

**Example 3.3.1**
The collection of groups and group homomorphisms forms a category where the objects are groups and $\operatorname{Mor}(A, B)$ is the set of groups homomorphisms from $A$ to $B$.

**Example 3.3.2**
The collection of vector spaces over $\mathbb{R}$ and $\mathbb{R}$-linear maps forms a category where objects are real vector spaces and $\operatorname{Mor}(V, W)$ is the set of linear maps from $V$ to $W$.

**Example 3.3.3 (Continuous Category)**
The collection of all topological spaces with continuous maps between them is a category.

**Example 3.3.4 (Smooth Category)**
The collection of smooth manifolds with smooth maps between them is a category.

**Example 3.3.5 (Category of Pointed Manifolds)**
Let $M$ be a smooth manifold and $q \in M$. $(M, q)$ is known as a *pointed manifold*. Given any two such pairs $(N, p), (M, q)$, let $\operatorname{Mor}((N, q), (M, q))$ be the set of all smooth maps $F : N \to M$ such that $F(p) = q$. This is a category.

**Definition 3.3.2 (Object Isomorphism)**
Two objects $A, B$ in a category are *isomorphic* if there are morphisms $f : A \to B$ and $g : B \to A$ such that

$$g \circ f = \mathrm{Id}_A, f \circ g = \mathrm{Id}_B \,.$$

In this case, both $f, g$ ar called *isomorphisms*.

The usual notation for an isomorphism is $\simeq$. Thus $A \simeq B$ can mean a group isomorphism, a vector space isomorphism, a homeomorphism, or a diffeomorphism, depending on the category and the context.

### 3.3.2 Functors

**Definition 3.3.3 ((Covariant) Functor)**
A *(covariant) functor* $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object $A$ in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f : A \to B$ a morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$. such that

(i) $\mathcal{F}(\mathrm{Id}_A) = \mathrm{Id}_{\mathcal{F}(A)}$

(ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$

**Example 3.3.6**
The tangent space construction is a functor fro the category of pointed manifolds to the category of vector spaces. To each pointed manifold $(N, p)$ we associate the tangent space $T_pN$ and to each smooth map $f : (N, p) \to (M, f(p))$ we associate the differential $f_{*,p} : T_pN \to T_{f(p)}M$.

The functorial property (i) holds because if $\mathrm{Id} : N \to N$ is the identity map, then its differential $\mathrm{Id}_{*,p} : T_pN \to T_pN$ is also the identity map. The functorial property (ii) holds because in this context it is simply the chain rule.

**Proposition 3.3.7**
Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If $f : A \to B$ is an isomorphism in $\mathcal{C}$, then $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ is an isomorphism in $\mathcal{D}$.

If in condition (ii) of the definition for a covariant functor we reverse the direction of the arrow for the morphism $\mathcal{F}(f)$, we obtain a *contravariant functor*.

**Definition 3.3.4 (Contravariant Functor)**
A *contravariant functor* from from category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object $A$ in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f : A \to B$ a morphism $\mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(A)$ such that

(i) $\mathcal{F}(\mathrm{Id}_A) = \mathrm{Id}_{\mathcal{F}(A)}$

(ii) $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$

**Example 3.3.8**
Smooth functions on a manifold give rise to a contravariant functor that associates to each manifold $M$ the algebra $\mathcal{F}(M) = C^\infty(M)$ of $C^\infty$ functions on $M$ and to each smooth map $F : N \to M$ of manifolds the pullback map $\mathcal{F}(F) = F^* : C^\infty(M) \to C^\infty(N)$ given by

$$F^*(h) = h \circ F$$

for $h \in C^\infty(M)$. It can be checked that the pullback satisfies the two functorial properties.

### 3.3.3 The Dual and Multicovector Functors

Another example of a contravariant functor is the dual of a vector space.

Let $V$ be a real vector space and recall $V^\vee$ is the vector space of all linear functionals on $V$, sometimes denoted

$$V^\vee = \mathrm{Hom}(V, \mathbb{R}).$$

If $V$ is a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$, reclal that $V^\vee$ has a basis of linear functionals $\{\alpha^1, \ldots, \alpha^n\}$ defined by

$$\alpha^i(e_j) = \delta^i_j$$

for $i, j \in [n]$.

A linear map $L : V \to W$ of vector spaces induces a linear map $L^\vee$, called the *dual of* $L$, as follows. For every linear functional $\alpha : W \to \mathbb{R}$, the dual map $L^\vee : W^\vee \to V^\vee$ associates the linear functional

$$L^\vee(\alpha) = \alpha \circ L.$$

Note that the dual of $L$ reverses the direction of the arrow.

**Proposition 3.3.9**
Suppose $V, W, S$ are real vector spaces.

(i) If $\mathrm{Id}_V : V \to V$ is the identity map, then $\mathrm{Id}_V^\vee : V^\vee \to V^\vee$ is the identity map on $V^\vee$.

(ii) If $f : V \to W$ and $g : W \to S$ are linear maps, then $(g \circ f)^\vee = f^\vee \circ g^\vee$.

This proposition shows that the dual construction $\mathcal{F} : () \mapsto ()^\vee$ is a contravariant functor from the category of vector spaces to itself: for a real vector space $V$, $\mathcal{F}(V) = V^\vee$ and for $f \in \operatorname{Hom}(V, W)$, $\mathcal{F}(f) = f^\vee \in \operatorname{Hom}(W^\vee, V^\vee)$. Consequently, if $f : V \to W$ is an isomorphism, then so is its dual $f^\vee : W^\vee \to V^\vee$.

Fix a positive integer $k \geq 1$. For any linear map $L : V \to W$ of vector spaces define the *pullback map* $L^* : A_k(W) \to A_k(V)$ that sends $f \in A_k(W)$ to

$$(L^* f)(v_1, \ldots, v_k) = f(L(v_1), \ldots, L(v_k)).$$

It can be checked from the definition that $L^*$ is a linear map.

**Proposition 3.3.10**
The pullback of covectors by a linear map satisfies the two functorial properties:

(i) If $\mathrm{Id}_V : V \to V$ is the identity map on $V$, then $\mathrm{Id}_V^* = \mathrm{Id}_{A_k(V)}$, the identity map on $A_k(V)$.

(ii) If $K : U \to V$ and $L : V \to W$ are linear maps of vector spaces, then $(L \circ K)^* = K^* \circ L^* : A_k(W) \to A_k(U)$.

To each vector space $V$, we associate the vector space $A_k(V)$ of all $k$-covectors on $V$, and to each linear map $L : V \to W$ of vector spaces, we associate the pullback $A_k(L) = L^* : A_k(W) \to A_k(V)$. Then $A_k()$ is a contravariant functor from the category of vector spaces and linear maps to itself.

When $k = 1$, the space $A_1(V)$ is simply the dual space, and for any linear map $L : V \to W$, the pullback map $A_1(L) = L^*$ is the dual map $L^\vee : W^\vee \to V^\vee$. Thus the multicovector functor $A_k()$ generalizes the dual functor $()^\vee$.

## 3.4 The Rank of a Smooth Map

Recall the rank of a smooth map $f : N \to M$ at a point $p \in N$ is defined to be the rank of its differential at $p$. Two cases are of special interest: that in which $f$ has maximal rank a point and in which it has constant rank in a neighborhood.

Let $n = \dim N, m = \dim M$. In the case $f : N \to M$ has maximal rank, there are three possibilities (not necessarily exclusive). If $n = m$, then $f$ is a local diffeomorphism at $p$ by the inverse function theorem. If $n \leq m$, then the maximal rank is $n$ and $f$ is an *immersion* at $p$. Finally, if $n \geq m$, the maximal rank is $m$ and $f$ is a *submersion* at $p$.

Since manifolds are locally Euclidean, theorems on the rank of a smooth map between Euclidean spaces translate easily to theorems about manifolds. This leads to the constant rank theorem for manifolds, which gives a simple normal form for a smooth map having constant rank on an open set. This result specializes to the immersion and submersion theorems.

The image of a smooth map does not in general have a nice structure. However, using the immersion theorem we derive conditions under which the image of a smooth map is a manifold.

### 3.4.1 Constant Rank Theorem

Suppose $f : N \to M$ is a smooth map of manifolds and we want to show that the level set $f^{-1}(c)$ is a manifold for some $c \in M$. A sufficient condition from the regular level set theorem is for the differential to have maximal rank at every point of $f^{-1}(c)$. However, even if this is true, it may be difficult to show at times. The constant rank has the comparative advantage in that it is not necessary to know the precise rank of $f$; it suffices that the rank be constant.

Let us recall the constant rank theorem for Euclidean spaces.

**Theorem 3.4.1 (Constant Rank in Euclidean Space)**
If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has constant rank $k$ in a neighborhood of a point $p \in U$, there are diffeomorphisms $G$ of a neighborhood of $p \in U$ sending $p \mapsto 0 \in \mathbb{R}^n$ and $F$ of a neighborhood of $f(p) \in \mathbb{R}^m$ sending $f \mapsto 0 \in \mathbb{R}^m$ such that

$$(F \circ f \circ G)^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

Thus after a suitable change of coordinates near $p \in U$ and $f(p) \in \mathbb{R}^m$, the map $f$ assumes the form

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

**Proof**
By reordering the $f^1, \dots, f^m$ and coordinates $x^1, \dots, x^n$ if necessary, we may assume without loss of generality that the first $k \times k$-submatrix of the Jacobian $JF(p)$ at the point $p$ is non-singular. Relabel the coordinates $(x, y) = (x^1, \dots, x^k, y^1, \dots, y^{n-k}) := x$ and the function $(f, g) = (f^1, \dots, f^k, g^1, \dots, g^{n-k}) := f$.

Define $G : U \to \mathbb{R}^m$ by

$$(u, v) = G(x, y) := (f(x, y), y).$$

The Jacobian of $G$ is the block matrix given by

$$JG = \begin{bmatrix} \partial f/\partial x & \partial f/\partial y \\ 0 & \mathrm{Id} \end{bmatrix}.$$

By construction, $\det JG = \det[\partial f/\partial x] \neq 0$ and so the inverse function theorem guarantees the existence of neighborhoods $U_1 \ni p \in \mathbb{R}^n$ and $V_1 \ni G(p) \in \mathbb{R}^m$ such that $G : U_1 \to V_1$ is a diffeomorphism. By shrinking $U_1$ if necessary, we may assume that $(f, g)$ has constant rank $k$ on $U_1$.

On $V_1$,

$$(u,v) = (G \circ G^{-1})(u,v) = (f \circ G^{-1}, y \circ G^{-1})(u,v).$$

Comparing the first components give $u = (f \circ G^{-1})(u,v)$. Hence

$$\begin{aligned}((f,g) \circ G^{-1})(u,v)) &= (f \circ G^{-1}, g \circ G^{-1})(u,v) \\ &= (u, g \circ G^{-1}(u,v)) \\ &= (u, h(u,v)). && h := g \circ G^{-1}\end{aligned}$$

Since $G^{-1} : V_1 \to U_1$ is a diffeomorphism and $(f,g)$ has constant rank $k$ on $U_1$, the composite $(f,g) \circ G^{-1}$ has constant rank $k$ on $V_1$. Its Jacobian matrix is given by

$$J((f,g) \circ G^{-1})(u,v)) = \begin{bmatrix} \mathrm{Id} & 0 \\ \partial h/\partial u & \partial h/\partial v \end{bmatrix}.$$

For this matrix to have constant rank $k$, $\partial g/\partial v$ must be identically zero on $V_1$. Thus $h$ is a function of $u$ alone and we can write

$$((f,g) \circ G^{-1})(u,v) = (u, h(u)).$$

Finally, let $F : \mathbb{R}^m \to \mathbb{R}^m$ be the change of coordinates given by

$$F(x,y) = (x, y - h(x)).$$

We have

$$\begin{aligned}(F \circ f \circ G^{-1})(u,v) &= F(u, h(u)) \\ &= (u, h(u) - h(u)) \\ &= (u, 0).\end{aligned}$$

The constan rank theorem for Euclidean spaces has an immediate analogue for manifolds.

**Theorem 3.4.2 (Constant Rank)**
Let $N, M$ be manifolds of dimensions $n, m$ respectively. Suppose $f : N \to M$ has constant rank $k$ in a neighborhood of $p \in N$. Then there are charts $(U, \phi)$ centered at $p \in N$ and $(V, \Psi)$ centered at $f(p) \in M$ such that for $(r^1, \ldots, r^n) \in \phi(U)$,

$$(\Psi \circ f \circ \phi^{-1})(r^1, \ldots, r^n) = (r^1, \ldots, r^k, 0, \ldots, 0).$$

**Proof**
Choose a chart $(\bar{U}, \bar{\phi})$ about $p \in N$ and $(\bar{V}, \bar{\Psi})$ about $f(p) \in M$. Then $\bar{\Psi} \circ f \circ \bar{\phi}^{-1}$ is a map between open subsets of Euclidean spaces. Because $\bar{\phi}, \bar{\Psi}$ are diffeomorphisms, $\bar{\Psi} \circ f \circ \bar{\phi}^{-1}$

has the same constant rank $k$ as $f$ in a neighborhood of $\bar{\phi}(p) \in \mathbb{R}^n$. By the constant rank theorem for Euclidean spaces, there are diffeomorphisms $G$ of a neighborhood of $\bar{\phi}(p) \in \mathbb{R}^n$ and $F$ of a neighborhood of $(\bar{\Psi} \circ f)(p) \in \mathbb{R}^m$ such that

$$(F \circ \bar{\Psi} \circ f \circ \bar{\phi}^{-1} \circ G^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

We can then take $\phi = G \circ \bar{\phi}$ and $\Psi = F \circ \bar{\psi}$.

The constant-rank level theorem follows easily. Recall a *neighborhood* of a subset $A \subseteq M$ is an open set containing $A$.

**Theorem 3.4.3 (Constant-Rank Level Set)**
Let $f : N \to M$ be a smooth map of manifolds and $c \in M$. If $f$ has constant rank $k$ in a neighborhood of the level set $f^{-1}(c) \in N$, then $f^{-1}(c)$ is a regular submanifold of $N$ with codimension $k$.

**Proof**
Let $p \in f^{-1}(c)$ be arbitrary. By the constant rank theorem, we can find a coordinate chart $(U, \phi) = (U, x^1, \dots, x^n)$ centered at $p \in N$ and a coordinate chart $(V, \Psi) = V(y^1, \dots, y^m)$ centered at $f(p) = c \in M$ such that

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0) \in \mathbb{R}^m.$$

This shows that the level set $(\Psi \circ f \circ \phi^{-1})^{-1}(0)$ is defined by the vanishing of the coordinates $r^1, \dots, r^k$. The image of the level set $f^{-1}(c)$ under $\phi$ is precisely the level set $(\Psi \circ f \circ \phi^{-1})^{-1}(0)$, since

$$\phi(f^{-1}(c)) = \phi(f^{-1}(\Psi^{-1}(0))) = (\Psi \circ f \circ \phi^{-1})^{-1}(0).$$

Thus the level set $f^{-1}(c)$ in $U$ is defined by the vanishing of the coordinate functions $x^1, \dots, x^k$ where $x^i := r^i \circ \phi$. This proves that $f^{-1}(c)$ is a regular submanifold of $N$ with codimension $k$.

**Example 3.4.4 (Orthogonal Group)**
Let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be given by

$$f(A) := A^T A.$$

The *orthogonal group* $O(n)$ is given by

$$O(n) = f^{-1}(I).$$

It can be checked that $f$ is of constant rank on $\mathrm{GL}(n, \mathbb{R})$ and hence $O(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Consider $f : N \to M$ a map with constant rank $k$ in a neighborhood of a point $p \in N$, with charts $(U, \phi) = (U, x^1, \dots, x^n)$ about $p$ and $(V, \Psi) = (V, y^1, \dots, y^m)$ about $f(p)$ from the constant rank theorem. Note that for any $q \in Q$,

$$\begin{aligned} \phi(q) &= (x^1(q), \dots, x^n(q)) \\ (y^1(f(q)), \dots, y^n(f(q))) &= \Psi(f(q)) \\ &= (\Psi \circ f \circ \phi^{-1})(\phi(q)) \\ &= (\Psi \circ f \circ \phi^{-1})(x^1(q), \dots, x^n(q)) \\ &= (x^1(q), \dots, x^k(q), 0, \dots, 0). \end{aligned}$$

As functions on $U$,

$$(y^1 \circ f, \dots, y^m \circ f) = (x^1, \dots, x^k, 0, \dots, 0).$$

The local normal form of $f$ relative to the charts above in the constant rank theorem can be expressed in terms of the local coordinates $x^1, \dots, x^n$ and $y^1, \dots, y^m$ as follows. The map $f$ is given by

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

### 3.4.2 The Immersion & Submersion Theorems

The constant rank theorem givrs local normal forms for immersions and submersions, called the immersion theorem and submersion theorem respectively.

Consider a smooth map $f : N \to M$ between manifolds of dimension $n, m$ respectively. Let $(U, (x^i)), (V, (y^j))$ be charts about $p \in N, f(p) \in M$ respectively and consider the Jacobian of $f_{*,p}$ with respect to these charts. Then for any $p \in N$, $f_{*,p}$ is injective if and only if $n \leq m$ and the Jacobian has rank $\operatorname{rank} J = n$. Similarly, $f_{*,p}$ is surjective if and only if $n \geq m$ and $\operatorname{rank} J = m$.

Having maximal rank at a point is an *open condition* in the sense that the set

$$D_{\max}(f) := \{p \in U : f_{*,p} \text{ has maximal rank at } p\}$$

is an open subset of $U$. Indeed, the complement

$$U - D_{\max}(f) = \{p \in U : \operatorname{rank} J < k\}$$

is equivalent to the vanishing of all $k \times k$ minors of the Jacobian. As the pullback of a closed (singleton) set of finitely many continuous functions, $U - D_{\max}(f)$ is closed and so $D_{\max}(f)$ is open. In particular, if $f$ has maximal rank at $p$, then it has maximal rank at all points in some neighborhood of $p$.

**Proposition 3.4.5**
Let $N$ be an $n$-manifold and $M$ be an $m$-manifold. If a smooth map $f : N \to M$ is an immersion at a point $p \in N$, then it has constant rank $n$ in a neighborhood of $p$. If a smooth map $f : N \to M$ is a submersion at a point $p \in N$, then it has constant rank $m$ in a neighborhood of $p$.

The following theorems follow immediately as special cases of the constant rank theorem.

**Theorem 3.4.6 (Immersion/Submersion)**

(i) (Immerision theorem) Suppose $f : N \to M$ is an immersion at $p \in N$. Then there are charts $(U, \phi)$ centered at $p \in N$ and $(V, \Psi)$ centered at $f(p) \in M$ such that in a neighborhood of $\phi(p)$,

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0).$$

(ii) (Submerision theorem) Suppose $f : N \to M$ is an submersion at $p \in N$. Then there are charts $(U, \phi)$ centered at $p \in N$ and $(V, \Psi)$ centered at $f(p) \in M$ such that in a neighborhood of $\phi(p)$,

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^m, r^{m+1}, \dots, r^n) = (r^1, \dots, r^m).$$

**Corollary 3.4.6.1**
A submersion $f : N \to M$ of manifolds is an open map.

**Proof**
Let $W \subseteq N$ be open and pick a point $f(p) \in f(W)$. By the submersion theorem, there are charts $(U, \phi), (V, \Psi)$ about $p, f(p)$ such that $(\Psi \circ f \circ \phi^{-1})$ is a projection. Let $U \supseteq B \ni p$ be an open neighborhood about $p$. Then $\phi(B)$ is open in $\mathbb{R}^n$. But projections are open maps and so

$$(\Psi \circ f)(B) = (\Psi \circ f \circ \phi^{-1})(\phi(B))$$

is open in $\mathbb{R}^m$. But then

$$f(B) = \Psi^{-1}[(\Psi \circ f)(B)]$$

is an open subset of $f(W)$ containing $f(p)$. This concludes the proof by the arbitrary choice of $f(p)$.

We now derive the regular level set theorem as corollaries of both the submersion and constant rank theorems. Indeed, for a smooth map $f : N \to M$ of manifolds, a level set $f^{-1}(c)$ is regular if and only if $f$ is a submersion at every point $p \in f^{-1}(c)$. Fix one such point $p \in f^{-1}(c)$ and let $(U, \phi), (V, \Psi)$ be charts in the submersion theorem. Then $\Psi \circ f \circ \phi^{-1} =$

$\pi : \phi(U) \to \mathbb{R}^m$ is the projection onto the first $m$ coordinates,

$$\pi(r^1, \dots, r^n) = (r^1, \dots, r^m).$$

It follows that on $U$,

$$\Psi \circ f = \pi \circ \phi = (r^1, \dots, r^m) \circ \phi = (x^1, \dots, x^m).$$

It follows that

$$f^{-1}(c) = f^{-1}(\Psi^{-1}(0)) = (\Psi \circ f)^{-1}(0) = Z(\Psi \circ f) = Z(x^1, \dots, x^m).$$

So $f^{-1}(c)$ is defined by the vanishing of the $m$ coordinate functions $x^1, \dots, x^m$ and $(U, x^1, \dots, x^n)$ is an adapted chart for $N$ relative to $f^{-1}(c)$.

A third proof of the regular level set theorem using the submersion theorem proceeds as follows. On a regular level set $f^{-1}(c)$, the map $f : N \to M$ has maximal rank $m$ at every point. Since the maximality of the rank is an open condition, a regular level set $f^{-1}(c)$ has a neighborhood on which $f$ has constant rank $m$. By the constant rank level set theorem above, $f^{-1}(c)$ is thus a regular submanifold of $N$.

### 3.4.3 Images of Smooth Maps

The following are all examples of smooth maps $f : N \to M$ with $N = \mathbb{R}, M = \mathbb{R}^2$.

> **Example 3.4.7**
> $f(t) = (t^2, t^3)$ is not an immersion since $f'(0) = (0, 0)$.

> **Example 3.4.8**
> $f(t) = (t^2 - 1, t^3 - t)$ is an immersion since the equation
>
> $$f'(t) = (2t, 3t^2 - 1) = (0, 0)$$
>
> has no solutions in $t$.

> **Example 3.4.9**
> Let $M$ be the union of the graph of $y = \sin(1/x)$ on $(0, 1)$ and a smooth curve joining $(0, 0)$ and $(1, \sin 1)$. $f : \mathbb{R} \to M$ in the intuitive way is a injective immersion whose image with the subspace topology is not homeomorphic to $\mathbb{R}$.

In the examples above, $f(N)$ is not a regular submanifold of $M = \mathbb{R}^2$. We would like conditions on $f$ so that its image would be a regular submanifold of $M$.

**Definition 3.4.1 (Embedding)**
A $C^\infty$ map $f : N \to M$ is called an *embedding* if

(i) it is an (injective) immersion

(ii) the image $f(N)$ with the subspace topology is homeomorphic to $N$ under $f$.

Note that the condition that $f$ is injective in (i) is redundant since a homeomorphism is necessarily a bijection.

**Remark 3.4.10** The word "submanifold" is used differently in many contexts. Some authors give the image $f(N)$ of an injective immersion the topology inherited from $f$ rather than the subspace topology of $M$. With this topology, $f(N)$ is by definition homeomorphic to $N$. These authors define a submanifold to be the image of any injective immersion with the topology and differentiable structure inherited from $f$. Such a set is sometimes called an *immersed submanifold*. Note that if the underlying set of an immersed submanifold is given the subspace topology, it need not be a manifold at all!

For us, a submanifold without any qualifying adjective is always a *regular submanifold*.

We use the phrase "near $p$" to mean "in a neighborhood of $p$."

**Theorem 3.4.11**
If $f : N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold of $M$.

**Proof**
Fix $p \in N$. We need to show that im some neighborhood of $f(p)$, the set $f(N)$ is defined by the vanishing of $m - n$ coordinates.

By the immersion theorem, there are local coordinates $(U, x^1, \dots, x^n)$ near $p$ and $(V, y^1, \dots, y^m)$ near $f(p)$ such that $f : U \to V$ has the form

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0).$$

Thus $f(U)$ is defined in $V$ by the vanishing of the coordinates $y^{n+1}, \dots, y^m$. This is not sufficient since we do not know that $f(N) \cap V = f(U) \cap V$.

Since $f(N)$ with the subspace topology is homeomorphic to $N$, the image $f(U)$ is open in $f(N)$. By the definition of the subspace topology, there is an open set $V' \subseteq M$ such that $V' \cap f(N) = f(U)$. In $V \cap V'$,

$$(V \cap V') \cap f(N) = V \cap f(U) = (V \cap V') \cap f(U)$$

and $f(U)$ is defined by the vanishing of $y^{n+1}, \dots, y^m$. Thus $(V \cap V', y^1, \dots, y^m)$ is an adapted chart containing $f(p)$ for $f(N)$. This concludes the proof by the arbitrary choice of $p$.

**Theorem 3.4.12**
If $N$ is a regular submanifold of $M$, then the inclusion $\iota : N \to M$ is an embedding.

**Proof**
Since a regular submanifold has the subspace topology and $\iota(N)$ also has the subspace topology, $\iota : N \to \iota(N)$ is certainly a homeomorphism. It remains to show that $\iota : N \to M$ is an immersion.

Fix $p \in N$. Choose an adapted chart $(V, y^1, \ldots, y^n, y^{n+1}, \ldots, y^m)$ for $M$ about $p$ such that $V \cap N$ is the zero set of $y^{n+1}, \ldots, y^m$. Relative to the charts $(V \cap N, y^1, \ldots, y^n)$ for $N$ and $(V, y^1, \ldots, y^m)$ for $M$, the inclusion $i$ is given by

$$(y^1, \ldots, y^n) \mapsto (y^1, \ldots, y^n, 0, \ldots, 0)$$

which shows that $\iota$ is an immersion.

The image of an embedding is often called an *embedded submanifold*. Our results above show that an embedded submanifold and a regular submanifold are the same thing.

### 3.4.4 Smooth Maps into Submanifold

Suppose $f : N \to M$ is a smooth map whose image $f(N)$ lies in a submanifold $S \subseteq M$. Is the induced map $\tilde{f} : N \to S$ also smooth? The answer depends on whether $S$ is a regular submanifold or an immersed submanifold of $M$.

**Theorem 3.4.13**
Suppose $f : N \to M$ is $C^\infty$ and the image of $f$ lies in a subset $S \subseteq M$. If $S$ is a regular submanifold of $M$, then the induced map $\tilde{f} : N \to S$ is also $C^\infty$.

**Proof**
Denote the dimensions of $N, M, S$ by $n, m, s$, respectively. Fix $p \in N$. Since $S$ is a regular submanifold of $M$, there is an adapted coordinate chart $(V, \Psi) = (V, y^1, \ldots, y^m)$ for $M$ about $p$ such that $S \cap V$ is the zero set of $y^{s+1}, \ldots, y^m$, with coordinate chart $(S \cap V, \Psi_S) = (S \cap V, y^1, \ldots, y^s)$. By the continuity of $f$, we can choose a neighborhood $U \ni p$ of $p$ such that $f(U) \subseteq V$. Then $f(U) \subseteq V \cap S$ by construction, so that for any $q \in U$,

$$(\Psi \circ f)(q) = (y^1(f(q)), \ldots, y^s(f(q)), 0, \ldots, 0).$$

It follows that on $U$,

$$\Psi_S \circ \tilde{f} = (y^1 \circ f, \ldots, y^s \circ f).$$

Since the coordinates $y^1 \circ f, \ldots, y^s \circ f$ are smooth on $U$, $\tilde{f}$ is also smooth on $U$ and in particular at $p$. We conclude the proof by the arbitrary choice of $p \in N$.

**Example 3.4.14 (Multiplication Map of SL($n, \mathbb{R}$))**
The multiplication mapp

$$\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

given by $(A, B) \mapsto (A, B)$ is smooth since each entry of the resulting matrix is a polynomial function of the entries of the input matrices. However, it is not immediately obvious that the induced map

$$\bar{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is smooth as the canonical coordinate system on $\mathrm{GL}(n, \mathbb{R})$ is not a coordinate system on $\mathrm{SL}(n, \mathbb{R})$.

Now, $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of the product manifold $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$. Thus the inclusion map

$$\iota : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$$

is smooth. It follows thaat the composition $\mu \circ \iota$ is smooth as well. Because the image of $\mu \circ \iota$ lies in $\mathrm{SL}(n, \mathbb{R})$, a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$, we can apply the previous theorem to deduce that the induced multiplication map is smooth.

### 3.4.5 The Tangent Plane to a Surface in $\mathbb{R}^3$

Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ has no critical points on its zero set $N = f^{-1}(0)$. By the regular level set theorem, $N$ is a regular submanifold of $\mathbb{R}^3$. Since the inclusion map $\iota : N \to \mathbb{R}^3$ on regular submanifolds is an embedding (immersion), $\iota_{*,p} : T_pN \to T_p\mathbb{R}^3$ is injective at every point $p \in N$. We can thus identify the tangent plane $T_pN$ as a plane in $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$. We would like to find the equation of this plane.

Supppose $v = \sum_i v^i \partial/\partial x^i|_p$ is a tangent vector in $T_pN$. Under the linear isomorphism $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$, we identify $v$ with the vector $\langle v^1, v^2, v^3 \rangle \in \mathbb{R}^3$. Let $c(t)$ be a curve in starting at $c(0) = p$ and initial velocity $c'(0) = \langle v^1, v^2, v^3 \rangle$. Since $c(t)$ lies in $N$, $f(c(t)) = 0$ for every $t$. It follows by the chain rule that

$$0 = \frac{d}{dt} f(c(t)) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(c(t))(c^i)'(t).$$

At $t = 0$,

$$0 = \frac{d}{dt} f(c(0)) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p) v^i.$$

Since the vector $v = \langle v^1, v^2, v^3 \rangle$ represents the arrow from $p = (p^1, p^2, p^3)$ to $x = (x^1, x^2 x^3) \in T_pN$, we usually make the substitution $v^i = x^i - p^i$. This amounts to translating the tangent plane from the origin to "$p$". Thus the tangent plane to $N$ at $p$ is defined by the equation

$$0 = \frac{d}{dt} f(c(0)) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p)(x^i - p^i).$$

One interpretation of this equation is that the gradient vector

$$\left\langle \partial f/\partial x^1(p), \partial f/\partial x^2(p), \partial f/\partial x^3(p) \right\rangle$$

of $f$ at $p$ is normal to any vector in the tangent plane.

We remark that to recover the tangent vector $v \in T_pN$, we do not use the substitution and instead directly compute the $v^i$'s.

**Example 3.4.15 (Tangent Plane to a Sphere)**
Let $f(x, y, z) := x^2 + y^2 + z^2 - 1$. To get the equation of the tangent plane to the unit sphere $S^2 = f^{-1}(0)$ in $\mathbb{R}^3$ at $(a, b, c) \in S^2$, we first compute the partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \\ \frac{\partial f}{\partial y} &= 2y \\ \frac{\partial f}{\partial z} &= 2z. \end{aligned}$$

At $p = (a, b, c)$,

$$\begin{aligned} \frac{\partial f}{\partial x}(p) &= 2a \\ \frac{\partial f}{\partial y}(p) &= 2b \\ \frac{\partial f}{\partial z}(p) &= 2c. \end{aligned}$$

The equation determining the tangent sphere that we derived above is given by

$$\begin{aligned} 2a(x-a) + 2b(y-b) + 2x(z-c) &= 0 \\ ax + by + cz &= 1. \qquad a^2 + b^2 + c^2 = 1 \end{aligned}$$

### 3.4.6 The Differential of an Inclusion Map

Let $\iota : S^1 \to \mathbb{R}^2$ be the inclusion map of the unit circle. Denote by $x, y$ the standard coordinates on $\mathbb{R}^2$ and $\bar{x} = i^*x, \bar{y} = i^*y$ their restrictions to $S^1$. On the upper semicircle

$$U = \{(a, b) \in S^1 : b > 0\},$$

$\bar{x}$ is a local coordinate, so that $\partial/\partial\bar{x}$ is defined.

We know that

$$u \left.\frac{\partial}{\partial x}\right|_p + v \left.\frac{\partial}{\partial y}\right|_p = \iota_* \left( \left.\frac{\partial}{\partial \bar{x}}\right|_p \right) = \left.\frac{\partial}{\partial \bar{x}}\right|_p (\cdot \circ \iota)$$

for some $u, v \in \mathbb{R}$. Evaluating at $\bar{x}, \bar{y}$ yields the result

$$\iota_* \left( \left.\frac{\partial}{\partial \bar{x}}\right|_p \right) = \left.\frac{\partial}{\partial x}\right|_p + \frac{\partial \bar{y}}{\partial \bar{x}} \cdot \left.\frac{\partial}{\partial y}\right|_p .$$

Thus although $\iota_* : T_pS^1 \to T_p\mathbb{R}^2$ is injective, we cannot identify $\partial/\partial\bar{x}|_p$ with $\partial/\partial x|_p$.

**Remark 3.4.16** The formula for the image of $\partial/\partial\bar{x}|_p$ holds for any smooth curve in $C \subseteq \mathbb{R}^2$, such that there is a chart in $C$ on which $\bar{x}$, the restriction of $x$ to $C$. is a local coordinate.

Now consider the unit sphere $S^2$. On the upper hemisphere, we have the coordinate map $\phi = (u, v)$ where $u, v$ are the first and second coordinate maps in $\mathbb{R}^3$. Thus the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of $S^2$ at any point $p = (a, b, c)$ on the upper hemisphere.

Let $\iota : S^2 \to \mathbb{R}^3$ be the inclusion map and $x, y, z$ the standard coordinates on $\mathbb{R}^3$. Then

$$\iota_* \left( \left.\frac{\partial}{\partial u}\right|_p \right) = \alpha^1 \left.\frac{\partial}{\partial x}\right|_p + \beta^1 \left.\frac{\partial}{\partial y}\right|_p + \gamma^1 \left.\frac{\partial}{\partial z}\right|_p$$
$$\iota_* \left( \left.\frac{\partial}{\partial v}\right|_p \right) = \alpha^2 \left.\frac{\partial}{\partial x}\right|_p + \beta^2 \left.\frac{\partial}{\partial y}\right|_p + \gamma^2 \left.\frac{\partial}{\partial z}\right|_p .$$

By a similar calculation to the above, we can check that $\alpha^1 = \beta^2 = 1$ and $\beta^1 = \alpha^2 = 0$. At the point $p = (a, b, c)$,

$$\begin{aligned}
\gamma^1 &= \iota_* \left( \frac{\partial}{\partial u}\bigg|_p \right)(z) \\
&= \frac{\partial}{\partial u}\bigg|_p (z \circ i) \\
&= \frac{\partial}{\partial u}\bigg|_p (\sqrt{1 - u^2 - v^2}) \\
&= \frac{-2u}{2\sqrt{1 - u^2 - v^2}}\bigg|_p \\
&= -\frac{a}{c}. \\
\gamma^2 &= \dots \\
&= -\frac{b}{c}.
\end{aligned}$$

### 3.4.7 Useful Results

**Proposition 3.4.17**
Every smooth map $f$ from a compact manifold $N \to \mathbb{R}^m$ has a critical point.

**Proof**
Suppose towards a contradiction that $f_{*,p}$ is surjective at every $p \in N$, ie it is a submersion. Let $\pi : \mathbb{R}^m \to \mathbb{R}$ denote the projection onto the first coordinate and consider $\pi \circ f : N \to \mathbb{R}$. It is clear that $\pi$ is a submersion. But then $\pi \circ f$ is a submersion as well.

Recall that continuous maps attain their extrema on compact sets. Such an extreme is necessarily a critical point of $\pi \circ f$, which contradicts that $\pi \circ f$ is a submersion.

**Proposition 3.4.18**
Any injective immersion $f : N \to M$ on a compact manifold $N$ is an embedding.

**Proof**
An injective map is bijective onto its image $f(N) \subseteq M$. Moreover, $f$ is an immersion by assumption. It suffices to check that $f^{-1}$ is continuous in order to conclude that $f$ is homeomorphic onto its image and thus is an embedding.

We check that $f$ is a closed map, so the pre-images of closed sets under $f^{-1}$ are closed and so $f^{-1}$ is continuous. This is purely a topological results. Any closed subset of a compact space $N$ is compact. Hence the continuous image under $f$ is also compact and therefore closed.

This concludes the proof.

## 3.5 The Tangent Bundle

A *smooth vector bundle* over a smooth manifold $M$ is a smoothly varying family of vector spaces, parametrized by points of $M$, that locally looks like a product. The collection of tangent spaces to a manifold has the structure of a vector bundle over the manifold, called the *tangent bundle*. A smooth map between two manifolds induces a bundle map between two manifolds. This the tangent bundle construction is a functor from the category of smooth manifolds to the category of vector bundles.

For us, the importance of the vector bundle point of view comes from its role in unifying concepts. A *section* of a vector bundle $\pi : M \to E$ is a mapping from each point of $M$ into the *fiber* of the *bundle* over the point. Both vector fields and differential forms on a manifold are sections of vector bundles over the manifold.

### 3.5.1 The Topology of the Tangent Bundle

Let $M$ be a smooth manifold. Recall that at each $p \in M$, the tangent space $T_pM$ is the vector space of all point-derivations of $C_p^\infty(M)$, the algebra of germs of $C^\infty$ functions at $p$.

**Definition 3.5.1 (Tangent Bundle)**
The *tangent bundle* of a smooth manifold $M$ is the (disjoint) union of all the tangent spaces of $M$

$$TM := \bigsqcup_{p\in M} T_pM.$$

In general, if $\{A_i\}_{i\in I}$ is an indexed collection of subsets of a set $S$, then their *disjoint union* is defined to be the set

$$\bigsqcup_{i\in I} A_i := \bigcup_{i\in I} \{i\} \times A_i.$$

Since the union $\cup_{p\in M} T_p$ is already disjoint, it is not so important to us whether to specify $\cup, \sqcup$ in the definition.

There is a natural map $\pi : TM \to M$ given by $\pi(v) = p$ where $v \in T_pM$. At times, we use the notation $(p, v)$ to denote a tangent vector $v \in T_pM$ to make explicit the point $p \in M$ at which $v$ is a tangent vector.

As defined, $TM$ has no topology or manifold structure. We will make it into a smooth manifold and further show that it is a smooth vector bundle over $M$. We focus for now on defining a topology on $TM$.

Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a coordinate chart on $M$. Define

$$TU = \bigcup_{p \in U} T_pU = \bigcup_{p \in U} T_pM.$$

Recall that a basis for $T_pU = T_pM$ is the set of partial derivatives of coordinate functions. Thus any $v \in T_pM$ can be uniquely written as

$$v = \sum_i c^i \left. \frac{\partial}{\partial x^i} \right|_p.$$

The coeficients $c^i = c^i(v)$ depend on $v$ and so are functions on $TU$. Let $\bar{x}^i = x^i \circ \pi$ where $\pi(v) = p$ for $v \in T_pM$ and define the map $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ by

$$v \mapsto (x^1(p), \dots, x^n(p), c^1(v), \dots, c^n(v)) = (\bar{x}^1, \dots, \bar{x}^n, c^1, \dots, c^n)(v).$$

Then $\tilde{\phi}$ has an inverse

$$(\phi(p), c^1, \dots, c^n) \mapsto \sum_i c^i \left. \frac{\partial}{\partial x^i} \right|_p.$$

This means we can use $\tilde{\phi}$ to transfer the topology of $\phi(U) \times \mathbb{R}^n$ to $TU$: a set in $TU$ is open if and only if $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$, under the standard topology of $\mathbb{R}^{2n}$. By construction, $TU$ is homeomorphic to $\phi(U) \times \mathbb{R}^n$. If $V \subseteq U$ is open, then $\phi(V) \times \mathbb{R}^n$ is an open subset of $\phi(U) \times \mathbb{R}^n$. Hence the relative topology on $TV$ as a subset of $TU$ is the same as the topology induced fron the bijection $\tilde{\phi}|_{TV} : TV \to \phi(V) \times \mathbb{R}^n$.

Let $\phi_* : T_pU \to T_{\phi(p)}\mathbb{R}^n$ be the differential of the coordinate map $\phi$ at $p$. We may identity $\phi_*(v)$ with some column vector $\langle c^1, \dots, c^n \rangle \in \mathbb{R}^n$ where $c^i$'s are the coefficients of the tangent vector $\phi_*(v)$ with respect to the standard basis of $T_{\phi(p)}\mathbb{R}^n$. Thus another way to describe $\tilde{\phi}$ is

$$\tilde{\phi} = (\phi \circ \pi, \phi_*).$$

Let $\mathcal{B}$ be the collection of all open subsets of $T(U_\alpha)$ for all coordinate open sets $U_\alpha$ in $M$. That is,

$$\mathcal{B} = \bigcup_\alpha \{A : A \text{ open in } T(U_\alpha) \text{ for the coordinate open set } U_\alpha \subseteq M\}.$$

**Lemma 3.5.1**

(i) For any manifold $M$, the set $TM$ is the union of all $A \in \mathcal{B}$.

(ii) Let $U, V$ be coordinate open sets in a manifold $M$. If $A$ is open in $TU$ and $B$ is oepn in $TV$, then $A \cap B$ is open in $T(U \cap V)$.

It follows from this lemma that $\mathcal{B}$ forms a basis for some topology on $TM$. We given the tangent bundle $TM$ the topology generated by the basis $\mathcal{B}$.

**Proof**
(i) Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas for $M$. Then

$$TM = \bigcup_\alpha T(U_\alpha) \subseteq \bigcup_{A \in \mathcal{B}} A \subseteq TM.$$

(ii) Since $T(U \cap V)$ is a subspace of $TU$, $A \cap T(U \cap V)$ must be open in $T(U \cap V)$. Similarly, $B \cap T(U \cap V)$ is open in $T(U \cap V)$. But then

$$A \cap B \subseteq TU \cap TV = T(U \cap V)$$

so that $A \cap B$ is open in $T(U \cap V)$.

**Lemma 3.5.2**
A topological manifold $M$ has a countable basis consisting of coordinate open sets.

**Proof**
Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on $M$ and $\mathcal{B} = \{B_i\}$ a countable basis for $M$. For each coordinate open set $U_\alpha$ and $p \in U_\alpha$, there is a basic open set $B_{p,\alpha} \in \mathcal{B}$ such that

$$p \in B_{p,\alpha} \subseteq U_\alpha.$$

The collection $\{B_{p,\alpha}\} \subseteq \mathcal{B}$ satisfies the desired properties.

**Proposition 3.5.3**
The tangent bundle $TM$ of a manifold $M$ is second countable.

**Proof**
Let $\{(U_i, \phi)\}_i$ be a countable atlas for $M$. Since $TU_i$ is homeomorphic to the open subset $\phi_i(U_i) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ and any subset of a Euclidean space is secound countable, $TU_i$ is second countable. Let $\{B_{i,j}\}_j$ be a countable basis for $TU_i$. Then $\{B_{i,j}\}_{i,j}$ is a countable basis for $TM$.

**Proposition 3.5.4**
The tangent bundle $TM$ of a manifold is Hausdorff.

**Proof**
Let $(p, X_p) \neq (q, X_q) \in TM$.

If $p \neq q$, then since $M$ is hausdorff, we can find open sets $U, V \subseteq M$ separating $p, q$. By shrinking $U, V$ if necessary, we may assume that $U, V$ are coordinate open sets. Then

> $TU, TV$ are disjoint open sets in $TM$ separating $(p, X_p), (q, X_q)$.
>
> Suppose now that $p = q$ and $X_p \neq X_q$. Let $(U, \phi)$ be a chart about $p$. Then $TU$ is homeomorphic to an open subset of $\phi(U) \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$ through the map $\tilde{\phi}$. But Euclidean space is Hausdorff so there are open sets $V_p, V_q$ separating $\tilde{\phi}(p, X_p), \tilde{\phi}(p, X_q)$. Since $\tilde{\phi}$ is a homeomorphism, $\tilde{\phi}^{-1}(V_p), \tilde{\phi}^{-1}(V_q)$ separate $(p, X_p), (p, X_q)$ as desired.

### 3.5.2 The Manifold Structure on the Tangent Bundle

The most natural set of charts to propose as an atlas is the set

$$\{(TU_\alpha, \tilde{\phi}_\alpha)\}.$$

Our goal is now to show that this is indeed a smooth structure on $TM$. We already have $TM = \cup_\alpha TU_\alpha$. It remains to check that $\tilde{\phi}_\alpha, \tilde{\phi}_\beta$ are smoothly compatible on $(TU_\alpha) \cap (TU_\beta)$.

Recall if $(U, x^1, \dots, x^n), (V, y^1, \dots, y^n)$ are two charts on $M$, then for any $p \in U \cap V$ there are two bases singled out for the tangent space $T_pM$: $\{\partial/\partial x^j|_p\}_j$ and $\{\partial/\partial y^i|_p\}_i$. So any tangent vector $v \in T_pM$ has two descriptions:

$$v = \sum_j a^j \left.\frac{\partial}{\partial x^j}\right|_p = \sum_i b^i \left.\frac{\partial}{\partial y^i}\right|_p.$$

We can translate the coefficients easily by applying both sides to $x^k$.

$$a^k = \left(\sum_j a^j \frac{\partial}{\partial x^j}\right) x^k = \left(\sum_i b^i \frac{\partial}{\partial y^j}\right) x^k = \sum_i b^i \frac{\partial x^k}{\partial y^i}.$$

Similarly, applying both sides to $y^k$ yields

$$b^k = \sum_j a^j \frac{\partial y^k}{\partial x^j}.$$

Write $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $\phi_\alpha = (x^1, \dots, x^n)$, and $\phi_\beta = (y^1, \dots, y^n)$. Then

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} : \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \phi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$$

is given by

$$\begin{aligned}(\phi_\alpha(p), a^1, \dots, a^n) &\mapsto \left(p, \sum_j a^j \left.\frac{\partial}{\partial x^j}\right|_p\right) \\ &\mapsto \left((\phi_\beta \circ \phi_\alpha)^{-1}(\phi_\alpha(p)), b^1, \dots, b^n\right).\end{aligned}$$

But recall we computed

$$\begin{aligned} b^i &= \sum_j a^j \frac{\partial y^i}{\partial x^j}(p) \\ &= \sum_j a^j \frac{\partial(\phi_\beta \circ \phi_\alpha^{-1})^i}{\partial r^j}(\phi_\alpha(p)). \end{aligned}$$

By the definition of a smooth atlas, $\phi_\beta \circ \phi_\alpha^{-1}$ is $C^\infty$. Therefore $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ is $C^\infty$. This completes the proof that the tangent bundle $TM$ is a smooth manifold with $\{(TU_\alpha, \tilde{\phi}_\alpha)\}$ as a smooth atlas.

### 3.5.3 Vector Bundles

On the tangent bundle $TM$ of a smooth manifold $M$, the natural projection map $\pi : TM \to M$ given by

$$\pi(p, v) := p$$

makes $TM$ into a $C^\infty$ *vector bundle* over $M$. We make this precise in this section.

**Definition 3.5.2 (Fiber)**
Given any map $\pi : E \to M$, we call the pre-image $\pi^{-1}(p)$ of $p \in M$ the *fiber at* $p$.

We usually denote the fiber at $p$ as $E_p$. For any two maps $\pi : E \to M, \pi' : E' \to M$, a map $\phi : E \to E'$ is said to be *fiber-preserving* if $\phi(E_p) \subseteq E'_p$ for all $p \in M$.

**Definition 3.5.3 (Locally Trivial)**
A surjective smooth map $\pi : E \to M$ of manifolds is said to be *locally trivial of rank* $r$ if

(i) each fiber $\pi^{-1}(p)$ has the structure of a vector space of dimension $r$

(ii) for each $p \in M$, there is an open neighborhood $U \ni p$ and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for every $q \in U$ the restriction

$$\phi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism.

Such an open set $U$ in (ii) is called a *trivializing open set* for $E$ and $\phi$ is a *trivialization* of $E$ over $U$. Also note that $\phi$ is fiber-preserving with respect to the projection map $U \times \mathbb{R}^r \to U$.

The collection $\{(U, \phi)\}$ with $\{U\}$ being an open cover of $M$, is called a *local trivialization* for $E$, and $\{U\}$ is a *trivializing open cover* of $M$ for $E$.

**Definition 3.5.4 (Smooth Vector Bundle)**
A $C^\infty$ *vector bundle of rank* $r$ is a triple $(E, M, \pi)$ consisting of manifolds $E, M$ and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank $r$.

The manifold $E$ is called the *total space* of the vector bundle and $M$ is the *base space*. By abuse of language, we say that $E$ is a *vector bundle over* $M$.

Let $(E, M, \pi)$ be a vector bundle of rank $r$. For any regular submanifold $S \subseteq M$, the triple $(\pi^{-1}S, S, \pi|_{\pi^{-1}(S)})$ is also a smooth vector bundle over $S$, called the *restriction* of $E$ to $S$. We often write the restriction as $E|_S$ instead of $\pi^{-1}S$.

Properly speaking, the tangent bundle of a manifold $M$ is a triple $(TM, M, \pi)$, and $TM$ is the total space of the tangent bundle. Here $\pi$ is the canonical projection as aforementioned. In common usage, $TM$ is often referred to as the tangent bundle.

**Example 3.5.5 (Product Bundle)**
Given a manifold $M$, let $\pi : M \times \mathbb{R}^r \to M$ be the projection onto the first factor. Then $M \times \mathbb{R}^r \to M$ is a vector bundle of rank $r$, called the *product bundle* of rank $r$ over $M$. The vector space structure on the fiber $\pi^{-1}(p)$ is the obvious one.

A local trivialization on $M \times \mathbb{R}$ is given by the identity map $\mathrm{Id}_{M\times\mathbb{R}}$. The infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the circle.

Let $\pi : E \to M$ be a smooth vector bundle. Suppose $(U, \Psi) = (U, x^1, \ldots, x^n)$ is a chart on $M$ and

$$\begin{aligned} \phi &: E|_U \to U \times \mathbb{R}^r \\ &\phi(e) = (\pi(e), c^1(e), \ldots, c^r(e)) \end{aligned}$$

is a trivialization of $E$ over $U$. Then

$$\begin{aligned} &(\Psi \times \mathrm{Id}) \circ \phi \\ &= (x^1, \ldots, x^n, c^1, \ldots, c^r) : E|_U \to U \times \mathbb{R}^r \to \Psi(U) \times \mathbb{R}^r \\ &\subseteq \mathbb{R}^n \times \mathbb{R}^r \end{aligned}$$

is a diffeomorphism of $E|_U$ onto its image and so is a chart on $E$. We call $x^1, \ldots, x^n$ the *base coordinates* and $c^1, \ldots, c^r$ the *fiber coordinates* of the chart

$$(E|_U, (\Psi \times \mathrm{Id}) \circ \phi)$$

on $E$. Note that the fiber coordinates $c^i$ depend only on the trivialization $\phi$ of the bundle $E|_U$ and not on the trivialization $\Psi$ of the base $U$.

Let $\pi_E : E \to M$ and $\pi_F : F \to N$ be two vector bundles, possibly of different ranks.

**Definition 3.5.5 (Bundle Map)**
A *bundle map* from $E$ to $F$ is a pair of maps $(f, \tilde{f})$ where $f : M \to N$ and $\tilde{f} : E \to F$ such that

(i) $\pi_F \circ \tilde{f} = f \circ \pi_E$

(ii) $\tilde{f}$ is linear on each fiber, ie for each $p \in M$, $\tilde{f} : E_p \to F_{f(p)}$ is a linear map of vector spaces.

The collection of all vector bundles together with bundle maps between them forms a category.

**Example 3.5.6**
A smooth map $f : N \to M$ between manifolds induces a bundle map $(f, \tilde{f})$, where $\tilde{f} : TN \to TM$ is given by

$$\tilde{f}(p, v) = (f(p), f_*(v)) \in \{f(p)\} \times T_{f(p)}M \subseteq TM$$

for all $v \in T_pN$. This gives rise to a covariant functor $T$ from the vategory of smooth manifolds and smooth maps to the cateogry of vector bundles and bundle maps: To each manifold $M$, we associate its tangent bundle $TM$, and to each smooth map $f : N \to M$ between manifolds, we associate the bundle map

$$Tf = (f : N \to M, \tilde{f} : TN \to TM).$$

If $E, F$ are two vector bundles over the same manifold $M$, then a bundle map from $E$ to $F$ over $M$ is a bundle map in which the base map is the identity $\mathrm{Id}_M$. For a fixed manifold $M$, we can also consider the category of all smooth vector bundles over $M$ and smooth vector bundles over $M$. In this category, it makes sense to speak of an isomorphism of vector bundles over $M$. Any vector bundle over $M$ isomorphic over $M$ to the product bundle $M \times \mathbb{R}^r$ is called a *trivial bundle*.

### 3.5.4 Smooth Sections

**Definition 3.5.6 (Section)**
A section of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{Id}_M$.

This condition simply means that for each $p \in M$, $s$ maps $p$ into the fiber $E_p$ above $p$. We can visualize a section as a "cross-section" of the bunndle. We say that a section is *smooth* if it is smooth as a map from $M \to E$.

**Definition 3.5.7 (Vector Field)**
A *vector field* $X$ on a manifold $M$ is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$.

In terms of the tangent bundle, a vector field on $M$ is imply a section of the tangent bundle $\pi : TM \to M$ and the vector field is *smooth* if it is smooth as a map from $M \to TM$.

**Example 3.5.7**
The formula

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

defines a smooth vector field on $\mathbb{R}^2$.

Let $s, t : M \to E$ be smooth sections of a smooth vector bundle $\pi : E \to M$ and $f \in C^\infty(M)$. We define the sum $s + t : M \to E$ and product $fs : M \to E$ as follows:

$$(s + t)(p) := s(p) + t(p) \in E_p \qquad p \in M$$
$$(fs)(p) := f(p)s(p) \in E_p. \qquad p \in M$$

**Proposition 3.5.8**
Let $s, t : M \to E$ be smooth sections of a smooth vector bundle $\pi : E \to M$ and $f \in C^\infty(M)$.

(i) $s + t$ is a smooth section of $E$

(ii) $fs$ is a smooth section of $E$

**Proof**
(i) It is clear that $s + t$ is a section of $E$. To show that it is smooth, fix a point $p \in M$ and let $V \ni p$ be a trivializing open set for $E$, with smooth trivialization

$$\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r.$$

Suppose that for $q \in V$,

$$(\phi \circ s)(q) = (q, a^1(q), \ldots, a^r(q))$$
$$(\phi \circ t)(q) = (q, b^1(q), \ldots, b^r(q)).$$

Since $s, t$ are smooth, the components $a^i, b^i$ are smooth on $V$. Since $\phi$ is linear on each fiber,

$$(\phi \circ (s + t))(q) = (q, a^1(q) + b^1(q), \ldots, a^r(q) + b^r(q)).$$

This proves that $s+t$ is a smooth map on $V$ and hence at $p$. Since $p$ is an arbitrary point of $M$, the section $s + t$ is smooth on $M$.

(ii) We again use the linearity of $\phi$ along each fiber to deduce that

$$(\phi \circ (fs))(q) = (q, f(q)a^1(q), \ldots, f(q)a^r(q)).$$

Since smoothness is preserved under multiplication, this function is smooth. We omit the details as it is similar to (i).

Denote the set of all smooth sections of $E$ by $\Gamma(E)$. The proposition above shows that $\Gamma(E)$ is not only a vector space over $\mathbb{R}$, but also a module over the ring $C^\infty(M)$. For any open subset $U \subseteq M$, one can also consider the vector space $\Gamma(U, E)$ of smooth sections of $E$ over $U$. Then $\Gamma(U, E)$ is both a vector space over $\mathbb{R}$ and a $C^\infty(U)$-module. Note that $\Gamma(M, E) = \Gamma(E)$. To contrast with sections over a proper subset $U$, a section over the entire manifold $M$ is called a *global section*.

### 3.5.5 Smooth Frames

**Definition 3.5.8 (Frame)**
A *frame* for a vector bundle $\pi : E \to M$ over an open set $U$ is a collection of sections $s_1, \ldots, s_r$ of $E$ over $U$ such that at each point $p \in U$, the elements $s_1(p), \ldots, s_r(p)$ form a basis for the fiber $E_p := \pi^{-1}(p)$.

A frame is said to be *smooth* if $s_1, \ldots, s_r$ are smooth as sections of $E$ over $U$. A frame for the tangent bundle $TM \to M$ over an open set $U$ is simply called a *frame on $U$*.

**Example 3.5.9**
Let $M$ be a manifold and $e_1, \ldots, e_r$ the standard basis for $\mathbb{R}^n$. Define $\bar{e}_i : M \to M \times \mathbb{R}^r$ by

$$\bar{e}_i(p) := (p, e_i).$$

Then $\bar{e}_1, \ldots, \bar{e}_r$ is a smooth frame for the product bundle $M \times \mathbb{R}^r \to M$.

**Example 3.5.10 (The Frame of a Trivialization)**
Let $\pi : E \to M$ be a smooth vector bundle of rank $r$. If $\phi : E|_U \to U \times \mathbb{R}^r$ is a trivialization of $E$ over an open set $U$, then $\phi^{-1}$ carries the smooth frame $\bar{e}_1, \ldots, \bar{e}_r$ of the product bundle $U \times \mathbb{R}^r$ to a smooth frame $t_1, \ldots, r_t$ for $E$ over $U$:

$$t_i(p) = \phi^{-1}(\bar{e}_i(p)) = \phi^{-1}(p, e_i)$$

for each $p \in U$. We call $t_1, \ldots, t_r$ the smooth frame over $U$ of the trivialization $\phi$.

**Lemma 3.5.11**
Let $\phi : E|_U \to U \times \mathbb{R}^r$ be a trivialization over an open set $U$ of a smooth vector bundle $E \to M$, and $t_1, \ldots, t_r$ the smooth frame over $U$ of the trivialization. Then a section

$$s = \sum_i b^i t_i$$

of $E$ over $U$ is smooth if and only if its coefficients $b^i$ relative to the frame $t_1, \ldots, t_r$ are smooth.

**Proof**
Suppose the section $s = \sum_i b^i t_i$ of $E$ over $U$ is smooth. Then $\phi \circ s$ is smooth. But

$$\begin{aligned}(\phi \circ s)(p) &= \sum_i b^i(p)\phi(t_i(p)) \\ &= \sum_i b^i(p)(p, e_i) \\ &= \left(p, \sum_i b^i(p)e_i\right).\end{aligned}$$

Thus $b^i(p)$ are simply the fiber coordinates of $s(p)$ relative to the trivialization $\phi$. Since $\phi \circ s$ is smooth, all the $b^i$'s must be smooth.

The converse holds since the collections of smooth sections is a module over $C^\infty(U)$.

**Proposition 3.5.12 (Characterization of Smooth Sections)**
Let $\pi : E \to M$ be a smooth vector bundle and $U \subseteq M$ an open subset. Suppose $s_1, \ldots, s_r$ is a smooth frame for $E$ over $U$. Then a section

$$s = \sum_j c^j s_j$$

of $E$ over $U$ is smooth if and only if the coefficients $c^j$ are smooth functions on $U$.

We have already proven the case where $s_1, \ldots, s_r$ is the frame of a trivialization of $E$ over $U$. Thus it suffices to reduce the general case to this one.

**Proof**
Suppose $s = \sum_j c^j s_j$ is a smooth section of $E$ over $U$. Fix a point $p \in U$ and choose a trivializating open set $V \subseteq U$ for $E$ containing $p$, with smooth trivialization $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r$. Let $t_1, \ldots, t_r$ be the smooth frame of the trivialization $\phi$. If we write $s, s_j$ in

terms of the frame $t_1, \dots, t_r$,

$$s = \sum_i b^i t_i$$
$$s_j = \sum_j a^i_j t_i,$$

the coefficients $b^i, a^i_j$ will all be smooth functions on $V$ by the previous lemma. Next we express $s = \sum_j c^j s_j$ in terms of the $t_i$'s:

$$\sum_i b^i t_i = s = \sum_j c^j s_j = \sum_{i,j} c^j a^i_j t_i.$$

Comparing the coefficients of $t_i$ gives $b^i = \sum_j c^j a^i_j$. In matrix notation,

$$b = \begin{bmatrix} b^1 \\ \vdots \\ b^r \end{bmatrix} = A \begin{bmatrix} c^1 \\ \vdots \\ c^r \end{bmatrix} = Ac.$$

At each point of $V$, being the transition matix between two bases, the matrix $A$ is invertible. By Cramer's rule, $A^{-1}$ is a matrix of smooth functions on $V$. Hence $c = A^{-1}b$ is a column vector of smooth functions on $V$. This proves that $c^1, \dots, c^r$ are smooth functions at $p \in U$. Since $p$ is an arbitrary point of $U$, the coefficients $c^j$ are smooth functions on $U$.

The converse holds similarly to the base case since the collection of smooth sections forms a $C^\infty(U)$-module.

**Remark 3.5.13** If one replaces "smooth" by "continuous" throughout, the discussion in this subsection remains valid in the continuous category.

## 3.6 Bump Functions and Partitions of Unity

A *partition of unity* on a manifold $M$ is a collection $\{\rho_\alpha\}$ of non-negative functions summing to 1. We typically ask that the partition of unity is indexed by the the same set as an open cover $\{U_\alpha\}$ and that the support of $\rho_\alpha$ is contained in $U_\alpha$ so that $\rho_\alpha$ vanishes outside of $U_\alpha$.

The existence of smooth partitions of unity is an important technical tool in the theory of smooth manifolds that distinguishes it from that of real-analytic or complex manifolds. We construct smooth ump functions on any manifold and prove the existence of a smooth partition of unity on a compact manifold. The general case is more technical and omitted.

If we have some objective locally defined for each $U_\alpha$, we have a generic way of extending it to all of $M$ as a "weighted sum". Conversely, we can decompose a global object on a manifold into a locally finite sum of local objects.

### 3.6.1 Smooth Bump Functions

Recall that $\mathbb{R}^\times$ denote the group fo nonzero real numbers under multiplication. Also, recall the *support* of a real-valued function $f : M \to \mathbb{R}$ is defined to be

$$\operatorname{supp} f := \overline{\{p \in M : f(p) \neq 0\}}.$$

Let $q \in M$ and $U \ni q$ a neighborhood of $q$. A *bump function at $q$ supported in $U$* is a continuous non-negative function $\rho$ on $M$ that is 1 in a neighborhood of $q$ with $\operatorname{supp} \rho \subseteq U$. We are only interested in smooth bump functions which always requires a formula for verification of smoothness.

The main challenge in constructing a smooth bump function is to obtain a smooth step function. Consider the smooth function

$$f(t) := \begin{cases} \exp(-1/t), & t > 0 \\ 0, & t \leq 0 \end{cases}.$$

We seek a smooth step function $g(t)$ by dividing $f(t)$ by some positive function $\ell(t)$. Such a $g(t)$ vanishes for $t \leq 0$. The denominator should be a positive function that agrees with $f(t)$ for $t \geq 1$ so that $g(t) = 1$ for $t \geq 1$. We can take $\ell(t) = f(t) + f(1-t)$ and consider

$$g(t) := \frac{f(t)}{f(t) + f(1-t)}.$$

$\ell(t) > 0$ for all $t \in \mathbb{R}$ by construction and $g(t)$ is smooth since it is the quotient of two smooth functions with a non-vanishing denominator.

Given $0 < a < b \in \mathbb{R}$, we make the linear change of variables to map $[a^2, b^2] \to [0, 1]$:

$$x \mapsto \frac{x - a^2}{b^2 - a^2}.$$

Then $g : \mathbb{R} \to [0, 1]$ given by

$$h(x) := g\left(\frac{x - a^2}{b^2 - a^2}\right)$$

is a smooth step function that vanishes for $x \leq a^2$ and is 1 for $x \geq b^2$. We then perform another change of variables within $h$ to make the function symmetric

$$k(x) := h(x^2).$$

Finally, set

$$\rho(x) := 1 - k(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2}\right).$$

This is a smooth bump function at $0 \in \mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$. For any $q \in \mathbb{R}$, $\rho(x - q)$ is a smooth bump functionat $q$.

In order to extend this construction to $\mathbb{R}^n$, consider

$$\sigma(x) := \rho(\|x\|) = 1 - g\left(\frac{\|x\|r^2 - a^2}{b^2 - a^2}\right).$$

This is a smooth bump function at $0 \in \mathbb{R}^n$ that is 1 on the closed ball $\bar{B}(0, a)$ and has support in the closed ball $\bar{B}(0, b)$. Smoothness is a result of composing smooth functions. Again, translating by $q$ yields a smooth bump function at $q \in \mathbb{R}^n$,

$$\sigma(x - q).$$

**Proposition 3.6.1**
Let $q \in M$ be a point in a manifold and $U \ni q$ a neighborhood of $q$. There is a smooth bump function at $q$ supported in $U$.

**Proof**
Let $V \subseteq U$ be a neighborhood of $q$ and $\varphi : V \to \mathbb{R}^n$ a coordinate map on $V$. Without loss of generality, by translating if necessary, we can assume $\varphi(q) = 0 \in \mathbb{R}^n$. Pick $\varepsilon > 0$ sufficiently small so that the open Euclidean ball $B(0; 3\varepsilon) \subseteq \varphi(V)$. Let $\sigma : \mathbb{R}^n \to \mathbb{R}$ be the smooth bump function that is 1 on $\bar{B}(0, \varepsilon)$ and has support in $\bar{B}(0, 2\varepsilon)$. Note that $\operatorname{supp} \sigma \subseteq B(0; 3\varepsilon)$. Then the function $\sigma \circ \varphi : V \to \mathbb{R}^n \to \mathbb{R}$ is the desired bump function.

In general, a smooth function on an open subset $U$ of a manifold $M$ cannot be extended to a smooth function on all $M$. An example is the secant function on the open interval $(-\pi/2, \pi/2)$ in $\mathbb{R}$. However, this is possible if we relax the condition so that the global function agrees with $M$ only on some neighborhood of a point in $U$.

**Proposition 3.6.2 (Smooth Extension of a Function)**
Suppose $f$ is a smooth function defined on a neighborhood $U \in p$ in a manifold $M$. Then there is a smooth function $\tilde{f}$ on $M$ that agrees with $f$ in some possibly smaller neighborhood of $p$.

**Proof**
Choose a smooth bump function $\rho : M \to \mathbb{R}$ supported in $U$ that is 1 on a neighborhood

$V$ of $p$. Define

$$\tilde{f}(q) := \begin{cases} \rho(q)f(q), & q \in U \\ 0, & q \notin U \end{cases}$$

As the product of two smooth functions on $U$, $\tilde{f}$ is smooth on $U$. If $q \notin U$, then $q \notin \operatorname{supp} \rho$ and by the closedness of $\operatorname{supp} \rho$, there is an open set containing $q$ on which $\tilde{f}$ is 0. Thus $\tilde{f}$ is smooth at every point $q \notin U$ as well.

Since $\rho \equiv 1$ on $V$, the function $\tilde{f}$ agrees with $f$ on $V$.

### 3.6.2 Partitions of Unity

A collection $\{U_i\}$ of subsets of a topological space $S$ is said to be *locally finite* if every point $q \in S$ has a neighborhood that meets only finite many of the sets $U_i$. In particular, every $q \in S$ is contained in only finitely many of the $U_i$'s.

**Definition 3.6.1 (Partition of Unity)**
A *smooth partition of unity* is a collection of non-negative smooth functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha \in A}$ such that

(i) The collection of supports $\{\operatorname{supp} \rho_\alpha\}$ is locally finite

(ii) $\sum_\alpha \rho_\alpha = 1$

Given an open cover $\{U_\alpha\}_{\alpha \in A}$ of $M$, we say that a partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ is *subordinate to the open cover* $\{U_\alpha\}$ if $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$ for every $\alpha \in A$.

The sum in (ii) makes sense as the locally finite condition ensures that it is a finite sum at any given point.

In general, consider $\{f_\alpha\}$ a collection of smooth functions on a manifold $M$ such that the collection of its supports $\{\operatorname{supp} f_\alpha\}$ is locally finite. Then every $q \in M$ has a neighborhood $W_q$ that intersects only finitely many $\operatorname{supp} f_\alpha$. Thus on $W_q$, the sum $\sum_\alpha f_\alpha$ is actully a finite sum. Thus shows that the functio is well-defined and smooth on the manifold $M$. We call such a sum a *locally finite* sum.

### 3.6.3 Existence of a Partition of Unity

In this subsection we begin a proof of the existence of a smooth partition of unity on a manifold. The case of compact manifolds is easier and already has some features of the general case. The proof of the general case requires an appropriate substitute for compactness that is not necessary elsewhere, hence we omit it.

**Lemma 3.6.3**
If $\rho_1, \ldots, \rho_m$ are real-valued functions on a manifold $M$, then

$$\operatorname{supp}\left(\sum_i \rho_i\right) \subseteq \bigcup_i \operatorname{supp} \rho_i.$$

**Proof**
Define

$$A := \{p \in M : \sum_i \rho_i(p) \neq 0\}$$

and $A_i := \{p \in M : \rho_i(p) \neq 0\}$. It is clear that $A \subseteq \cup_i A_i$. Then

$$\begin{aligned}\operatorname{supp} \sum_i \rho_i &:= \bar{A} \\ &\subseteq \overline{\bigcup_i A_i} \\ &= \bigcup_i \overline{A_i} \\ &= \bigcup_i \operatorname{supp} \rho_i.\end{aligned}$$

Here we used the fact that $\overline{B \cup C} = \bar{B} \cup \bar{C}$. A fact that can be proved using the definition of the closure of $B$ as the smallest closed set containing $B$.

**Proposition 3.6.4**
Let $M$ be a compact manifold and $\{U_\alpha\}_{\alpha \in A}$ an open cover of $M$. There is a smooth partition of unity $\{\rho_\alpha\}_{\alpha \in A}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$.

**Proof**
For each $q \in M$, find an open set $U_\alpha \ni q$ and let $\Psi_q$ be a smooth bump function at $q$ supported in $U_\alpha$. Because $\Psi_q(q) > 0$, there is a neighborhood $W_q$ of $q$ on which $\Psi_q > 0$. By the compactness of $M$, we can find a subcover

$$\{W_{q_1}, \ldots, W_{q_m}\} \subseteq \{W_q : q \in M\}.$$

Consider

$$\Psi := \sum_{i=1}^{m} \Psi_{q_i}.$$

This is positive at every point $q \in M$ as $q \in W_{q_i}$ for some $i$.

Define

$$\varphi_i := \frac{\Psi_{q_i}}{\Psi}$$

for $i \in [m]$. We have $\sum_i \varphi_i = 1$ by construction. Also, this is a finite sum and hence locally finite. Hence we already have a partition of unity and it remains to show that it is subordinate to $\{U_\alpha\}$ by re-indexing.

Since $\Psi > 0$, $\varphi_i(q) \neq 0$ if and only if $\Psi_{q_i}(q) \neq 0$ so that

$$\operatorname{supp} \varphi_i = \operatorname{supp} \Psi_{q_i} \subseteq U_\alpha$$

for some $\alpha \in A$. For each $i \in [m]$, choose $\tau(i) \in A$ to be an index such that $\operatorname{supp} \varphi_i \subseteq U_{\tau(i)}$. Group the functions $\{\varphi_i\}$ by $\tau(i)$ and define

$$\rho_\alpha := \sum_{i \in [m] : \tau(i) = \alpha} \varphi_i$$

for each $\alpha \in A$. The empty sum is taken to zero. Then

$$\sum_{\alpha \in A} \rho_\alpha = \sum_{\alpha \in A} \sum_{\tau(i) = \alpha} \varphi_i = \sum_i \varphi_i = 1.$$

Moreover, the lemma above guarantees that

$$\operatorname{supp} \rho_\alpha \subseteq \bigcup_{\tau(i) = \alpha} \operatorname{supp} \varphi_i \subseteq U_\alpha.$$

This concludes the proof.

The statement of the existence of smooth partition of unity is as follows.

**Theorem 3.6.5 (Existence of Smooth Partition of Unity)**
Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of a manifold $M$.

(i) There is a smooth partition of unity $\{\varphi_k\}_{k=1}^\infty$ such that for every $k$, $\varphi_k$ has compact support and $\operatorname{supp} \varphi_k \subseteq U_\alpha$ for some $\alpha \in A$.

(ii) If we do not require compact support, there is a smooth partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.

## 3.7 Vector Fields

A vector field $X$ on a manifold $M$ assigns a tangent vector $X_p \in T_pM$ to each $p \in M$. More formally, a vector field on $M$ is a section of the tangent bundle $TM$ of $M$.

Vector fields are abundant in nature. For instance, the velocity vector field of a fluid flow, the electric field of a charge, the gravitation field of a mass, and so on. The fluid flow model is quite natural, as every smooth vector field may be viewed locally as the velocity vector field of a fluid flow. The path traced out by a point under this flow is called the *integral curve* of the vector field. Integral are curves whose velocity vector field is the restriction of the given vector field to the curve. The theory of ODEs guarantee the existence of integral curves.

### 3.7.1 Smoothness of Vector Field

We previously defined a vector field $X : M \to TM$ to be *smooth* if it is smooth as a section of the tangent bundle $\pi : TM \to M$. In a coordinate chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on $M$, the value of the vector field at $p \in U$ is a linear combination

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

As $p$ varies in $U$, the coefficients $a^i$ become functions on $U$.

Recall the chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on $M$ induces a chart on the tangle bundle

$$(TU, \tilde{\phi}) = (TU, \bar{x}^1, \ldots, \bar{x}^n, c^1, \ldots, c^n)$$

where $\bar{x}^i = \pi^* x^i = x^i \circ \pi$ and the $c^i$ are defined by

$$v = \sum_i c^i(v) \left.\frac{\partial}{\partial x^i}\right|_p$$

for $v \in T_pM$.

Comparing coefficients of $X_p$

$$X_p = \sum_i a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p = \sum_i c^i(X_p) \left.\frac{\partial}{\partial x^i}\right|_p$$

for $p \in U$ yields the equality $a^i = c^i \circ X$ as functions on $U$. The $c^i$'s are smooth functions on $TU$ as they are coordinates. Thus if $X$ is smooth and $(U, x^1, \ldots, x^n)$ is a chart on $M$, then the coefficients $a^i$ of $X$ relative to the frame $\partial/\partial x^i$ are smooth on $U$.

The following lemma shows that the converse also holds.

**Lemma 3.7.1 (Smoothness of a Vector Field on a Chart)**
Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold $M$. A vector field $X = \sum_i a^i \partial/\partial x^i$ on $U$ is smooth if and only if the coefficient functions $a^i$ are all smooth on $U$.

**Proof**

We have proven a more general fact: For any smooth vector bundle $\pi : E \to M$ and open subset $U \subseteq M$. If $s_1, \dots, s_r$ is a smooth frame for $E$ over $U$, then a section $s = \sum_j c^j s_j$ of $E$ over $U$ is smooth if and only if the $c^j$'s are smooth functions on $U$.

We let $\pi$ be the tangent bundle and $s_i = \partial/\partial x^i$ be the coordinate section to complete the proof.

We can now characterize the smoothness of a vector field on a manifold in terms of its coefficients relative to coordinate frames.

**Proposition 3.7.2 (Smoothness of a Vector Field in terms of Coefficients)**
Let $X$ be a vector field on a manifold $M$. The following are equivalent:

(i) $X$ is smooth on $M$

(ii) $M$ has an atlas such that on any chart $(U, x^1, \dots, x^n)$ of the atlas, the coefficients $a^i$ of $X = \sum_i a^i \partial/\partial x^i$ relative to the frame $\partial/\partial x^i$ are all smooth

(iii) On any chart $(U, x^1, \dots, x^n)$ for $M$, the coefficients $a^i$ of $X = \sum_i a^i \partial/\partial x^i$ relative to the frame $\partial/\partial x^i$ are all smooth

Just as in the Euclidean case, a vector field $X$ on a manifold $M$ induces a linear map on the algebra $C^\infty(M)$ of smooth functions on $M$. For $f \in C^\infty(M)$, define $Xf$ to be the function

$$(Xf)(p) = X_p f.$$

We can now state an alternative characterization of a smooth vector field in terms of its action as an operator on smooth functions.

**Proposition 3.7.3 (Smoothness of a Vector Field in terms of Functions)**
A vector field $X$ on $M$ is smooth if and only if for every smooth function $f$ on $M$, the function $Xf$ is smooth on $M$.

**Proof**

<u>$(\Longrightarrow)$</u> Suppose $X$ is smooth so that on any chart $(U, x^1, \dots, x^n)$ of $M$, the coefficients $a^i$ of $X = \sum_i a^i \partial/\partial x^i$ are smooth. For any $f \in C^\infty(M)$, it follows that $Xf = \sum_i a^i \partial f/\partial x^i$ is smooth on $U$. Since $M$ can be covered by charts, $Xf$ is smooth on $M$.

<u>$(\Longleftarrow)$</u> Let $(U, x^1, \dots, x^n)$ be any chart on $M$. Suppose $X = \sum_i a^i \partial/\partial x^i$ on $U$ and $p \in U$. Each of the coordinate functions $x^k$ can be extended to a smooth function $\tilde{x}^k$ on $M$ that agrees with $x^k$ in a neighborhood $V$ of $p \in U$. Thus on $V$,

$$X\tilde{x}^k = \left(\sum_i a^i \frac{\partial}{\partial x^i}\right) \tilde{x}^k = \left(\sum_i a^i \frac{\partial}{\partial x^i}\right) x^k = a^k.$$

By assumption, each $a^k$ is smooth on $V$ and in particular at $p$. But $p \in M$ was arbitrary, concluding the proof.

The proposition above shows that we can view a smooth vector field $X$ as a linear operator on $C^\infty(M)$. Similar to the Euclidean case, $X$ is a derivation: for all $f, g \in C^\infty(M)$,

$$X(fg) = (Xf)g + f(Xg).$$

An alternative view of smooth vector fields on a manifold $M$ can be taken as smooth sections of the tangent bundle $TM$ and as derivations on the algebra $C^\infty(M)$ of smooth functions. In fact, it can be shown that these two descriptions of smooth vector fields are equivalent.

Similary to smooth extensions of smooth functions, we can smoothly extend vector fields using bump functions.

**Proposition 3.7.4**
Suppose $X$ is a smooth vector field defined on a neighborhood $U \ni p$ in a manifold $M$. Then there is a smooth vector field $\tilde{X}$ on $M$ that agrees with $X$ on some (possibly smaller) neighborhood of $p$.

### 3.7.2 Integral Curves

**Definition 3.7.1 (Integral Curve)**
Let $X$ be a smooth vector field on a manifold $M$ and $p \in M$. An *integral curve* of $X$ is a smooth curve $c : (a, b) \to M$ such that $c'(t) = X_{c(t)}$ for all $t \in (a, b)$.

We typically assume that $0 \in (a, b)$. If we furthermore have $c(0) = p$, then we say that $c$ is an integral curve *starting at* $p$ and call $p$ the *initial point* of $c$. To show the dependence of such an integral curve on the initial point $p$, we also write $c(t, p)$ instead of $c(t)$.

We say that an integral curve is *maximal* if its domain cannot be extended to a larger interval.

**Example 3.7.5**
Let $X$ be the vector field $x^2 d/dx$ on the real line $\mathbb{R}$. We wish to determine the maximal integral curve of $X$ starting at $x = 2$.

Denote the integral curve by $x(t)$. Then

$$x'(t) = X_{x(t)} \iff \dot{x}(t)\frac{d}{dt} = x^2 \frac{d}{dt}.$$

Thus $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = x^2$$

subject to $x(0) = 2$. Solving the above by separation of variables yields

$$\begin{aligned}\frac{dx}{x^2} &= dt\\ -\frac{1}{x} &= t + C\\ x &= -\frac{1}{t+C}.\end{aligned}$$

The initial conditions forces $C = -1/2$. Hence

$$x(t) = \frac{2}{1-2t}.$$

The maximal interval containing 0 on which $x(t)$ is defined is $(-\infty, 1/2)$.

This example shows that it may not be possible to extend the domain of an integral curve to the entire real line.

### 3.7.3 Local Flows

Finding an integral curve of a vector field locally amounts to solving a system of first-order ODE with initial conditions. Suppose we wish to find an integral curve $c(t)$ of a smooth vector field $X$ on a manifold $M$ in general. We first choose a coordinate chart $(U, \phi) = (U, x^1, \ldots, x^n)$ about $p$. In terms of the local coordinates,

$$X_{c(t)} = \sum_i a^i(c(t)) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}$$

and

$$c'(t) = \sum_i \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)},$$

where $c^i(t) = x^i \circ c(t)$ is the $i$-th component of $c(t)$ in the chart $(U, \phi)$. The condition $c'(t) = X_{c(t)}$ is thus equivalent to

$$\dot{c}^i(t) = \dot{a}^i(c(t))$$

for each $i \in [n]$. This is an ODE with initial condition $c(0) = p$ translating to

$$c^i(0) = p^i$$

for each $i \in [n]$.

**Remark 3.7.6** We should think of elements of the tangent space as an infinitessimal direction and the differential of a map encoding how an infinitessimal direction in the domain corresponds to an infinitessimal direction in the image of the map.

Also recall that partial derivatives simplify to the calculus derivative for maps between Euclidean spaces.

By the existence and uniqueness of solutions to ODEs, the system above always has a unique local solution.

**Theorem 3.7.7**
Let $V \subseteq \mathbb{R}^n$ be open, $p_0 \in V$, and $f : V \to \mathbb{R}^n$ a smooth function. Then the differential equation

$$\frac{dy}{dt} = f(y(t))$$

with initial conditions $y(0) = p_0$ has a unique smooth solution $y : (a(p_0), b(p_0)) \to V$ where $(a(p_0), b(p_0))$ is the maximal open interval containing 0 on which $y$ is defined.

Uniqueness above means that if $z : (\delta, \varepsilon) \to V$ satisfies the same ODE, then $(\delta, \varepsilon) \subseteq (a(p_0), b(p_0))$ and $z, y$ agree on their common domain.

This theorem guarantees the existence and uniquness of a maximal integral curve starting at $p$ for a vector field $X$ on a chart $U$ of a manifold and a point $p \in U$.

Next we study the dependence of an integral curve on its initial point. Again we begin with the problem locally on $\mathbb{R}^n$. The function $y$ will now be a function of two arguments $t, q$, and the condition for $y$ to be an integral curve starting at $q$ is

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q))$$

with initial conditions $y(0, q) = q$.

The following theorem from ODE theory guarantees the smooth dependence of the solution on the initial point.

**Theorem 3.7.8**
Let $V \subseteq \mathbb{R}^n$ be open and $f : V \to \mathbb{R}^n$ be smooth on $V$. For each $p_0 \in V$, there is a neighborhood $W \ni p_0$ in $V$, some $\varepsilon > 0$, and a smooth function $y : (-\varepsilon, \varepsilon) \times W \to V$ such that

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q))$$

and $y(0, q) = q$ for all $(t, q) \in (-\varepsilon, \varepsilon) \times W$.

It follows from this theorem that if $X$ is a any smooth vector field on a chart $U$ with $p \in U$, then there is a neighorhood $W \ni p$ in $U$, $\varepsilon > 0$, and a smooth map $F : (-\varepsilon, \varepsilon) \times W \to U$ such that for each $q \in W$, the function $F(\cdot, q)$ is an integral curve of $X$ starting at $q$. In particular, $F(0, q) = q$. We usually write $F_t(q) = F(t, q)$.

Suppose $s, t \in (-\varepsilon, \varepsilon)$ are such that both $F_t(F_s(q))$ and $F_{t+s}(q)$ are defined. Then both $F_t(F_s(q))$ and $F_{t+s}(q)$ as functions of $t$ are integral curves of $X$ with initial point $F_s(q)$. By the uniqueness of the integral curve starting at a point,

$$F_t(F_s(q)) = F_{t+s}(q).$$

The map $F$ is called a *local flow generated by* $X$. For each $q \in U$, the function $F_t(q)$ of $t$ is called a *flow line* of the local flow. Each flow line is an integral curve of $X$. If a local flow $F$ is defined on $\mathbb{R} \times M$, it is called a *global flow.* Every smooth vector field has a local flow about any point, but no necessarily a global flow. A vector field having a global flow is called a *complete vector field.* If $F$ is a global flow, then for every $t \in \mathbb{R}$,

$$F_t \circ F_{-t} = F_{-t} \circ F_t = F_0 = \mathbb{1}_M$$

and so $F_t : M \to M$ is a diffeomorphism. Hence a global flow on $M$ gives rise to a one-parameter group of diffeomorphisms of $M$.

The discussion above suggests the following formal definition.

**Definition 3.7.2 (Local Flow)**
A *local flow* about a point $p \in U$ in a open subset of a manifold is a smooth function

$$F : (-\varepsilon, \varepsilon) \times W \to U,$$

where $\varepsilon > 0$, $W \ni p$ is a neighborhood within $U$, such that writing $F_t(q) = F(t, q)$ we have
(i) $F_0(q) = q$ for all $q \in W$
(ii) $F_t(F_s(q)) = F_{t+s}(q)$ whenever both sides are defined

If $F(t, q)$ is a local flow of the vector field $X$ on $U$, then

$$F(0, q) = q \qquad \frac{\partial F}{\partial t}(0, q) = X_{F(0,q)} = X_q$$

Thus we can recover the vector field from its flow. If we do not know a priori that $F(t, q)$ is the local flow of some vector field $X$, we can still define $q \mapsto \partial F / \partial t(0, q) =: X_q$. This is known as the *infinitessimal generator of* $F$.

**Proposition 3.7.9**
Let $F : (-\varepsilon, \varepsilon) \times W \to U$ be a local flow about $p \in U$. The infinitessimal generator $X$ of $F$ is a smooth vector field on $W$ and each curve $F(\cdot, p)$ is an integral curve of $X$.

**Proof**
We wish to show that $Xf$ is smooth for every smooth real-valued function $f$ on an open

subset $V \subseteq W$. For any such $f$ and $p \in U$, write $c(t) = F(t,p)$ so that $c'(0) = X_p$

$$\begin{aligned}(Xf)(p) &= X_p f \\ &= c'(0) f \\ &= c_* \left( \left.\frac{d}{dt}\right|_0 \right) f \\ &= \left.\frac{d}{dt}\right|_0 f(c(t)) \\ &= \left.\frac{\partial}{\partial t}\right|_{(0,p)} f(F(t,p)).\end{aligned}$$

Since compositions preserve smoothness, $f(F(t,p))$ is a smooth function of $(t,p)$ and so is its partial derivative with respect to $t$. Thus $(Xf)(p)$ depends smoothly on $p$ and $X$ is smooth.

We wish to show that $X_{F_{t_0}(p)} = d/dt F(t_0, p)$ for all $p \in W$ and $t_0 \in (-\varepsilon, \varepsilon)$. Define $q := F_{t_0}(p)$ and we show that $d/dt F(t_0, p) = X_q$. By the group law,

$$\begin{aligned}F(t,q) &= F(t, F_{t_0}(p)) \\ &= F(t_0 + t, p).\end{aligned}$$

Thus for any smooth real-valued function $f$ defined in a neighborhood of $q$,

$$\begin{aligned}X_q f &= \left.\frac{d}{dt}\right|_0 f(F(t,q)) \\ &= \left.\frac{d}{dt}\right|_0 f(F(t_0 + t, p)) \\ &= \left.\frac{d}{dt}\right|_{t_0} f(F(t,p))\end{aligned}$$

as desired.

**Remark 3.7.10** In general, A smooth vector field $X$ on a manifold $M$ generates a local flow whose domain at each point $p \in M$ is an interval containing 0. This can be proven by "piecing together" the maximal integral curves starting at $p$. Indeed, regardless of the chosen chart, any two integral curves starting at $p$ must agree on a small interval about 0. This shows that $F(0,q) = q$ for each $q$. The group laws can then be verified using properties of solutions of ODEs.

Conversely, we have essentially shown that such a local flow is generated by some smooth vector field $X$.

### 3.7.4 The Lie Bracket

Suppose $X, Y$ are smooth vector fields on an open subset $U$ of a manifold $M$. We view $X, Y$ as derivation operators on $C^\infty(U)$. The map $XY$ is an $\mathbb{R}$-linear operator but does not satisfy the Leibniz rule. If we consider $XY - YX$ however, this will be a derivation.

**Definition 3.7.3 (Lie Bracket)**
The *Lie bracket* of two smooth vector fields $X, Y$ on $U \ni p$ is defined to be

$$[X, Y]_p f := (X_p Y - Y_p X) f.$$

Here $f$ is a germ of a smooth function at $p$.

As $p$ varies over $U$, $[X, Y]$ becomes a vector field on $U$.

The Lie bracket provides a product operation on the vector space $\mathfrak{X}(M)$ of all smooth vector fields on $M$. Clearly $[Y, X] = -[X, Y]$.

**Proposition 3.7.11 (Jacobi Identity)**
Let $X, Y, Z \in \mathfrak{X}(M)$. Then

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

**Proposition 3.7.12 (Lie Bracket in Local Coordinates)**
Consider two smooth vector fields $X, Y$ on a manifold $M$ and $(U, x^1, \dots, x^n)$ a coordinate chart. $X, Y$ have local expressions

$$X = \sum_i a^i \frac{\partial}{\partial x^i} \qquad Y = \sum_j b^j \frac{\partial}{\partial x^j}.$$

Then

$$[X, Y] = \sum_k c^k \frac{\partial}{\partial x^k}$$

where

$$c^k = \sum_i \left( a^i \frac{\partial b^k}{\partial x^i} - b^i \frac{\partial a^k}{\partial x^i} \right).$$

**Proof**
Fix $p \in U$. Evaluating $[X, Y]_p$ at the coordinate functions $x^k$'s yields

$$\begin{aligned} c^k(p) &= [X, Y]_p x^k \\ &= (X_p Y - Y_p X) x^k \\ &= X_p b^k - Y_p a^k \\ &= \sum_i \left( a^i \frac{\partial b^k}{\partial x^i} - b^i \frac{\partial a^k}{\partial x^i} \right). \end{aligned}$$

**Definition 3.7.4 (Lie Algebra over a Field)**
Let $\mathbb{K}$ be a field. A *Lie algebra* over $\mathbb{K}$ is a vector space $V$ over $\mathbb{K}$ together with a product $[,] : V \times V \to V$ called the *bracket*, satisfying the following properties for all $a, b \in \mathbb{K}$ and $X, Y, Z \in V$:

(i) (bilinearity) $[aX + bZ, Z] = a[X, Z] = b[Y, Z]$
and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$

(ii) (anticommutativity) $[Y, X] = -[X, Y]$

(iii) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

In practice, we only concern ourselves with *real Lie algebras*, i.e. Lie algebras over $\mathbb{R}$. From hereonforth, a Lie algebra means a real Lie algebra.

**Example 3.7.13 (Abelian Lie Algebra)**
On any vector space $V$, the trivial bracket $[X, Y] = 0$ makes $V$ into a Lie algebra known as an *abelian Lie algebra*.

Note that our definition of an algebra requires the product be associative. In general, the bracket of a Lie algebra need not be associate. Thus despite its name, a Lie algebra is in general not an algebra.

**Example 3.7.14**
If $M$ is a manifold, the vector space $\mathfrak{X}(M)$ of smooth vector fields on $M$ is a real Lie algebra with the Lie bracket as the bracket.

**Example 3.7.15**
Let $\mathbb{K}^{n \times n}$ be the vector space of all $n \times n$ matrices over a field $\mathbb{K}$. Define for $X, Y \in \mathbb{K}^{n \times n}$,

$$[X, Y] = XY - YX,$$

where $XY$ is the matrix product. With this bracket, $\mathbb{K}^{n \times n}$ becomes a Lie algebra.

More generally, if $A$ is any algebra over a field $\mathbb{K}$, then the product

$$[x, y] = xy - yx$$

makes $A$ into a Lie algebra over $\mathbb{K}$.

**Definition 3.7.5 (Derivation of a Lie Algebra)**
A *derivation* of a Lie algebra $V$ over a field $\mathbb{K}$ is a $\mathbb{K}$-linear map $D : V \to V$ satisfying the product rule:

$$D[Y, Z] = [DY, Z] + [Y, DZ]$$

for $Y, Z \in V$.

**Example 3.7.16**
Let $V$ be a Lie algebra over a field $\mathbb{K}$. For each $X \in V$, define $\mathrm{ad}_X : V \to V$ by

$$\mathrm{ad}_X(Y) := [X, Y].$$

The Jacobi identity ensures that $\mathrm{ad}_X$ is a derivation of $V$

$$\begin{aligned} \mathrm{ad}_X[Y, Z] &= [X, [Y, Z]] \\ &= [[X, Y], Z] + [Y, [X, Z]] \\ &= [\mathrm{ad}_X Y, Z] + [Y, \mathrm{ad}_X Z]. \end{aligned}$$

### 3.7.5 The Pushforward of Vector Fields

Let $F : N \to M$ be a smooth map between manifolds and $F_* : T_pN \to T_{F(p)}M$ be its differential at a point $p \in N$. If $X_p \in T_pN$, we call $F_*(X_p)$ the *pushforward* of the vector $X_p$ at $p$. This notion does not extend in general to vector fields, since $F$ is not necessarily injective.

Consider the special case when $F : N \to M$ is a diffeomorphism. then the pushforward $F_*X$ of any vector field on $X$ always makes sense. There is no ambiguity about the meaning of

$$(F_*X)_{F(p)} = F_{*,p}(X_p).$$

Moreover, since $F$ is surjective, $F_*X$ is defined everywhere on $M$.

### 3.7.6 Related Vector Fields

Under a smooth map $F : N \to M$, we cannot in general push forward a vector field on $N$. There is nonetheless a useful notion of a *related vector field*.

**Definition 3.7.6 (Related Vector Field)**
Let $F : N \to M$ be a smooth map between manifolds. A vector field $X$ on $N$ is $F$-related to a vector field $\bar{X}$ on $M$ if for all $p \in N$,

$$F_{*,p}(X_p) = \bar{X}_{F(p)}.$$

**Example 3.7.17**
If $F : N \to M$ is a diffeomorphism and $X$ is a vector field on $N$, then the pushforward $F_* X$ on $M$ is defined. By definition, $X$ is $F$-related to the vector field $F_* X$ on $M$.

An equivalent condition of $F$-relatedness is as follows.

**Proposition 3.7.18**
Let $F : N \to M$ be a smooth map of manifolds. A vector feld $X$ on $N$ and a vector field $\bar{X}$ on $M$ are $F$-related if and only if for all $g \in C^\infty(M)$,

$$X(g \circ F) = (\bar{X} g) \circ F.$$

**Proof**
$X, \bar{X}$ are $F$-related if and only if

$$\begin{aligned} F_{*,p}(X_p) g &= \bar{X}_{F(p)} g && \forall p \in N, g \in C^\infty(M) \\ X_p(g \circ F) &= (\bar{X} g)(F(p)) && \forall p, g \\ (X(g \circ F))(p) &= (\bar{X} g)(F(p)) && \forall p, g \\ X(g \circ F) &= (\bar{X} g) \circ F && \forall g \end{aligned}$$

**Proposition 3.7.19**
Let $F : N \to M$ be a smooth map of manifolds. If the smooth vector fields $X, Y$ on $N$ are $F$-related to the smooth vector fields $\bar{X}, \bar{Y}$, respectively, on $M$, then the Lie bracket $[X, Y]$ on $N$ is $F$-related to the Lie bracket $[\bar{X}, \bar{Y}]$ on $M$.

**Proof**
For any $g \in C^\infty(M)$,

$$\begin{aligned} [X, Y](g \circ F) &= XY(g \circ F) - YX(g \circ F) \\ &= X((\bar{Y} g) \circ F) - Y((\bar{X} g) \circ F) \\ &= (\bar{X}\bar{Y} g) \circ F - (\bar{Y}\bar{X} g) \circ F \\ &= ((\bar{X}\bar{Y} - \bar{Y}\bar{X}) g) \circ F \\ &= ([\bar{X}, \bar{Y}] g) \circ F. \end{aligned}$$

# Chapter 4

# Lie Groups and Lie Algebras

A Lie group is a manifold equipped with smooth group operations. The invertible matrix groups form important and interesting examples of Lie groups. The left translation by a group element $g$ is a diffeomorphism from the group to itself that maps the identity to $g$. Thus the group locally looks the same around any point. To study the local structure of a Lie group, it suffices to examine a neighborhood of the identity element. It is not surprising that the tangent space at the identity of a Lie group should play a role.

The tangent space at the identity of a Lie group turns out to have a canonical bracket that makes it into a Lie algebra. This encodes within it much information about the group.

The interplay of group theory, topology, and linear algebra makes the theory of Lie groups and Lie algebras a particularly rich and vibrant branch of mathematics. Our humble goal is to examine Lie groups as an important class of manifolds and Lie algebras as examples of tangent spaces.

## 4.1 Lie Groups

### 4.1.1 Lie Groups & Examples

We begin with some methods for recognizing a Lie group.

Recall the definition of a Lie group.

**Definition 4.1.1 (Lie Group)**
A *Lie group* is a smooth manifold $G$ that is also a group such that the two group operations, multiplication and inverse are smooth.

$$\mu : G \times G \to G \qquad \mu(a,b) = ab$$
$$\iota : G \to G \qquad \iota(a) = a^{-1}.$$

for $a \in G$, we denote by $\ell_a(x) = ax$ the *left multiplication* operation by $a$, and by $r_a(x) = xa$ the *right multiplication* operation by $a$. We also refer to left/right multiplication as *left/right translation*.

**Proposition 4.1.1**
For an element $a$ in a Lie group $G$, the left multiplication $\ell_a : G \to G$ is a diffeomorphism.

**Proof**
Consider the inclusion map $I : G \to G \times G$ given by $I(x) = (a, x)$. This is certainly smooth. Then the composition

$$\ell_a = I \circ \mu$$

is smooth.

**Definition 4.1.2 (Lie Group Homomorphism)**
A map $F : H \to G$ between Lie groups $H, G$ is a *Lie group homomorphism* if it is a smooth map and a group homomorphism.

Recall a group homomorphism is a map that preserves group operations

$$F(hx) = F(h)F(x).$$

This may be rewritten in functional notation as

$$F \circ \ell_h = \ell_{F(h)} \circ F$$

for all $h \in H$. Recall that group homomorphisms always map the identity to itself.

We use capital letters to denote matrices, but generally lowercase letters to denote their entries.

**Example 4.1.2 (General Linear Group)**
We previously proved that the general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n \times n} : \det A \neq 0\}$$

is a Lie group.

**Example 4.1.3 (Special Linear Group)**
The special linear group $\mathrm{SL}(n,\mathbb{R})$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of matrices with determinant 1. We know that $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of dimension $n^2-1$.

Recall that smooth maps $f : N \to M$ whose image lie in a regular submanifold $S \subseteq$ induces a smooth map $\tilde{f} : N \to S$. Thus multiplication and inverse operations from $\mathrm{GL}(n,\mathbb{R})$ induce smooth multiplication and inverse operations on $\mathrm{SL}(n,\mathbb{R})$.

An analogous argument proves that the complex special linear group $\mathrm{SL}(n,\mathbb{C})$ is also a Lie group.

**Example 4.1.4 (Orthogonal Group)**
The orthogonal group $O(n)$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ of matrices $A$ such that $A^TA = \mathrm{Id}$. Thus $O(n)$ is the inverse image of Id under the map $f(A) = A^TA$. We previous showed that $f : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ has constant rank. Hence by the constant-rank level set theorem, $O(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$. By a similar argument to the previous example, it is hence a Lie group.

The constant rank level set theorem is more general than the regular level set theorem, at the cost that we do not directly know the co-dimension of the regular submanifold. We wish to directly apply the regular level set theorem to determine the dimension of $O(n)$.

**Lemma 4.1.5 (Space of Symmetric Matrices)**
The vector space $S_n$ of $n \times n$ real symmetric matrices has dimension

$$\frac{n(n+1)}{2} = \frac{n^2+n}{2}.$$

Consider the map $f : \mathrm{GL}(n,\mathbb{R}) \to S_n$ given by $f(A) = A^TA$. The tangent space of $S_n$ at any point is canonically isomorphic to $S_n$ itself, as $S_n$ is a vector space. Thus the image of the differential

$$f_{*,A} : T_A \mathrm{GL}(n,\mathbb{R}) \to T_{f(A)}S_n \simeq S_n.$$

While it is true that $f$ also maps $\mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ or $\mathbb{R}^{n\times n}$, we cannot hope for the differential to be surjective. This illustrates a general principle: for the differential to be surjective, we should restrict the target space of $f$ to be as small as possible.

Now we explicitly compute $f_{*,A}$ to show that it is surjective. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, its tangent space at any $A \in \mathrm{GL}(n,\mathbb{R})$ is

$$T_A \mathrm{GL}(n,\mathbb{R}) = T_A\mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}.$$

For any matrix $X \in \mathbb{R}^{n\times n}$, we know that there is a curve $c(t)$ in $\mathrm{GL}(n,\mathbb{R})$ with $c(0) = A$

and $c'(0) = X$. Then

$$\begin{aligned} f_{*,A}(X) &= \frac{d}{dt} f(c(t)) \Big|_{t=0} \\ &= \frac{d}{dt} c(t)^T c(t) \Big|_{t=0} \\ &= (c'(t)c(t) + c(t)^T c'(t)) \Big|_{t=0} && \text{matrix product rule} \\ &= X^T A + A^T X. \end{aligned}$$

Fix $A \in O(n)$ and $B \in S_n$. To show surjectivity, it suffice to find some $X \in \mathbb{R}^{n \times n}$ such that

$$X^T A + A^T X = B.$$

This equation has the solution

$$X = \frac{1}{2}(A^T)^{-1} B.$$

Thus $f_{*,A} : T_A \operatorname{GL}(n, \mathbb{R}) \to S_n$ is surjective for all $A \in O(n)$ and $O(n)$ is a regular level set of $f$. By the regular level set theorem, $O(n)$ is a regular submanifold of $\operatorname{GL}(n, \mathbb{R})$ of dimension

$$n^2 - \dim S_n = \frac{n^2 - n}{2}.$$

### 4.1.2 Lie Subgroups

Recall that a smooth map $F : N \to M$ between manifolds is an *immersion* if its differential is everywhere injective. If $F$ is injective, we refer to the image $F(N)$ under the topology inherited from $F$ as an *immersed submanifold*.

**Definition 4.1.3 (Lie Subgroup)**
A subgroup $H$ of a Lie group $G$ is a *Lie subgroup* if

(i) $H$ is an *immersed* submanifold via the inclusion map

(ii) The group operations on $H$ are smooth

The group operations on $H$ must be the restrictions of the operations inherited from $G$. However, since a Lie group is an immersed submanifold instead of a regular submanifold, it need not have the relative topology. Let $\iota : H \to G$ be the inclusion map. The composition of smooth maps

$$\mu \circ (\iota \times \iota) : H \times H \to G \times G \to G$$

is smooth. If we took $H$ to be a regular submanifold, then the multiplication map $H \times H \to H$ and inverse map $H \to H$ would automatically be smooth and condition (ii) is redundant.

**Proposition 4.1.6**
If $H$ is a subgroup and regular submanifold of a Lie group $G$, then it is a Lie subgroup of $G$.

A subgroup such as in the proposition above is called an *embedded Lie subgroup*, as the inclusion map $\iota : H \to G$ of a regular submanifold is an embedding.

**Example 4.1.7**
$\mathrm{SL}(n, \mathbb{R})$ and $O(n)$ are embedded Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$.

We now state a powerful theorem without proof.

**Theorem 4.1.8 (Closed Subgroup)**
A closed subgroup of a Lie group is an embedded Lie subgroup.

**Example 4.1.9**
$\mathrm{SL}(n, \mathbb{R})$ and $O(n)$ are the zero sets of polynomial equations on $\mathrm{GL}(n, \mathbb{R})$ and hence are closed subsets of $\mathrm{GL}(n, \mathbb{R})$. It follows that they are embedded Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$.

### 4.1.3 The Matrix Exponential

In order to compute the differential of a map on a subgroup of $\mathrm{GL}(n, \mathbb{R})$, we need a curve of nonsingular matrices. Since the matrix exponential is always nonsingular, it is suitable for this purpose.

The vector space $\mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$ of $n \times n$ matrices can be given the Euclidean norm

$$\|X\| = \sum_{ij} \left(x_{ij}^2\right)^2 .$$

The *matrix exponential* $e^X$ of a matrix $X \in \mathbb{R}^{n \times n}$ is defined by

$$e^X := \mathrm{Id} + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \dots$$

For this formula to make sense, we need to check that it converges in the normed linear space $\mathbb{R}^{n \times n}$.

**Definition 4.1.4 (Normed Algebra)**
A *normed algebra* $V$ is a normed vector space that is also an algebra over $\mathbb{R}$ with the submultiplicative property: for all $v, w \in V$,

$$\|vw\| \le \|v\|\|w\|.$$

Matrix multiplication makes $\mathbb{R}^{n\times n}$ into a normed algebra.

**Proposition 4.1.10**
For $X, Y \in \mathbb{R}^{n\times n}$,
$$\|XY\| \leq \|X\|\|Y\|.$$

**Proof**
Cauchy-Schwartz inequality.

In a normed algebra, multiplication distributes over a finite sum. The distributivity of multiplication over an infinite sum requires proof.

**Proposition 4.1.11**
Let $V$ be a normed algebra.

(i) If $a \in$ and $(s_m)$ is a sequence in $V$ that converges to $s$, then $as_m$ converges to $as$.

(ii) If $a \in V$ and $\sum_{k\geq 0} b_k$ is a convergent series in $V$, then $a\sum_k b_k = \sum_k ab_k$.

Recall a series $\sum_k a_k$ in a normed vector space $V$ is said to *converge absolutely* if the series $\sum_k \|a_k\|$ of norms converges in $\mathbb{R}$. In a complete normed vector space (*Banach space*), absolute convergence implies convergence. Hence to show that a series of matrices converges, it suffices to show absolute convergence.

For any $X \in \mathbb{R}^{n\times n}$ and $k > 0$, repeated application of submultiplicativity yields $\left\|X^k\right\| \leq \|X\|^k$. Thus the series $\sum_{k\geq 0}\left\|X^k/k!\right\|$ is bounded term by term in absolute value by the convergent series

$$\sqrt{n} + \|X\| + \frac{1}{2!}\|X\|^2 + \frac{1}{3!}\|X\|^3 + \cdots = (\sqrt{n} - 1) + e^{\|X\|}.$$

Hence the matrix exponential converges absolutely for any $n \times n$ matrix $X$.

We write $e$ or both the exponential map and for the identity element of a general Lie group. The context should prevent any confusion. We sometimes write $\exp(X) = e^X$.

Unlike the exponential of real numbers, when $A, B$ are $n \times n$ matrices, it is not necessarily true that

$$e^{A+B} = e^A e^B.$$

**Proposition 4.1.12**
If $A, B \in \mathbb{R}^{n\times n}$ commute, then
$$e^{A+B} = e^A e^B.$$

**Proposition 4.1.13**
For $X \in \mathbb{R}^{n\times n}$,

$$\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X.$$

**Proof**
Since each $(i,j)$-th entry of the series for the exponential function $e^tX$ is a convergent power series in $t$, it is justified to differentiate term by term.

$$\begin{aligned}\frac{d}{dt}e^{tX} &= \frac{d}{dt}\left(\operatorname{Id} + tX + \frac{1}{2!}t^2X^2 + \frac{1}{3!}t^3X^3\right)\\ &= X + tX^2 + \frac{1}{2!}t^2X^3\\ &= X\left(\operatorname{Id} + tX + \frac{1}{2!}t^2X^2 + \dots\right)\\ &= Xe^tX.\end{aligned}$$

By factoring the $X$ to the RHS, we obtain the second equality.

The definition of the matrix exponential $e^X$ makes sense even for complex matrices. We need only replace the Euclidean norm by the Hermitian norm.

### 4.1.4 The Trace of a Matrix

The *trace* of a matrix is the sum of its diagonal entries

$$\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}.$$

**Lemma 4.1.14**
(i) For any $X, Y \in \mathbb{R}^{n\times n}$, $\operatorname{tr}(XY) = \operatorname{tr}(YX)$.

(ii) For $X \in \mathbb{R}^{n\times n}$ and $A \in \mathrm{GL}(n,\mathbb{R})$, $\operatorname{tr}(AXA^{-1}) = \operatorname{tr}(X)$.

We now recall some facts from linear algebra.

**Proposition 4.1.15**
The trace of a real or complex matrix is equal to the sum of its (complex) eigenvalues.

**Proposition 4.1.16**
For any $X \in \mathbb{R}^{n\times n}$, $\det\left(e^X\right) = e^{\operatorname{tr} X}$.

**Proof**
Suppose first that $X$ is upper triangular. Then the diagonal entries of $e^X$ is $e^{x_{ii}}$ for $i \in [n]$. Thus

$$\det e^X = \prod_i e^{x_{ii}} = e^{\operatorname{tr} X}.$$

If $X$ is not upper triangular, we can find a nonsingular matrix such that $AXA^{-1}$ is triangular. Thus

$$e^{AXA^{-1}} = \sum_{k\geq 0} \frac{1}{k!}(AXA^{-1})^k = \sum_{k\geq 0} A\left(\frac{1}{k!}X^k\right)A^{-1} = Ae^X A^{-1}.$$

But then

$$\det e^X = \det\bigl(Ae^X A^{-1}\bigr) = \det\left(e^{AXA^{-1}}\right) = e^{\operatorname{tr}(AXA^{-1})} = e^{\operatorname{tr} X}$$

by the special case.

It follows that the matrix exponential $e^X$ is never singular regardless of the $X$, as its determinant is strictly positive.

This enables us to write down an explicit curve in $\mathrm{GL}(n,\mathbb{R})$ with given initial point and given iniital velocity. For example, $c(t) = e^{tX} : \mathbb{R} \to \mathrm{GL}(n,\mathbb{R})$ is a curve in $\mathrm{GL}(n,\mathbb{R})$ with initial point Id and initial velocity $X$, since

$$c(0) = e^{0X} = I$$

and

$$c'(0) = \left.\frac{d}{dt}\right|_{t=0} e^{tX} = \left. Xe^{tX}\right|_{t=0} = X.$$

Similarly, $c(t) = Ae^{tX}$ is a curve in $\mathrm{GL}(n,\mathbb{R})$ with initial point $A$ and initial velocity $AX$.

### 4.1.5 The Differential of Det at the Identity

The tangent space $T_{\mathrm{Id}}\,\mathrm{GL}(n,\mathbb{R})$ at the identity matrix Id is the vector space $\mathbb{R}^{n\times n}$ and the tangent space $T_1\mathbb{R}$ is $\mathbb{R}$. Thus we can view the differential of the determinant map

$$\det_{*,\mathrm{Id}} : \mathbb{R}^{n\times n} \to \mathbb{R}$$

as a map between Euclidean spaces under the identifications above.

**Proposition 4.1.17**
For any $X \in \mathbb{R}^{n\times n}$,

$$\det_{*,\mathrm{Id}}(X) = \operatorname{tr} X.$$

**Proof**
Choose the matrix exponential $c(t) = e^{tX}$ so that $c(0) = \mathrm{Id}$ and $c'(0) = X$. Then

$$\begin{aligned}
\det_{*,\mathrm{Id}}(X) &= \frac{d}{dt}\det\left(e^{tX}\right)\bigg|_{t=0} \\
&= \frac{d}{dt}e^{t\operatorname{tr}X}\bigg|_{t=0} \\
&= (\operatorname{tr}X)e^{t\operatorname{tr}X}\bigg|_{t=0} \\
&= \operatorname{tr}X.
\end{aligned}$$

## 4.2 Lie Algebras

In a Lie group $G$, left translation by an element $e \in G$ is a diffeomorphism that maps a neighborhood of the identity to a neighborhood of $g$. Thus all the local information about the group is concentrated in a neighborhood of the identity, and the tangent space at the identity is especially important.

We can give the tangent space $T_eG$ a Lie bracket $[,]$ so that it becomes a Lie algebra, called the *Lie algebra* of the Lie group. Our goal is to define the Lie algebra and identity the Lie algebras of a few important groups.

The Lie bracket on the tangent space $T_eG$ is defined using a canonical isomorphism between the tangent space at the identity and the vector space of left-invariant vector fields on $G$. With respect to this Lie bracket, the differential of a Lie group homomorphism becomes a Lie algebra homomorphism. We thus obtain a functor from the category of Lie groups and Lie group homomorphisms to the category of Lie algebras and Lie algebra homomorphisms.

### 4.2.1 Tangent Space at the Identity of a Lie Group

The existence of a smooth multiplication and smooth inverse makes a Lie group a very special manifold. For any $g \in G$, left translation $\ell_g : G \to G$ by $g$ is a diffeomorphism with inverse $\ell_{g^{-1}}$. The diffeomorphism $\ell_g$ takes the identity element $e$ to the element $g$ and induces an isomorphism of tangent spaces

$$\ell_{g*} = (\ell_g)_{*,e} : T_eG \to T_gG.$$

Thus if we describe the tangent space $T_eG$, then $\ell_{g*}T_eG$ will give a description of the tangent space $T_gG$ at any $g \in G$.

**Example 4.2.1 (Tangent space to $\mathbf{GL}(n, \mathbb{R})$ at Id)**
We know $T_g \operatorname{GL}(n, \mathbb{R}) \simeq \mathbb{R}^{n\times n}$. We also identified the isomorphism $\ell_{g*} : T_{\mathrm{Id}} \operatorname{GL}(n, \mathbb{R}) \to T_g \operatorname{GL}(n, \mathbb{R})$ as left multiplication by $g : X \mapsto gX$.

**Example 4.2.2 (Tangent space to $\mathbf{SL}(n, \mathbb{R})$ at Id)**
We begin by finding a condition that a tangent vector $X \in T_{\mathrm{Id}} \operatorname{SL}(n, \mathbb{R})$ must satisfy. We know there is a curve $c : (-\varepsilon, \varepsilon) \to \operatorname{SL}(n, \mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. Being in $\operatorname{SL}(n, \mathbb{R})$, this curve has constant determinant 1. We now take the derivative at $t = 0$.

$$\begin{aligned}
0 &= \frac{d}{dt} \det(c(t))\Big|_{t=0} \\
&= (\det \circ c)_* \left( \frac{d}{dt}\Big|_0 \right) \\
&= \det_{*,I} \left( c_* \frac{d}{dt}\Big|_0 \right) \\
&= \det_{*,I}(c'(0)) \\
&= \det_{*,I}(X) \\
&= \operatorname{tr}(X). && \text{previous proposition}
\end{aligned}$$

Thus the tangent space is contained in the subspace of trace 0 matrices. But this subspace has dimension $n^2 - 1 = \dim T_{\mathrm{Id}} \operatorname{SL}(n, \mathbb{R})$ and the two spaces must be equal.

**Proposition 4.2.3**
$T_{\mathrm{Id}} \operatorname{SL}(n, \mathbb{R})$ can be identified with the subspace of trace 0 $n \times n$ matrices.

**Example 4.2.4 (Tangent Space to $\boldsymbol{O(n)}$ at Id)**
Let $X \in T_{\mathrm{Id}}O(n)$ and choose a curve in $O(n)$ defined on a small interval about 0 such that $c(0) = I$ and $c'(0) = X$. Since $c(t) \in O(n)$,

$$c(t)^T c(t) = \mathrm{Id}\,.$$

Differentiating both sides with respect to $t$ using the matrix product rule yields

$$c'(t)^T c(t) + c(t)^T c'(t) = 0.$$

Evaluating at $t = 0$ gives

$$X^T + X = 0.$$

Thus $X$ is skew-symmetric. The subspace of skew-symmetric matrices must have zero on the diagonal entries and non-diagonal entries indexed by $i, j$ is the negation of the $j, i$-th entry. Hence this subspace has dimension

$$\frac{1}{2}(n^2 - n) = \dim T_{\mathrm{Id}}O(n)$$

and these vector spaces must be equal.

**Proposition 4.2.5**
The tangent space $T_{\mathrm{Id}}O(n)$ can be identified with the $n \times n$ skew-symmetric matrices.

### 4.2.2 Left-Invariant Vector Fields on a Lie Group

Let $X$ be any not necessarily smooth vector field on a Lie group $G$. Fro any $g \in G$, since left multiplication $\ell_g : G \to G$ is a diffeomorphism, the pushforward $\ell_{g*}X$ is a well-defined vector field on $G$. We say that the vector field $X$ is *left-invariant* if

$$\ell_{g*}(X) = X$$

for every $g \in G$. This means that for any $h \in G$,

$$\ell_{g*}(X_h) = X_{gh}.$$

In other words, $X$ is left-invariant if and only if it is $\ell_g$-related to itself for all $g \in G$. Thus a left-invariant vector field $X$ is completely determined by its value $X_e$ at the identity, since

$$X_g = \ell_{g*}(X_e).$$

Conversely, given a tangent vector $A \in T_eG$, we can define a vector field $\tilde{A}$ on $G$ by

$$(\tilde{A})_g := \ell_{g*}A.$$

As defined, the vector field $\tilde{A}$ is by construction left-invariant, since

$$\begin{aligned}\ell_{g*}(\tilde{A}_h) &= \ell_{g*}\ell_{h*}A \\ &= (\ell_g \circ \ell_h)_* A \\ &= (\ell_{gh})_*(A) \\ &= \tilde{A}_{gh}.\end{aligned}$$

We say that $\tilde{A}$ is the *left-invariant vector field on $G$ generated y $A \in T_eG$*. Let $L(G)$ be the vector space of all left-invariant vector fields on $G$. Then there is a bijective correspondence

$$\begin{aligned}T_eG &\leftrightarrow L(G) \\ X_e &\leftarrow\!\shortmid X \\ A &\mapsto \tilde{A}.\end{aligned}$$

It can be shown that this is in fact a vector space isomorphism.

**Example 4.2.6 (Left-Invariant Vector Fields on $\mathbb{R}$)**
On the Lie group $R$, the group operation is addition and the identity element is 0. Thus "left multiplication" $\ell_g$ is actually left addition

$$\ell_g(x) = g + x.$$

Let us compute $\ell_{g*}(d/dx|_0)$. Since $\ell_{g*} * (d/dx|_0)$ is a tangent vector at $g$, it is a scalar multiple of $d/dx|_g$:

$$\ell_{g*}\left(\left.\frac{d}{dx}\right|_0\right) = a\left.\frac{d}{dx}\right|_g.$$

In order to compute $a$, we can evaluate both sides at the function $f(x) = x$ to see that $a = 1$. Thus

$$\ell_{g*}\left(\left.\frac{d}{dx}\right|_0\right) = \left.\frac{d}{dx}\right|_g.$$

This shows that $d/dx$ is a left-invariant vector field on $\mathbb{R}$. Therefore, the left-invariant vector fields on $\mathbb{R}$ are constant multiples of $d/dx$.

**Example 4.2.7 (Left-Invariant Vector Fields on $\mathbf{GL}(n, \mathbb{R})$)**
Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, at any $g \in \mathrm{GL}(n, \mathbb{R})$ there is a canonical identification of the tangent space $T_g \mathrm{GL}(n, \mathbb{R})$ with $\mathbb{R}^{n\times n}$, under which a tangent vector corresponds to an $n \times n$ matrix

$$\sum_{ij} a_{ij} \left.\frac{\partial}{\partial x_{ij}}\right|_g \leftrightarrow [a_{ij}].$$

We use the same letter $B$ to denote a tangent vector $B \in T_{\mathrm{Id}} \mathrm{GL}(n, \mathbb{R})$ and a matrix $B = [b_{ij}]$. Let $B = \sum_{ij} b_{ij}\partial/\partial x_{ij}|_{\mathrm{Id}} \in T_{\mathrm{Id}} \mathrm{GL}(n, \mathbb{R})$ and let $\tilde{B}$ be the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by $B$. Then

$$\tilde{B}_g = (\ell_g)_* B \leftrightarrow gB$$

under this identification.

**Proposition 4.2.8**
Any left-invariant vector field $X$ on a Lie group $G$ is smooth.

**Proof**
We show that for any $f \in C^\infty(G)$, the function $Xf$ is also smooth. Choose a smooth curve $c : I \to G$ on some interval about 0 such that $c(0) = e$ and $c'(0) = X_e$. If $g \in G$, then $gc(t)$ is a curve starting at $g$ with initial vector $X_g$, since $gc(0) = ge = g$ and

$$(gc)'(0) = \ell_{g*}c'(0) = \ell_{g*}X_e = X_g.$$

Then

$$(Xf)(g) = X_g f = \frac{d}{dt}\bigg|_{t=0} f(gc(t)).$$

The function $F(g,t) := f(gc(t))$ is a composition of smooth functions and is thus smooth:

$$\begin{array}{ccccccc} G \times I & \xrightarrow{\mathrm{Id}\times c} & G \times G & \xrightarrow{\mu} & G & \xrightarrow{f} & \mathbb{R} \\ (g,t) & \mapsto & (g, c(t)) & \mapsto & gc(t) & \mapsto & f(gc(t)). \end{array}$$

Its partial derivative $\partial F(g,t)/\partial t$ with respect to $t$ is therefore also smooth. But then $\partial F(g,t)/\partial t|_{t=0} = (Xf)(g)$ is thus also smooth. This shows that $X$ is indeed a smooth vector field on $G$.

This proposition shows that the vector space $L(G)$ of left-invariant vector fields on $G$ is a subspace of the vector space $\mathfrak{X}(G)$ of all smooth vector fields on $G$.

**Proposition 4.2.9**
If $X, Y$ are left-invariant vector fields on $G$, then so is $[X, Y]$.

**Proof**
For any $g \in G$, $X$ is $\ell_g$-related to itself, and $Y$ is $\ell_g$-related to itself. But then we know that $[X, Y]$ is $\ell_g$-related to itself.

### 4.2.3 The Lie Algebra of a Lie Group

Recall that a *Lie algebra* is a vector space $\mathfrak{g}$ together with a *bracket*, i.e. an anticommutative bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity. A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ that is closed under the bracket $[,]$. We know that the space $L(G)$ of left-invariant vector fields on a Lie group $G$ is closed under the Lie bracket $[,]$ and is therefore a Lie subalgebra of the Lie algebra $\mathfrak{X}(G)$ of all smooth vector fields on $G$.

The linear isomorphism $\varphi : T_eG \simeq L(G)$ is mutually beneficial to the two vector spaces, for each space has somehting the other lacks. The vector space $L(G)$ has a natural Lie algebra structure given by the Lie bracket of vector fields, while the tangent space at the identity has a natural notion of pushforward, given by the differential of a Lie group homomorphism. The linear isomorphism $\varphi : T_eG \simeq L(G)$ allows us to define a Lie bracket on $T_eG$ and to push forward left-invariant vector fields under a Lie group homomorphism.

We begin with the Lie bracket on $T_eG$. Given $A, B \in T_eG$, we first map them via $\varphi$ to the left-invariant vector fields $\tilde{A}, \tilde{B}$, take the Lie bracket $[\tilde{A}, \tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$, and then map it back to $T_eG$ via $\varphi^{-1}$. Thus the definition of the Lie algebra $[A, B] \in T_eG$ should be

$$[A, B] = [\tilde{A}, \tilde{B}]_e.$$

**Proposition 4.2.10**
If $A, B \in T_e G$ and $\tilde{A}, \tilde{B}$ are the left-invariant vector fields they generate, then

$$[\tilde{A}, \tilde{B}] = \widetilde{[A, B]}.$$

**Proof**
Applying $\widetilde{()}$ to both sides of the equation $[A, B] = [\tilde{A}, \tilde{B}]_e$ yields

$$\widetilde{[A, B]} = \widetilde{[\tilde{A}, \tilde{B}]_e} = [\tilde{A}, \tilde{B}],$$

since $\widetilde{()}$ and $()_e$ are inverse to each other.

With the Lie bracket $[,]$, the tangent space $T_e G$ becomes a Lie algebra, called the *Lie algebra* of the Lie group $G$. As a Lie algebra, $T_e G$ is usually denoted by $\mathfrak{g}$.

### 4.2.4 The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$

For $\mathrm{GL}(n, \mathbb{R})$, the tangent space at Id can be identified with the vector space of $n \times n$ real matrices. We make the identification

$$\sum_{ij} a_{ij} \left. \frac{\partial}{\partial x_{ij}} \right|_{\mathrm{Id}} \leftrightarrow [a_{ij}].$$

The tangent space $T_{\mathrm{Id}} \mathrm{GL}(n, \mathbb{R})$ with its Lie algebra structure is denoted by $\mathfrak{gl}(n, \mathbb{R})$. Let $\tilde{A}$ be the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by some $A \in \mathfrak{gl}(n, \mathbb{R})$. Then on the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ we have the Lie bracket $[A, B] = [\tilde{A}, \tilde{B}]_{\mathrm{Id}}$ coming from the Lie bracket of left-invariant vector fields.

In the following proposition, we identify the Lie bracket in terms of matrices.

**Proposition 4.2.11**
Let $A, B \in T_{\mathrm{Id}} \mathrm{GL}(n, \mathbb{R})$. If

$$[A, B] = [\tilde{A}, \tilde{B}]_{\mathrm{Id}} = \sum_{ij} c_{ij} \left. \frac{\partial}{\partial x_{ij}} \right|_{\mathrm{Id}},$$

then

$$c_{ij} = \sum_k a_{ik} b_{kj} - b_{ik} a_{kj}.$$

Thus if derivations are identified with matrices, then

$$[A, B] = AB - BA.$$

**Proof**
We evaluate both sides of

$$[A,B] = \sum_{ij} c_{ij} \left.\frac{\partial}{\partial x_{ij}}\right|_{\mathrm{Id}}$$

at $x_{ij}$. By the definition of the Lie bracket at the point Id,

$$\begin{aligned} c_{ij} &= [\tilde{A}, \tilde{B}]_{\mathrm{Id}} x_{ij} \\ &= \tilde{A}_{\mathrm{Id}} \tilde{B} x_{ij} - \tilde{B}_{\mathrm{Id}} \tilde{A} x_{ij} \\ &= A\tilde{B} x_{ij} - B\tilde{A} x_{ij}. \end{aligned}$$

In order to compute $\tilde{B}x_{ij}$, recall that the left-invariant vector field $\tilde{B}$ on $\mathrm{GL}(n,\mathbb{R})$ is given by

$$\tilde{B}_g = \sum_{ij} (gB)_{ij} \left.\frac{\partial}{\partial x_{ij}}\right|_g.$$

Hence

$$\begin{aligned} \tilde{B}_g x_{ij} &= (gB)_{ij} \\ &= \sum_k g_{ik} b_{kj} \\ &= \sum_k b_{kj} x_{ik}(g). \end{aligned}$$

Since this formula holds for all $g \in \mathrm{GL}(n,\mathbb{R})$, the function $\tilde{B}x_{ij}$ must be

$$\tilde{B}x_{ij} = \sum_k b_{kj} x_{ik}.$$

But then

$$\begin{aligned} A\tilde{B}_{ij} &= \sum_{pq} a_{pq} \left.\frac{\partial}{\partial x_{pq}}\right|_{\mathrm{Id}} \left( \sum_k b_{kj} x_{ik} \right) \\ &= \sum_{pqk} a_{pq} b_{kj} \delta_{ip} \delta_{kq} \\ &= \sum_k a_{ik} b_{kj} \\ &= (AB)_{ij} \end{aligned}$$

and

$$B\tilde{A}x_{ij} = (BA)_{ij}.$$

It follows that

$$c_{ij} = (AB - BA)_{ij}$$

as desired.

### 4.2.5 The Pushforward of Left-Invariant Vector Fields

Recall that if $F : N \to M$ is a smooth map of manifolds and $X$ is a smooth vector field on $N$, the pushforward $F_*X$ is not necessarily well-defined unless $F$ is a diffeomorphism. In the case of Lie groups, due to the correspondence between left-invariant vector fields and tangent vectors at the identity, it is possible to push forward left-invariant vector fields under a Lie group homomorphism.

Let $F : H \to G$ be a Lie group homomorphism. A left-invariant vector field $X$ on $H$ is generated by its value $A = X_e \in T_eH$at the identity, so that $X = \tilde{A}$. Since the Lie group homomorphism $F$ maps the identity of $H$ to the identity of $G$, its differential $F_{*,e}$ at the identity is a linear map from $T_eH \to T_eG$. The commutative diagram below shows the existence of an induced linear map $F_* : L(G) \to L(G)$ on left-invariant vector fields as well as a way to define it.

$$\begin{array}{ccc} T_eH & \xrightarrow{F_{*,e}} & T_eG \\ \simeq\uparrow & & \simeq\uparrow \\ L(H) & \dashrightarrow & L(G) \end{array} \qquad\qquad \begin{array}{ccc} A & \longmapsto & F_{*,e}A \\ \updownarrow & & \updownarrow \\ \tilde{A} & \dashmapsto & \widetilde{(F_{*,e}A)} \end{array}$$

**Definition 4.2.1**
Let $F : H \to G$ be a Lie group homomorphism. Define $F_* : L(G) \to L(H)$ by

$$F_*(\tilde{A}) := \widetilde{(F_{*,e}A)}$$

for all $A \in T_eH$.

**Proposition 4.2.12**
If $F : H \to G$ is a Lie group homomorphism and $X$ is a left-invariant vector field on $H$, then $X$ is $F$-related to the left-invariant vector field $F_*X$ on $G$.

**Proof**
Fix $h \in H$. It suffices to check that

$$F_{*,h}(X_h) = (F_*X)_{F(h)}.$$

The LHS is equal to

$$F_{*,h}(X_h) = F_{*,h}(\ell_{h*,e}X_e) = (F \circ \ell_h)_{*,e}(X_e),$$

while the RHS is equal to

$$\begin{aligned}(F_*X)_{F(h)} &= (\widetilde{F_{*,e}X_e})_{F(h)} \\ &= \ell_{F(h)*}F_{*,e}(X_e) && \text{left-invariance} \\ &= (\ell_{F(h)} \circ F)_{*,e}(X_e).\end{aligned}$$

But $F$ is a Lie group homomorphism, thus we have $F \circ \ell_h = \ell_{F(h)} \circ F$ and the two sides are equal.

Thus if $F : H \to G$ is a Lie group homomorphism and $X$ is a left-invariant vector field on $H$, we will call $F_*X$ the *pushforward of* $X$ *under* $F$.

### 4.2.6 The Differential as a Lie Algebra Homomorphism

**Proposition 4.2.13**
If $F : H \to G$ is a Lie group homomorphism, then its differential at the identity

$$F_* := F_{*,e} : T_eH \to T_eG$$

is a *Lie algebra homomorphism*, ie a linear map that for all $A, B \in T_eH$,

$$F_*[A, B] = [F_*A, F_*B].$$

**Proof**
By a previous proposition, the vector field $F_*\tilde{A}$ on $G$ is $F$-related to the vector field $\tilde{A}$ on $H$, and the vector field $F_*\tilde{B}$ is $F$-related to $\tilde{B}$ on $H$. Hence the bracket $[F_*\tilde{A}, F_*\tilde{B}]$ on $G$ is $F$-related to the bracket $[\tilde{A}, \tilde{B}]$ on $H$. In other words,

$$F_*([\tilde{A}, \tilde{B}]_e) = [F_*\tilde{A}, F_*\tilde{B}]_{F(e)=e}.$$

The LHS is by definition $F_*[A, B]$. The RHS is given by

$$\begin{aligned}[F_*\tilde{A}, F_*\tilde{B}]_e &= [\widetilde{(F_*A)}, \widetilde{(F_*B)}]_e \\ &= [F_*, F_*B].\end{aligned}$$

Equating the two sides concludes the proof.

Suppose $H$ is a Lie subgroup of a Lie group, with inclusion map $\iota : H \to G$. Since $\iota$ is an immersion, its differential

$$\iota_* : T_eH \to T_eG$$

is injective. To distinguish the Lie bracket on $T_eH$ from the Lie bracket on $T_eG$, we temporarily attach subscripts $T_eH$ and $T_eG$ to the two Lie brackets respectively. By the previous proposition,

$$\iota_*([X,Y]_{T_eH}) = [\iota_*X, \iota_*Y]_{T_eG}.$$

Thus if $T_eH$ is identified with a subspace of $T_eG$ via $\iota_*$, then the bracket on $T_eH$ is simply the restriction of the bracket on $T_eG$ to $T_eH$. Thus the Lie algebra of a Lie subgroup $H$ may be identified with a Lie subalgebra of the Lie algebra of $G$.

We typically denote the Lie algebras of the classical groups by gothic letters. The Lie algebras of $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$, $O(n)$, $U(n)$ are denoted by $\mathfrak{gl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{R}), \mathfrak{o}(n), \mathfrak{u}(n)$, respectively. Moreover, the Lie algebra structurse on $\mathfrak{sl}(n,\mathbb{R}), \mathfrak{o}(n), \mathfrak{u}(n)$ are given by

$$[A,B] = AB - BA$$

as on $\mathfrak{gl}(n,\mathbb{R})$.

**Remark 4.2.14** A fundamental theorem in Lie theory asserts the existence of a bijective correspondence between the connected Lie subgroups of a Lie group $G$ and the Lie subalgebras of its Lie algebra $\mathfrak{g}$. It is because of our desire for such a correspondence that a Lie subgroup of a Lie group is defined to be a subgroup that is also an *immersed* submanifold rather than a regular submanifold.

# Chapter 5

# Differential Forms

## 5.1 Differential 1-Forms

Let $M$ be a smooth manifold and $p \in M$. The *cotangent space of $M$ at $p$*, denoted $T_p^*M = T_p^*M$, is defined to be the dual space of the tangent space $T_pM$:

$$T_p^*M = (T_pM)^\vee = \mathrm{Hom}(T_pM, \mathbb{R}).$$

An element of $T_p^*M$ is called a *covector at $p$*. Thus a covector $\omega_p$ at $p$ is a linear function

$$\omega_p : T_pM \to \mathbb{R}.$$

A *covector field / differential 1-form / 1-form on $M$* is a function $\omega$ that assigns to each point $p \in M$ a covector $\omega_p$ at $p$. It is in this sense dual to a vector field on $M$.

Covector fields arise naturally even when we are only interested in vector fields. If $X$ is a smooth vector field on $\mathbb{R}^n$, then at each $p \in \mathbb{R}^n$, $X_p = \sum_i a^i \partial/\partial x^i|_p$. Thus the coefficient $a^i$ is a function of $X_p \in T_pM$. In fact, it is a linear function, ie a covector. As $p$ varies over $\mathbb{R}^n$, $a^i$ becomes a covector field on $\mathbb{R}^n$. It is none other than the 1-form $dx^i$ that picks out the $i$-th coefficient of a vector field relative to the standard frame $\partial/\partial x^1, \dots, \partial/\partial x^n$.

### 5.1.1 The Differential of a Function

**Definition 5.1.1 (Differential)**
If $f \in C^\infty(M)$, its *differential* is defined to be the 1-form $df$ on $M$ given by

$$(df)_p(X_p) = X_p f.$$

We may also write $df|_p$ for the value of the 1-form $df$ at $p$. This is in parallel to the notation for tangent vector $d/dt|_p$.

Recall our other notion of the differential $f_*$ for a smooth function $f : N \to M$ as a linear function between tangent spaces.

**Proposition 5.1.1**
If $f : M \to \mathbb{R}$ is a smooth function, then for $p \in M$ and $X_p \in T_pM$,

$$f_*(X_p) = (df)_p(X_p) \left.\frac{d}{dt}\right|_{f(p)}.$$

**Proof**
Evaluate both sides of

$$f_*(X_p) = a \left.\frac{d}{dt}\right|_{f(p)}$$

at $x$.

This proposition shows that under the canonical identification

$$a \left.\frac{d}{dt}\right|_{f(p)} \leftrightarrow a,$$

$f_*$ is the same as $df$. Hence we are justified in calling both of them the *differential* of $f$.

In terms of $df$, a smooth function $f : M \to \mathbb{R}$ has a critical point at $p \in M$ if and only if $(df)_p = 0$.

### 5.1.2 Local Expression for a Differential 1-Form

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a coordinate chart on a manifold $M$. Then the differentials $dx^1, \ldots, dx^n$ are 1-forms on $U$.

**Proposition 5.1.2**
At each $p \in U$, the covectors $(dx^1)_p, \ldots, (dx^n)_p$ form a basis for the cotangent space $T_p^*M$ dual to the basis $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ for $T_pM$.

**Proof**
The proof is identical to the Euclidean case:

$$(dx^i)_p \left( \left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j.$$

We can thus write every 1-form $\omega$ on $U$ as a linear combination

$$\omega = \sum_i a_i dx^i,$$

where the coefficients $a_i$ are functions on $U$. In particular, if $f$ is a smooth function on $M$, then the restriction of the 1-form $df$ to $U$ must be a linear combination

$$df = \sum_i a_i dx^i.$$

We can isolate $a_i$ by the usual trick of evaluating at both sides on $\partial/\partial x^j$to see that

$$a_j = \frac{\partial f}{\partial x^j}.$$

Thus we have the following local expression for $df$:

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

### 5.1.3 The Cotangent Bundle

The underlying set of the *cotangent bundle* $T^*M$ of a manifold $M$ is the (disjoint) union of the cotangent spaces at all points of $M$:

$$T^*M := \bigsqcup_{p\in M} T_p^*M.$$

Just as in the case of the tangent bundle, there is a natural map $\pi : T^*M \to M$ given by $\pi(\alpha) = p$ if $\alpha \in T_p^*M$. Mimicking the construction of the tangent bundle, we give $T^*M$ a topology as follows: If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a chart on $M$ and $p \in U$, then each $\alpha \in T_p^*M$ can be written uniquely as a linear combination

$$\alpha = \sum_i c_i(\alpha) dx^i|_p.$$

This gives rise to a bijection

$$\begin{aligned} \tilde{\phi} : T * U &\to \phi(U) \times \mathbb{R}^n \\ \alpha &\mapsto (\phi(p), c_1(\alpha), \ldots, c_n(\alpha)) = (\phi \circ \pi, c_1, \ldots, c_n)(\alpha). \end{aligned}$$

We can transfer the topology of $\phi(U) \times \mathbb{R}^n$ to $T^*U$ through this bijection.

For each domain $U$ of a chart in the maximal atlas of $M$, let $\mathcal{B}_U$ be the collection of all open subsets of $T * U$, and $\mathcal{B}$ the union of all $\mathcal{B}_U$'s. As before, $\mathcal{B}$ satisfies the conditions for a collection of subsets of $T^*M$ to be a basis. We give $T^*M$ the topology generated by the basis $\mathcal{B}$. As for the tangent bundle, with the maps $\tilde{\phi} = (x^1 \circ \pi, \dots, x^n \circ \pi, c_1, \dots, c_n)$ as coordinate maps, $T * M$ becomes a smooth manifold, and the projection map $\pi : T^*M \to M$ becomes a vector bundle of rank $n$ over $M$, justifying the "bundle" in the name "cotangent bundle". If $x^1, \dots, x^n$ are coordinates on $U \subseteq M$, then $\pi^*x^1, \dots, \pi^*x^n, c_1, \dots, c_n$ are coordinates on $\pi^{-1}U \subseteq T^*M$. Properly speaking, the *cotangent bundle* of a manifold $M$ is the triple $(T * M, M, \pi)$, while $T^*M$ and $M$ are the *total space* and *base space* of the cotangent bundles respectively. By abuse of language, it is customary to call $T^*M$ the cotangent bundle of $M$.

In terms of the cotangent bundle, a 1-form on $M$ is simply a section of the cotangent bundle $T^*M$, ie it is a map $\omega : M \to T^*M$ such that $\pi \circ \omega = \mathrm{Id}_M$. We say that a 1-form $\omega$ is *smooth* if it is smooth as a map $M \to T^*M$.

> **Example 5.1.3 (Liouville Form on the Cotangent Bundle)**
> Let $M$ be an $n$-manifold. The total space $T^*M$ of its cotangent bundle $\pi : T^*M \to M$ is a manifold of dimension $2n$. Remarkably, on $T^*M$ there is a 1-form $\lambda$, called the *Liouville form* (*Poincaré form*), defined independently of charts as follows.
>
> A point in $T^*M$ is a covector $\omega_p \in T_p^*M$ at some point $p \in M$. If $X_{\omega_p}$ is a tangent vector to $T^*M$ at $\omega_p$, then the pushforward $\pi_*(X_{\omega_p})$ is a tangent vector to $M$ at $p$. Thus one can pair up $\omega_p$ and $\pi_*(X_{\omega_p})$ to obtain a real number $\omega_p(\pi_*(X_{\omega_p}))$. Define
>
> $$\lambda_{\omega_p}(X_{\omega_p}) := \omega_p(\pi_*(X_{\omega_p})).$$
>
> The cotangent bundle and the Liouville form on it play an important role in classical mechanics.

### 5.1.4 Characterization of Smooth 1-Forms

We define a 1-form $\omega$ on a manifold $M$ to be *smooth* if $\omega : M \to T^*M$ is smooth as a section of the cotangent bundle $\pi : T * M \to M$. The set of all smooth 1-forms on $M$ has the structure of a vector space, denoted by $\Omega^1(M)$. In a coordinate chart $(U, \phi) = (U, x^1, \dots, x^n)$ on $M$, the value of the 1-form $\omega$ at $p \in U$ is a linear combination

$$\omega_p = \sum_i a_i(p) dx^i|_p.$$

As $p$ varies in $U$, the coefficients $a_i$ become functions on $U$. We no derive smoothness ceriteria for a 1-form in terms of the coefficient functions $a_i$. The development is parallel to that for a vector field.

Recall the chart $(U, \phi)$ induces a chart

$$(T^*U, \tilde{\phi}) = (T * U, \bar{x}^1, \dots, \bar{x}^n, c_1, \dots, c_n)$$

on $T^*M$, where $\bar{x}^i := \pi^*x^i := x^i \circ \pi$ and the $c_i$'s are the coefficient functions of $\alpha \in T_p^*M$.

Comparing the coefficients in

$$\omega_p = \sum_i a_i(p)dx^i|_p = \sum_i c_i(\omega_p)dx^i|_p,$$

we see that $a_i = c_i \circ \omega$, where $\omega$ is viewed as a map from $U$ to $T^*U$. Being coordinate functions, the $c_i$'s are smooth on $T^*U$. Thus if $\omega$ is smooth, then the coefficients $a_i$ of $\omega$ relative to the frame $dx^i$ must be smooth on $U$. The following lemma shows that the converse is also true.

**Lemma 5.1.4**
Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold $M$. A 1-form $\omega = \sum_i a_i dx^i$ on $U$ is smooth if and only if the coefficient functions $a_i$ are all smooth.

**Proof**
Recall the characterization of smooth sections over $U$ which states that a section over $U$ is smooth if and only if its coefficient functions with respect to any smooth frame over $U$ is smooth. This is simply a special case with the cotangent bundle as the vector bundle and the coordinate 1-forms $dx^j$ as the smooth frame.

**Proposition 5.1.5 (Smoothness of a 1-Form in terms of Coefficients)**
Let $\omega$ be a 1-form on a manifold $M$. The following are equivalent:

(i) $\omega$ is smooth on $M$

(ii) The manifold $M$ has an atlas such that on any chart $(U, x^1, \ldots, x^n)$ of the atlas, the coefficients $a_i$ of $\omega = \sum_i a_i dx^i$ relative to the frame $dx^i$ are all smooth

(iii) On any chart $(U, x^1, \ldots, x^n)$ on the manifold, the coefficients of $\omega$ relative to the frame $dx^i$ are all smooth

**Corollary 5.1.5.1**
If $f \in C^\infty(M)$, then its differential $df$ is a smooth 1-form on $M$.

**Proof**
On any chart $(U, x^1, \ldots, x^n)$ on $M$,

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

Smoothness follows from the smoothness of coefficient functions.

If $\omega$ is a 1-form and $X$ a vector field on $M$, we define a function $\omega(X)$ on $M$ given by

$$\omega(X)_p := \omega_p(X_p) \in \mathbb{R}$$

for each $p \in M$.

**Proposition 5.1.6 (Linearity of 1-Forms over Functions)**
Let $\omega$ be a 1-form on a manifold $M$. If $f$ is a function and $X$ a vector field on $M$, then $\omega(fX) = f\omega(X)$.

**Proof**
At each $p \in M$,

$$\omega(fX)_p := \omega_p(f(p)X_p) = f(p)\omega_p(X_p) =: (f\omega(X))_p.$$

**Proposition 5.1.7 (Smoothness of a 1-Form in terms of Vector Fields)**
A 1-form $\omega$ on a manifold $M$ is smooth if and only if for every smooth vector field $X$ on $M$, the function $\omega(X)$ is smooth on $M$.

**Proof**
<u>$(\implies)$</u> Let $\omega$ be a smooth 1-form and $X$ a smooth vector field on $M$. On any chart $(U, x^1, \dots, x^n)$ on $M$, we have $\omega = \sum_i a_i dx^i$ and $X = \sum_j b^j \frac{\partial}{\partial x^j}$. But then by the linearity of 1-forms over functions,

$$\omega(X) = \left(\sum_i a_i dx^i\right)\left(\sum_j b^j \frac{\partial}{\partial x^j}\right) = \sum_{i,j} a_i b^j \delta^i_j = \sum_i a_i b^i.$$

This is a smooth function on $U$.

<u>$(\impliedby)$</u> Conversely, suppose that $\omega$ is a 1-form on $M$ such that $\omega(X)$ is smooth for every smooth vector field $X$. Given $p \in M$, choose a coordinate neighborhood $(U, x^1, \dots, x^n)$ about $p$ so that $\omega = \sum_i a_i dx^i$ on $U$. For any $j \in [n]$, we can extend the smooth vector field $X := \partial/\partial x^j$ on $U$ to a smooth vector field $\bar{X}$ on $M$ that agrees with $\partial/\partial x^j$ in a neighborhood $V_p^j$ of $p$ in $U$. Restricted to $V_p^j$,

$$\omega(\bar{X}) = \left(\sum_i a_i dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = a_j.$$

Thus $a_j$ is smooth on a neighborhood of $p$. Repeat over all $j$ so that all coordinate functions are simultaneously smooth on the neighborhood $V_p := \cap_j V_p^j$ about $p$. We conclude the proof by the arbitrary choice of $p$.

Let $\mathcal{F} := C^\infty(M)$ be the ring of all smooth functions on $M$. A 1-form $\omega$ on $M$ defines a map $(M) \to \mathcal{F}$ given by $X \mapsto \omega(X)$. This map is both $\mathbb{R}$-linear and $\mathcal{F}$-linear.

### 5.1.5 Pullback of 1-Forms

If $F : N \to M$ is a smooth map between manifolds, then at each $p \in N$, $F_{*,p}$ is a linear map that pushes forward tangent vectors at $p$ from $N$ to $M$.

Recall for a linear map $T \in \mathcal{L}(V, W)$, its *dual* $T^\vee \in \mathcal{L}(W^\vee, V^\vee)$ is defined to be

$$T^\vee(\varphi) := \varphi \circ T.$$

**Definition 5.1.2 (Codifferential)**
The *codifferential* (dual of the differential),

$$(F_{*,p})^\vee : T^*_{F(p)}M \to T^*_p N$$

pulls back a covector at $F(p)$ from $M$ to $N$. This means that if $\omega_{F(p)} \in T^*_{F(p)}M$ is a covector at $F(p)$ and $X_p \in T_pN$ is a tangent vector at $p$, then

$$\left((F_{*,p})^\vee \omega_{F(p)}\right)(X_p) = \omega_{F(p)}(F_{*,p}X_p).$$

Another notation for the codifferential is $F^* = (F_{*,p})^\vee$. With this notation,

$$F^*(\omega_{F(p)})(X_p) = \left((F_{*,p})^\vee \omega_{F(p)}\right)(X_p).$$

We call $F^*(\omega_{F(p)})$ the *pullback* of the covector $\omega_{F(p)}$ by $F$. Thus the pullback of covectors is simply the codifferential.

Unlike vectgor fields which in general cannot be pushed forward under a smooth map, every covector field can be pulled back by a smooth map. If $\omega$ is a 1-form on $M$, its *pullback* $F^*\omega$ is the 1-form on $N$ defined pointwise as expected:

$$(F^*\omega)_p := F^*(\omega_{F(p)})$$

for $p \in N$. Thus for $X_p \in T_pN$,

$$(F^*\omega)_p(X_p) = \omega_{F(p)}(F_*(X_p)).$$

Having defined the pullback of a 1-form, we turn to the natural question of whether this operation preserves smoothness. First, we establish three commutation properties of the pullback: differential, sum, and product.

Recall that functions can also be pulled back: if $F$ is a smooth map from $N$ to $M$ and $g \in C^\infty(M)$, the $F^*g := g \circ F \in C^\infty(N)$.

**Proposition 5.1.8 (Commutation of the Pullback with the Differential)**
Let $F : N \to M$ be a smooth map of manifolds. For any $h \in C^\infty(M)$,

$$F^*(dh) = d(F^*h).$$

**Proof**
Fix $p \in N$ and $X_p \in T_pN$. We check that

$$(F^*dh)_p(X_p) = (dF^*h)_p(X_p).$$

The LHS is equal to

$$\begin{aligned}(F^*dh)_p(X_p) &= (df)_{F(p)}(F_*X_p)\\ &= (F_*X_p)h\\ &= X_p(h \circ F).\end{aligned}$$

The RHS is equal to

$$\begin{aligned}(dF^*h)_p(X_p) &= X_p(F^*h)\\ &= X_p(h \circ F).\end{aligned}$$

We proceed to check that pullbacks of functions and 1-forms respect addition and scalar multiplication.

**Proposition 5.1.9 (Pullbacks of Sums and Products)**
Let $F : N \to M$ be a smooth map of manifolds. Suppose $\omega, \tau \in \Omega^1(M)$ and $g \in C^\infty(M)$. Then

1. $F^*(\omega + \tau) = F^*\omega + F^*\tau$
2. $F^*(g\omega) = (F^*g)(F^*\omega)$

**Proof**
(i) Fix $p \in N$ and $X_p \in T_pN$.

$$\begin{aligned}(F^*(\omega + \tau))_pX_p &= (\omega + \tau)_{F(p)}(F_*X_p)\\ &= \omega_{F(p)}(F_*X_p) + \tau_{F(p)}(F^*X_p)\\ &= (F^*(\omega))_pX_p + (F^*(\tau))_pX_p.\end{aligned}$$

(ii)

$$\begin{aligned}(F^*(g\omega))_pX_p &= (g\omega)_{F(p)}(F_*X_p)\\ &= g(F(p))\omega_{F(p)}(F_*X_p)\\ &= (F^*g)_p(F^*\omega)_pX_p.\end{aligned}$$

We can now verify that pullbacks preserve smoothness.

**Proposition 5.1.10 (Pullback of a Smooth 1-Form)**
The pullback $F^*\omega$ of a smooth 1-form $\omega$ on $M$ under a smooth map $F : N \to M$ is a smooth 1-form on $N$.

**Proof**
Fix $p \in N$. It suffices to check that $F^*\omega$ is smooth at $p$. Choose a chart $(V, y^1, \dots, y^n)$ on $M$ about $F(p)$. By the continuity (smoothness) of $F$, there is a chart $(U, x^1, \dots, x^n)$ about $p$ in $N$ such that $F(U) \subseteq V$. On $V$, $\omega = \sum_i a_i dy^i$ for some $a_i \in C^\infty(V)$. On $U$,

$$\begin{aligned} F^*\omega &= \sum_i F^*(a_i dy^i) \\ &= \sum_i F^*(a_i) F^*(dy^i) \\ &= \sum_i (a_i \circ F) d(F^* y^i) \\ &= \sum_i (a_i \circ F) d(y^i \circ F) \\ &= \sum_i (a_i \circ F) \sum_j \frac{\partial F^i}{\partial x^j} dx^j \\ &= \sum_{i,j} (a_i \circ F) \frac{\partial F^i}{\partial x^j} dx^j. \end{aligned}$$

Now, each $a_i \circ F$ is a smooth composition of smooth functions at $p$ and hence $F^*\omega$ is smooth at $p$ as desired.

**Example 5.1.11 (Liouville Form on the Cotangent Bundle)**
Let $M$ be a manifold, The Liouville form $\lambda$ on the cotangent bundle $T^*M$ can be expressed as

$$\lambda_{\omega_p} = \pi^*(\omega_p)$$

at any $\omega_p \in T^*M$.

### 5.1.6 Restriction of 1-Forms to Immersed Submanifolds

Let $S \subseteq M$ be an immersed submanifold and $\iota : S \to M$ the inclusion map. As a reminder this means that $\iota(S)$ inherits the topology from $S$ and hence we automatically have $S \simeq \iota(S)$. At any $p \in S$, since the differential $\iota_* : T_pS \to T_pM$ is injective, we can view the tangent space $T_pS$ as a subspace of $T_pM$. If $\omega$ is a 1-form on $M$, the the *restriction* of $\omega$ to $S$ is the 1-form $\omega|_S$ defined by

$$(\omega|_S)_p(v) = \omega_p(v)$$

for every $p \in S$ and $v \in T_pS$.

**Proposition 5.1.12**
If $\iota : S \to M$ is the inclusion map of an immersed submanifold $S$ and $\omega$ is a 1-form on $M$, then

$$\iota^*\omega = \omega|_S.$$

**Proof**
For $p \in S$ and $v \in T_pS$,

$$\begin{aligned}(\iota^*\omega)_p(v) &= \omega_{\iota(p)}(\iota_* v) \\ &= \omega_p(v) \\ &= (\omega|_S)_p(v).\end{aligned}$$

For simplicity of notation, we sometimes write $\omega$ to mean $\omega|_S$.

**Example 5.1.13 (A 1-Form on the Circle)**
The velocity vector field of the unit circle $c(t) = (x, y) = (\cos t, \sin t)$ in $\mathbb{R}^2$ is given by

$$c'(t) = (-\sin t, \cos t) = (-y, x).$$

Thus

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

is a smooth vector field on the unit circle $S^1$.

This notation means that if $x, y$ are the standard coordinates on $\mathbb{R}^2$ and $\iota : S^1 \to \mathbb{R}^2$ is the inclusion map, then at a point $p = (x, y) \in S^1$, one has

$$\iota_* X_p = -y \left.\frac{\partial}{\partial x}\right|_p + x \left.\frac{\partial}{\partial y}\right|_p,$$

where $\partial/\partial x|_p$ and $\partial/\partial|_p$ are tangent vectors at $p$ in $\mathbb{R}^2$.

**Example 5.1.14**
The 1-form $\omega = -ydx + xdy$ satisfies

$$\omega(X) \equiv 1$$

on the unit circle $S^1$ where $X$ is the velocity vector field of the unit circle.

**Remark 5.1.15** If we wish to be pedantic, a 1-form $\omega = -\bar{y}d\bar{x} + \bar{x}d\bar{y}$ should be explicitly written as restrictions $\bar{x}, \bar{y}$ to $S^1$. However, since $\iota^*x = \bar{x}$ and $\iota^*dx = d\bar{x}$, there is little chance of confusion and we typically omit the bar.

This is in contrast to the situation for vector fields where $\iota_*(\partial/\partial\bar{x}_p) \neq \partial/\partial x|_p$.

> **Example 5.1.16 (Pullback of a 1-Form)**
> Let $h : \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$ be given by $h(t) = (x, y) = (\cos t, \sin t)$. If $\omega$ is the 1-form $-ydx + xdy$ on $S^1$, the pullback $h^*\omega$ is given by
>
> $$\begin{aligned} h^*(-ydx + xdy) &= -(h^*y)d(h^*x) + (h^*x)d(h^*y) \\ &= -(\sin t)d(\cos t) + (\cos t)d(\sin t) \\ &= \sin^2 t dt + \cos^2 dt \\ &= dt. \end{aligned}$$

## 5.2 Differential $k$-Forms

Similar to the Euclidean setting, we now generalize the construction of 1-forms on a manifold to that of $k$-forms. In parallel to the construction of the tangent and cotangent bundles on a manifold, we construct the $k$-th exterior power $\bigwedge^k(T^*M)$ of the tangent bundle. This yields a natural notion of smoothness of differential forms as smooth sections of the vector bundle $\bigwedge^k(T^*M)$. The pullback and wedge product of differential forms are defined pointwise. We consider left-invariant forms on a Lie group as examples of differential forms.

### 5.2.1 Differential Forms

Recall that a *$k$-tensor* on a vector space $V$ is a $k$-linear function

$$f : V \times \cdots \times V \to \mathbb{R}.$$

We say that the *$k$-tensor* $f$ is *alternating* if for any permutation $\sigma \in S_k$,

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) f(v_1, \ldots, v_k).$$

Note that all 1-tensors are alternating. An alternating $k$-tensor on $V$ is also called a *$k$-covector* on $V$.

For any vector space $V$, denote by $A_k(V)$ the vector space of alternating $k$-tensors on $V$. Another common notation is $\bigwedge^k(V^\vee)$. There is a purely algebraic construction of $\bigwedge^k(V)$, called the *$k$-th exterior power* of the vector space $V$, with the property that $\bigwedge^k(V^\vee) \simeq A_k(V)$. We ignore this construction for simplicity.

We apply the function $A_k()$ to the tangent space $T_pM$ of a manifold $M$ at a point $p$. The vector space $A_k(T_pM)$, typically denoted $\bigwedge^k(T_p^*M)$, is the space of all alternating $k$-tensors on the tangent space $T_pM$. A *$k$-covector field* on $M$ is a function $\omega$ that assigns to each

$p \in M$ a $k$-covector $\omega_p \in \bigwedge^k(T_p^*M)$. A $k$-covector field is also called a *differential $k$-form*, a *differential form of degree $k$*, or simply a *$k$-form*. A *top form* on a manifold is a differential form whose degree is the dimension of the manifold.

If $\omega$ is a $k$-form on a manifold $M$ and $X_1, \dots, X_k$ are vector fields on $M$, then $\omega(X_1, \dots, X_k)$ is the function on $M$ defined by

$$(\omega(X_1, \dots, X_k))(p) := \omega_p((X_1)_p, \dots, (X_k)_p).$$

**Proposition 5.2.1 (Multilinearity of a Form over Functions)**
Let $\omega$ be a $k$-form on a manifold $M$. For any vector fields $X_1, \dots, X_k$ and any function $h$ on $M$,

$$\omega(X_1, \dots, hX_i, \dots, X_k) = h\omega(X_1, \dots, X_i, \dots, X_k).$$

The proof follows by the point-wise $\mathbb{R}$-multilinearity of $\omega$.

**Example 5.2.2**
Let $(U, x^1, \dots, x^n)$ be a coordinate chart on a manifold. At each $p \in U$, a basis for the tangent space $T_pU$ is $\partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p$. Recall the dual basis for the cotangent space $T_p^*U$ is

$$(dx^1)_p, \dots, (dx^n)_p.$$

As $p$ varies over $U$, we get differential 1-forms $dx^1, \dots, dx^n$ on $U$.

Recall from the general theory of alternating $k$-tensors that a basis for $\bigwedge^k(T_p^*U)$ is

$$(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p,$$

for $1 \le i_1 < \cdots < i_k \le n$. If $\omega$ is a $k$-form on $U$, then at each $p \in U$, $\omega_p$ is a linear combination

$$\omega_p = \sum a_{i_1 \dots i_k}(p)(dx^{i_1})_p \wedge \cdots \wedge (dx^{i_k})_p.$$

Omitting the point $p$, we write

$$\omega = \sum a_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

In this expression, the coefficients $a_{i_1 \dots i_k}$ are functions on $U$ as they vary with the point $p$. For brevity, we write

$$\mathcal{J}_{k,n} := \{I = (i_1, \dots, i_k) : 1 \le i_1 < \cdots < i_k \le n\}$$

to indicate the set of all strictly ascending multi-indices between 1 and $n$ of length $k$, and write

$$\omega = \sum_{I \in \mathcal{J}_{k,n}} a_I dx^I,$$

where $dx^I := dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

### 5.2.2 Local Expression for a $k$-Form

We know that on a coordinate chart $(U, x^1, \dots, x^n)$ of a manifold $M$, a $k$-form on $U$ is a linear combination $\omega = \sum_I a_I dx^I$ where $I \in \mathcal{J}_{k,n}$ and the $a_I$'s are functions on $U$. We write $\partial_i := \partial/\partial x^i$ for the $i$-th coordinate vector field. Evaluating pointwise, we obtain the following equality on $U$ for $I, J \in \mathcal{J}_{k,n}$:

$$dx^I(\partial_{j_1}, \dots, \partial_{j_k}) = \delta^I_J = \begin{cases} 1, & I = J, \\ 0, & I \neq J. \end{cases}$$

Recall the notation

$$\frac{\partial(f^1, \dots, f^k)}{\partial(x^{i_1}, \dots, x^{i_k})} := \det \left[ \frac{\partial f^i}{\partial x^{i_j}} \right].$$

**Proposition 5.2.3 (A Wedge of Differentials in Local Coordinates)**
Let $(U, x^1, \dots, x^n)$ be a chart on a manifold and $f^1, \dots, f^k$ smooth functions on $U$. Then

$$df^1 \wedge \cdots \wedge df^k = \sum_{I \in \mathcal{J}_{k,n}} \frac{\partial(f^1, \dots, f^k)}{\partial(x^{i_1}, \dots, x^{i_k})} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

**Proof**
On $U$,

$$df^1 \wedge \cdots \wedge df^k = \sum_{J \in \mathcal{J}_{k,n}} c_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

for some functions $c_J$. For the LHS, recall that the wedge product of functions applied $k$ vectors satisfies

$$\begin{aligned}(df^1 \wedge \cdots \wedge df^k)(\partial_{i_1}, \dots, \partial_{i_k}) &= \det \left[ \frac{\partial f^i}{\partial x^{i_j}} \right] \\ &= \frac{\partial(f^1, \dots, f^k)}{\partial(x^{i_1}, \dots, x^{i_k})}.\end{aligned}$$

On the other hand, the RHS is equal to

$$\sum_J c_J dx^J(\partial_{i_1}, \dots, \partial_{i_k}) = \sum_J c_J \delta^J_I = c_I.$$

If $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ are two overlapping charts on a manifold, then on the intersection $U \cap V$, the proposition above yields the transition formula for $k$-forms:

$$dy^J = \sum_{I \in \mathcal{J}_{k,n}} \frac{\partial(y^{j_1}, \dots, y^{j_k})}{\partial(x^{i_1}, \dots, x^{i_k})} dx^I.$$

Two cases of the proposition above are of special interest:

**Corollary 5.2.3.1**
Let $(U, x^1, \dots, x^n)$ be a chart on a manifold and $f, f^1, \dots, f^n \in C^\infty(U)$. Then

(i) (1-forms) $df = \sum_i (\partial f / \partial x^i) dx^i$

(ii) (top forms) $df^1 \wedge \cdots \wedge df^n = \det[\partial f^j / \partial x^i] dx^1 \wedge \cdots \wedge dx^n$

**Proposition 5.2.4 (Transition Formula for a 2-Form)**
If $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^n)$ are two overlapping coordinate charts on $M$, then a smooth 2-form $\omega$ on $U \cap V$ has two local expressions

$$\omega = \sum_{i<j} a_{ij} dx^i \wedge dx^j = \sum_{k<\ell} b_{k\ell} dy^k \wedge dy^\ell.$$

Then

$$a_{ij} = \sum_{k<\ell} b_{k\ell} \frac{\partial(y^k, y^\ell)}{\partial(x^i, x^j)}.$$

**Proof**
By an earlier remark,

$$dy^k \wedge dy^\ell = \sum_{i<j} \frac{\partial(y^k, y^\ell)}{\partial(x^i, x^j)} dx^i \wedge dx^j.$$

So that

$$\omega = \sum_{i<j} a_{ij} dx^i \wedge dx^j = \sum_{k<\ell} b_{k\ell} \sum_{i<j} \frac{\partial(y^k, y^\ell)}{\partial(x^i, x^j)} dx^i \wedge dx^j.$$

Matching coefficients yields the desired result.

### 5.2.3 The Bundle Point of View

Let $M$ be a manifold of dimension $n$. We mimic the construction of the tangent and cotangent bundles and form the set

$$\bigwedge^k(T^*M) := \bigsqcup_{p \in M} \bigwedge^k(T_p^*M) = \bigsqcup_{p \in M} A_k(T_pM)$$

of all alternating $k$-tensors at all points of the manifold $M$. This set is called the *$k$-th exterior power* of the cotangent bundle. There is a canonical projection map $\pi : \bigwedge^k(T^*M) \to M$ given by $\pi(\alpha) = p$ for $\alpha \in \bigwedge^k(T_p^*M)$.

If $(U, \phi)$ is a coordinate chart on $M$, then there is a bijection

$$\bigwedge^k(T^*U) = \bigcup_{p\in U} \bigwedge^k(T_p^*U) \simeq \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$
$$\alpha \in \bigwedge^k(T_p^*U) \mapsto (\phi(p), \{c_I(\alpha)\}_I),$$

where $\alpha = \sum_I c_I(\alpha) dx^I|_p \in \bigwedge^k(T_p^*U)$ and $I = (1 \leq i_1 < \cdots < i_k \leq n)$. Hence we can given $\bigwedge^k(T^*U)$ and hence $\bigwedge^k(T^*M)$ a topology and even a differentiable structure. The details are just like those for the construction of the tangent bundle, so we omit them. The upshot is that the projection map $\pi : \bigwedge^k(T^*M) \to M$ is a smooth vector bundle of rank $\binom{n}{k}$ and that a differential $k$-form is simply a section of this bundle. We define a $k$-form to be *smooth* if it is smooth as a section of the bundle $\pi : \bigwedge^k(T^*M) \to M$.

If $E \to M$ is a smooth vector bundle, then the vector space of smooth sections of $E$ is denoted by $\Gamma(E)$ or $\Gamma(M, E)$. The vector space of all smooth $k$-forms on $M$ is usually denoted by $\Omega^k(M)$. Thus

$$\Omega^k(M) = \Gamma\left(\bigwedge^k(T^*M)\right) = \Gamma\left(M, \bigwedge^k(T^*M)\right).$$

### 5.2.4 Smooth $k$-Forms

We state several characterizations of a smooth $k$-form. The proofs are omitted since they are similar but more tedious than those for 1-forms.

**Lemma 5.2.5 (Smoothness of a $k$-Form on a Chart)**
Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold $M$. A $k$-form $\omega = \sum_I a_I dx^I$ on $U$ is smooth if and only if the coefficient functions $a_I$ are all smooth on $U$.

**Proposition 5.2.6 (Characterization of a Smooth $k$-Form)**
Let $\omega$ be a $k$-form on a manifold $M$. The following are equivalent:

(i) $\omega$ is smooth on $M$

(ii) $M$ has an atlas such that on every chart $(U, x^1, \ldots, x^n)$, the coefficients $a_I$ of $\omega = \sum_I a_I dx^I$ relative to the coordinate frame $\{dx^I\}_{I\in\mathcal{J}_{n,k}}$ are all smooth

(iii) On every chart $(U, x^1, \ldots, x^n)$ in the maximal atlas, the coefficients $a_I$ of $\omega = \sum_I a_I dx^I$ relative to the coordinate frame $\{dx^I\}_{I\in\mathcal{J}_{n,k}}$ are all smooth

(iv) For any $k$ smooth vector fields $X_1, \ldots, X_k$ on $M$, the function $\omega(X_1, \ldots, X_k)$ is smooth on $M$

We defined 0-tensors and 0-covectors to be the constant functions, so

$$L_0(V) = A_0(V) = \mathbb{R}.$$

Thus the bundle $\bigwedge^0(T^*M) \simeq M \times \mathbb{R}$ and a 1-form on $M$ is just a function on $M$. A smooth 0-form is thus the same as a smooth function on $M$. In our new notation,

$$\Omega^0(M) = \Gamma\left(\bigwedge^0(T^*M)\right) = \Gamma(M \times \mathbb{R}) = C^\infty(M).$$

Similar to smooth functions, smooth differential forms can also be smoothly extended.

**Proposition 5.2.7 (Smooth Extension of a Form)**
Suppose $\tau$ is a smooth differential form defined on a neighborhood $U \ni p$ in a manifold $M$. There is a smooth form $\tilde{\tau}$ on $M$ that agrees with $\tau$ on a possibly smaller neighborhood of $p$.

We note that the extension $\tilde{\tau}$ is not unique. It depends on $p$ as well as the choice of a bump function at $p$.

### 5.2.5 Pullback of $k$-Forms

We have defined the pullback of 0-forms and 1-forms under a smooth map $F : N \to M$. For a smooth 0-form on $M$, ie a smooth function on $M$,

$$\begin{aligned} F^*f : (N \xrightarrow{F} M) &\xrightarrow{f} \mathbb{R} \\ F^*(f) &= f \circ F \in \Omega^0(N). \end{aligned}$$

To generalize the pullback to $k$-forms for $k \geq 1$, we first recall the pullback of $k$-covectors. A linear map $L : V \to W$ of vector spaces induces a pullback map

$$\begin{aligned} L^* : A_k(W) &\to A_k(V) \\ (L^*\alpha)(v_1, \dots, v_k) &= \alpha(L(v_1), \dots, L(v_k)) \end{aligned}$$

for $\alpha \in A_k(W)$ and $v_1, \dots, v_k \in V$.

Suppose $F : N \to M$ is a smooth map of manifolds. At each $p \in N$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is a linear map of tangent spaces. Hence there is a pullback map

$$(F_{*,p})^* : A_k(T_{F(p)}M) \to A_k(T_pN).$$

This ugly notation is usually simplified to $F^*$. Thus if $\omega_{F(p)}$ is a $k$-covector at $F(p) \in M$, then its *pullback* $F^*(\omega_{F(p)})$ is the $k$-covector at $p \in N$ given by

$$F^*(\omega_{F(p)})(v_1, \dots, v_k) = \omega_{F(p)}(F_{*,p}(v_1), \dots, F_{*,p}(v_k))$$

for $v_i \in T_pN$.

For a $k$-form $\omega$ on $M$, its *pullback* $F^*\omega$ is the $k$-form on $N$ defined pointwise by

$$(F^*\omega)_p := F^*(\omega_{F(p)}) = \omega_{F(p)}(F_{*,p}(\cdot), \dots, F_{*,p}(\cdot)).$$

When $k = 1$, this formula specializes to the definition of the pullback of a 1-form. The pullback of a $k$-form can be viewed as a composition

$$T_pN \times \cdots \times T_pN \xrightarrow{F_* \times \cdots \times F_*} T_{F(p)}M \times \cdots \times T_{F(p)}M \xrightarrow{\omega_{F(p)}} \mathbb{R}.$$

Similar to the linearity of the pullback of a 0-form or 1-form, we can prove the following.

**Proposition 5.2.8 (Linearity of the Pullback)**
Let $F : N \to M$ be a smooth map. If $\omega, \tau$ are $k$-forms on $M$ and $a \in \mathbb{R}$, then

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$

(ii) $F^*(a\omega) = aF^*\omega$

We defer the basic question of whether the pullback of a smooth $k$-form undder a smooth map remains smooth for $k \geq 2$.

### 5.2.6 The Wedge Product

Recall that a $(k, \ell)$-shuffle is a permutation $\sigma$ such that

$$\sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+\ell).$$

We now that if $\alpha, \beta$ are alternating tensors of degree $k, \ell$ respectively on a vector space $V$, then their wedge product $\alpha \wedge \beta$ is the alternating $(k+\ell)$-tensor on $V$ given by

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+\ell}) = \sum_{\sigma} (\operatorname{sgn} \sigma) \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}).$$

Here $v_i \in V$ and $\sigma$ runs over all $(k, \ell)$-shuffles of $[k+\ell]$. For 1-covectors $\alpha, \beta$,

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1).$$

The wedge product extends pointwise to differential forms on a manifold: for a $k$-form $\omega$ and an $\ell$-form $\tau$ on $M$, define their *wedge product* $\omega \wedge \tau$ to be the $(k+\ell)$-form on $M$ such that

$$(\omega \wedge \tau)_p = \omega_p \wedge \tau_p$$

at all $p \in M$.

**Proposition 5.2.9**
If $\omega, \tau$ are smooth forms on $M$, then $\omega \wedge \tau$ is also smooth.

**Proof**
Let $(U, x^1, \dots, x^n)$ be a chart on $M$. On $U$,

$$\omega = \sum_I a_I dx^I, \qquad \tau = \sum_J b_J dx^J$$

for smooth functions $a_I, b_J$ on $U$. Their wedge product is given by

$$\begin{aligned}
\omega \wedge \tau &= \left(\sum_I a_I dx^I\right) \wedge \left(\sum_J b_J dx^J\right) \\
&= \sum_{I,J} a_I b_J dx^I \wedge dx^J \\
&\in \sum_K \left(\sum_{I \cup J = K, I \cap J = \varnothing} \pm a_I b_J\right) dx^K.
\end{aligned}$$

The last equality results from the observation that $dx^I \wedge dx^J = 0$ if $I, J$ have a common index. If $I, J$ are disjoint, then $dx^I \wedge dx^J \in \pm dx^K$ where $K = I \cup J$ but reordered as an increasing sequence. Since the coefficients of $dx^K$ are smooth on $U$, we conclude the proof.

**Proposition 5.2.10 (Pullback of a Wedge Product)**
If $F : N \to M$ is a smooth map of manifolds and $\omega, \tau$ are differential forms on $M$, then

$$F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau.$$

**Proof**
Let $(U, x^1, \dots, x^n)$ be a chart about a point $p \in M$ and $v_1, \dots, v_{k+\ell} \in T_pN$. We have the local expressions $\omega = \sum_I a_I dx^I$ and $\tau = \sum_J b_J dx^J$. We have

$$\begin{aligned}
&(F^*(\omega \wedge \tau))_p(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) \\
&= (\omega \wedge \tau)_p(F_{*,p}(v_1), \dots, F_{*,p}(v_k), F_{*,p}(v_{k+1}), \dots, F_{*,p}(v_{k+\ell})) \\
&= (\omega_p \wedge \tau_p)(F_{*,p}(v_1), \dots, F_{*,p}(v_k), F_{*,p}(v_{k+1}), \dots, F_{*,p}(v_{k+\ell})) \\
&= \sum_\sigma \omega_p(F_{*,p}(v_1), \dots, F_{*,p}(v_k))\tau_p(F_{*,p}(v_{k+1}), \dots, F_{*,p}(v_{k+\ell})) \\
&= \sum_\sigma (F^*\omega)_p(v_1, \dots, v_k)(F^*\tau)_p(v_{k+1}, \dots, v_{k+\ell}) \\
&= ((F^*\omega)_p \wedge (F^*\tau)_p)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) \\
&= (F^*\omega \wedge F^*\tau)_p(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}).
\end{aligned}$$

Define the vector space $\Omega^*(M)$ of smooth differential forms on a manifold $M$ of dimension

$n$ to be the direct sum

$$\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M).$$

This means each element $\Omega^*(M)$ is uniquely a sum $\sum_{k=0}^n \omega_k$, where $\omega_k \in \Omega^k(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a graded algebra, with the grading being the degree of differential forms.

### 5.2.7 Differential Forms on a Circle

Consider the map

$$h : \mathbb{R} \to S^1, \qquad h(t) = (\cos t, \sin t).$$

Since the derivative $\dot{h}(t) = (-\sin, \cos t)$ is nonzero for all $t$, the map $h : \mathbb{R} \to S^1$ is a submersion. It can be shown in this case that the pullback by a surjective submersion is an injective algebra homomorphism. Hence $h^* : \Omega(S^1) \to \Omega^*(\mathbb{R})$ is injective and we can identify the differential forms on $S^1$ with a subspace of differential forms on $\mathbb{R}$.

Let $\omega = -ydx + xdy$ be the nowhere-vanishing form on $S^1$. Recall that $h^*\omega = dt$. Since $\omega$ is nowhere vanishing, it is a frame for the cotangent bundle $T^*S^1$ over $S^1$, and every smooth 1-form $\alpha$ on $S^1$ can be written as $\alpha = f\omega$ for some smooth $f \in C^\infty(S^1)$. Its pullback $\bar{f} := h^*f$ is a smooth function on $\mathbb{R}$. Since pulling back preserves multiplication,

$$h^*\alpha = (h^*f)(h^*\omega) = \bar{f}dt.$$

We say that a function $g$ or a 1-form $gdt$ on $\mathbb{R}$ is *periodic* of period $a$ if $g(t+a) = g(t)$ for all $t \in \mathbb{R}$.

**Proposition 5.2.11**
For $k = 0, 1$, under the pullback map $h^*\Omega^*(S^1) \to \Omega^*(\mathbb{R})$, smooth $k$-forms on $S^1$ are identified with smooth periodic $k$-forms of period $2\pi$ on $\mathbb{R}$.

### 5.2.8 Invariant Forms on a Lie Group

Just as there are left-invariant vector fields on a Lie group $G$, there are also left invariant differential forms.

**Definition 5.2.1 (Left-Invariant Differential Form)**
Let $G$ be a Lie group and $\ell_g : G \to G$ denote left multiplication by $g \in G$. A $k$-form on $G$ is *left-invariant* if $\ell_g^*\omega = \omega$ for all $g \in G$. This means that for all $g, x \in G$,

$$\ell_g^*(\omega gx) = \omega_x.$$

By definition, a left-invariant $k$-form is uniquely determined by its value at the identity, since for any $g \in G$,

$$\omega_g = \ell^*_{g^{-1}}(\omega_e).$$

**Example 5.2.12**
$\omega = -ydx + xdy$ is a left-invariant 1-form on $S^1$.

**Proposition 5.2.13**
Every left-invariant $k$-form $\omega$ on a Lie group $G$ is smooth.

**Proof**
It suffices to show that for any $k$ smooth vector fields $X_1, \ldots, X_k$ on $G$, the function $\omega(X_1, \ldots, X_k)$ is smooth on $G$. Let $(Y_1)_e, \ldots, (Y_k)_e$ be a basis for $T_eG$ and $Y_1, \ldots, Y_n$ the left-invariant vector fields they generate. Then $Y_1, ;Y_n$ is a smooth frame on $G$ as any left-invariant vector fields are smooth. Each $X_j$ can be written as a linear combination $X_j = \sum_i a^i_j Y_i$ for some smooth functions $a^i_j$. It suffices thus to show that $\omega(Y_1, \ldots, Y_k)$ is smooth for any left-invariant vector fields $Y_1, \ldots, Y_k$.

We have

$$\begin{aligned} &(\omega(Y_1, \ldots, Y_k))(g) \\ &= \omega_g((Y_1)_g, \ldots (Y_k)_g) \\ &= (\ell^*_{g^{-1}}(\omega_e))(\ell_{g^*}(Y_1)_e, \ldots, \ell_{g*}(Y_k)_e) && \text{left-invariance of both } \omega, Y_i \\ &= \omega_e((Y_1)_e, \ldots, (Y_k)_e). \end{aligned}$$

This is a constant (smooth) function of $g$ as desired.

Similarly, a $k$-form $\omega$ on $G$ is said to be *right-invariant* if $r^*_g\omega = \omega$ for all $g \in G$. We can analogously prove that every right-invariant form on a Lie group is smooth.

Let $\Omega^k(G)^G$ denote the vector space of left-invariant $k$-forms on $G$. The linear map

$$\begin{aligned} \Omega^k(G)^G &\to \bigwedge^k(\mathfrak{g}^\vee) \\ \omega &\mapsto \omega_e \end{aligned}$$

has an inverse defined by the left-invariant differential form generated by $\omega_e$ and is therefore an isomorphism. It follows that

$$\dim \Omega^k(G)^G = \binom{n}{k}.$$

## 5.3 The Exterior Derivative

In contrast to standard calculus, the basic objects in calculus on manifolds are differential forms rather than functions.

Recall that an *antiderivation* on a graded algebra $A = \oplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D : A \to A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot (D\tau)$$

for all $\omega \in A^k, \tau \in A^\ell$. In the graded algebra $A$, and element of $A^k$ is called a *homogeneous element of degree k*. The antiderivation is *of degree m* if

$$\deg D\omega = \deg \omega + m.$$

for all homogeneous elements $\omega \in A$.

Let $M$ be a manifold and $\Omega^*(M)$ the graded algebra of smooth differential forms on $M$. On the graded algebra $\Omega^*(M)$, there is a uniquely and intrinsically defined antiderivation called the *exterior derivative*. The process of applying the exterior derivative is called *exterior differentiation*.

**Definition 5.3.1 (Exterior Derivative)**
An *exterior derivative* on a manifold $M$ is an $\mathbb{R}$-linear map

$$D : \Omega^*(M) \to 0\Omega^*(M)$$

such that

(i) $D$ is an antiderivative of degree 1

(ii) $D \circ D = 0$

(iii) For any $f \in C^\infty(M)$ and $X \in (M)$, $(Df)(X) = Xf$

Condition (iii) states that on 0-forms (functions), an exterior derivative agrees with the differential $df$ of a function $f$. Hence on a coordinate chart $(U, x^1, \dots, x^n)$,

$$Df = df = \sum_i \frac{\partial f}{\partial x^i} dx^i.$$

Our goal is to prove the existence and uniqueness of an exterior derivative on a manifold. Using its defining properties, we can then show that the exterior derivative commutes with the pullback. As a corollary, the pullback of a smooth form by a smooth map is smooth.

### 5.3.1 Exterior Derivative on a Coordinate Chart

We showed the existence and uniqueness of an exterior derivative on an open subset of $\mathbb{R}^n$. The same proof carries over to any coordinate chart on a manifold. Indeed, suppose $(U, x^1, \dots, x^n)$ is a coordinate chart on a manifold $M$. Then any $k$-form $\omega$ on $U$ is uniquely a linear combination

$$\omega = \sum_I a_I dx^I$$

for some $a_I \in C^\infty(U)$.

If $D$ is an exterior derivative on $U$,

$$\begin{aligned}
&D(dx^I) \\
&= D(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&= (Ddx^{i_1}) \cdot dx^{i_2} \wedge \cdots \wedge dx^{i_k} + (-1)^1 dx^{i_1} \cdot D(dx^{i_2} \wedge \cdots \wedge dx^{i_k}) \\
&= 0 - dx^{i_1} \cdot D(dx^{i_2} \wedge \cdots \wedge dx^{i_k}). && D \circ d = 0
\end{aligned}$$

By an inductive argument on $k$, we see that $Ddx^I = 0$ for all $I \in \mathcal{J}_{n,k}, k \geq 0$. It follows that

$$\begin{aligned}
D\omega &= \sum_I D(a_I dx^I) && \text{linearity} \\
&= \sum_I (Da_I) \wedge dx^I + \sum_I a_I (Ddx^I) && \text{antiderivation} \\
&= \sum_I (Da_I \wedge dx^I) && \text{above} \\
&= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I.
\end{aligned}$$

Hence if any exterior derivative $D$ exists, then it is uniquely defined by the expression above.

To show existence, we define $D$ as above and show that it satisfies the three conditions. The proof is exactly the same as in the Euclidean case and is thus omitted. We denote the unique exterior derivative on a chart $(U, \phi)$ by $d_U$.

### 5.3.2 Local Operators

Similar to the derivative of a function on $\mathbb{R}^n$, an antiderivation $D$ on $\Omega^*(M)$ has the property that for a $k$-form $\omega$, the value of $D\omega$ at a point $p$ depends only on the values of $\omega$ in a neighborhood of $p$. We formalize this under the notion of local operators.

An endomorphism of a vector space $W$ is often called an *operator* on $W$. For example, if $W = C^\infty(\mathbb{R})$ is the vector space of smooth functions on $\mathbb{R}$, then the derivative $d/dx$ is an operator on $W$:

$$\frac{d}{dx} f(x) := f'(x).$$

The derivative has the property that the value of $f'(x)$ at a point $p$ depends only on the values of $f$ in a small neighborhood of $p$. More precisely, if $f = g$ on an open set $U \subseteq \mathbb{R}$, then $f' = g'$ on $U$. We say that the derivative is a *local operator* on $C^\infty(\mathbb{R})$.

**Definition 5.3.2 (Local Operator)**
An operator $D : \Omega^*(M) \to \Omega^*(M)$ is said to be *local* if for all $k \geq 0$, whenever a $k$-form $\omega \in \Omega^k(M)$ restricts to 0 on an open set $U$ in $M$, then $D\omega \equiv 0$ on $U$.

An equivalent criterion is that for all $k \geq 0$, whenever two $k$-forms $\omega, \tau \in \Omega^k(M)$ agree on an open set $U$, then $D\omega \equiv D\tau$ on $U$.

**Example 5.3.1 (Integral Operator)**
Define the integral operator

$$\begin{aligned} I : C^\infty[a, b] &\to C^\infty[a, b] \\ f &\mapsto \int_a^b f(t)dt. \end{aligned}$$

We consider $I(f)$ as a constant function over $[a, b]$. This is not a local operator since $I(f)$ depends on the value of $f$ over all $[a, b]$.

**Proposition 5.3.2**
Any antiderivation $D$ on $\Omega^*(M)$ is a local operator.

**Proof**
Suppose $\omega \in \Omega^k(M)$ and $\omega \equiv 0$ on a open subset $U$. Let $p \in U$ be arbitrary. We claim that $(D\omega)_p = 0$.

Choose a smooth bump function $f$ at $p$ supported in $U$. In particular, $f \equiv 1$ in a neighborhood of $p$ within $U$. Then $f\omega \equiv 0$ on $M$, since for a point $q \in U$, then $\omega_q = 0$ and if $q \notin U$, we have $f(q) = 0$. By the antiderivation property of $D$ to $f\omega$,

$$\begin{aligned} 0 &= D(0) \\ &= D(f\omega) \\ &= (Df) \wedge \omega + (-1)^0 \wedge f(D\omega). \end{aligned}$$

Evaluating at the RHS at $p$ yields

$$0 = (Df) \wedge 0 + 1 \wedge (D\omega) = (D\omega)_p.$$

We remark that the same proof shows that a derivation on $\Omega^*(M)$ is also a local operator.

### 5.3.3 Existence of an Exterior Derivative on a Manifold

To define an exterior derivative on a manifold $M$, consider a $k$-form $\omega$ on $M$ and some point $p \in M$. Choose a chart $(U, x^1, \dots, x^n)$ about $p$. We can locally express $\omega = \sum_I a_I dx^I$ on $U$. We know that there is an exterior derivative $d_U$ on $U$ given by

$$d_U \omega = \sum_I da_I \wedge dx^I$$

on $U$. Define $(d\omega)_p = (d_U \omega)_p$. We need to show that $(d_U \omega)_p$ is independent of the chart $U$ about $p$. If $(V, y^1, \dots, y^n)$ is another chart about $p$ and $\omega = \sum_J b_J dy^J$ on $V$, then on $U \cap V$,

$$\sum_I a_I dx^I = \sum_J dy^J.$$

On $U \cap V$, there is a unique exterior derivative

$$d_{U \cap V} : \Omega^*(U \cap V) \to \Omega^*(U \cap V).$$

By the properties of the exterior derivative on $U \cap V$,

$$\begin{aligned} d_{U\cap V}\left(\sum_I a_I dx^I\right) &= d_{U\cap V}\left(\sum_J b_J dy^J\right) \\ \sum_I da_I \wedge dx^I &= \sum_J db_J \wedge dy^J \\ \left(\sum_I da_I \wedge dx^I\right)_p &= \left(\sum_J db_J \wedge dy^J\right)_p. \end{aligned}$$

Thus $(d\omega)_p = (d_U \omega)_p$ is well defined independently of the chart $(U, x^1, \dots, x^n)$.

As $p$ varies over all points of $M$, this defines an operator $d : \Omega^*(M) \to \Omega^*(M)$. In order to check that $d$ satisfies the defining properties of an exterior derivative, it suffices to check them at each point $p \in M$, which we have already done.

### 5.3.4 Uniqueness of the Exterior Derivative

Suppose $D : \Omega^*(M) \to \Omega^*(M)$ is an exterior derivative. We show that $D$ necessarily coincides with the exterior derivative defined above.

If $f$ is a smooth function and $X$ a smooth vector field on $M$, then by the defining conditions,

$$(Df)(X) = Xf = (df)(X).$$

Thus $Df = df$ on functions $f \in \Omega^0(M)$.

Consider now a wedge product of exact 1-forms $df^1 \wedge \cdots \wedge df^k$:

$$
\begin{aligned}
&D(df^1 \wedge \cdots \wedge df^k) \\
&= D(Df^1 \wedge \cdots \wedge Df^k) && Df^i = df^i \\
&= \sum_{i=1}^k (-1)^{i-1} Df^1 \wedge \cdots \wedge DDf^i \wedge \cdots \wedge Df^k && \text{antiderivation} \\
&= 0. && D^2 = 0
\end{aligned}
$$

We now show that $D$ agrees with $d$ on any $k$-form $\omega \in \Omega^k(M)$. Fix $p \in M$ and choose a chart $(U, x^1, \dots, x^n)$ about $p$ so that $\omega = \sum_I a_I dx^I$ on $U$. Extend the functions $a_I, x^i$ on $U$ to smooth functions $\tilde{a}_I, \tilde{x}^i$ on $M$ that agree with $a_I, x^i$ on a neighborhood $V \ni p$. Define

$$
\tilde{\omega} = \sum_I \tilde{a}_I d\tilde{x}^I \in \Omega^k(M).
$$

Then $\omega \equiv \tilde{\omega}$ on $V$.

Since $D$ is a local operator, $D\omega = D\tilde{\omega}$ on $V$. Hence

$$
\begin{aligned}
&(D\omega)_p \\
&= (D\tilde{\omega})_p \\
&= (D \sum_I \tilde{a}_I d\tilde{x}^I)_p \\
&= (\sum_I D\tilde{a}_I \wedge d\tilde{x}^I \wedge \sum_I \tilde{a}_I \wedge Dd\tilde{x}^I)_p \\
&= (\sum_I d\tilde{a}_I \wedge d\tilde{x}^I)_p && Dd = DD = 0 \\
&= (\sum_I da_I \wedge dx^I)_p \\
&= (d\omega)_p.
\end{aligned}
$$

This yields the following theorem.

> **Theorem 5.3.3**
> On any manifold $M$, there exists an exterior derivative $d : \Omega^*(M) \to \Omega^*(M)$ characterized uniquely by the three defining properties.

### 5.3.5 Exterior Differentiation Under a Pullback

We now show that the pullback of differential forms commutes with the exterior derivative. Combined with the fact that the pullback preserves the wedge product, this is a cornerstone of calculations involving the pullback. We use these two properties to show that the pullback of a smooth form under a smooth form remains smooth.

**Proposition 5.3.4 (Commutation of the Pullback with $d$)**
Let $F : N \to M$ be a smooth map of manifolds. If $\omega \in \Omega^k(M)$, then $dF^*\omega = F^*d\omega$.

**Proof**
We have already proven the case of $k = 0$ where $\omega = h \in C^\infty(M)$ by checking that $(F^*dh)_p(X_p) = (dF^*h)_p(X_p)$ for any $X_p \in T_pN$. Consider now the case of $k \geq 1$. We check that $dF^*\omega = F^*d\omega$ at every point $p \in N$. This reduces the proof to a local computation.

If $(V, y^1, \ldots, y^m)$ is a chart on $M$ about $F(p)$, then on $V$

$$\omega = \sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

where $I = (i_1 < \cdots < i_k)$ and $a_I \in C^\infty(V)$. Since the pullback distributes across the wedge product,

$$\begin{aligned} F * \omega &= \sum_I (F^*a_I) F^*dy^{i_1} \wedge \cdots \wedge F^*dy^{i_k} \\ &= \sum_I (a_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k} && \text{base case} \\ dF^*\omega &= \sum_I d(a_I \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}. \end{aligned}$$

On the other hand, again by distributivity,

$$\begin{aligned} F^*d\omega &= F^* \left( \sum_I da_I \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_k} \right) \\ &= \sum_I F^*da_I \wedge F^*dy^{i_1} \wedge \cdots \wedge F^*dy^{i_k} \\ &= \sum_I d(F^*a_I) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k} && \text{base case} \\ &= \sum_I d(a_I \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}. \end{aligned}$$

By computation,

$$dF^*\omega = F^*d\omega.$$

**Corollary 5.3.4.1**
If $U \subseteq M$ is open and $\omega \in \Omega^k(M)$, then

$$(d\omega)|_U = d(\omega|_U).$$

**Proof**
Let $\iota : U \to M$ be the inclusion map. We have $\omega|_U = \iota^*\omega$ so that

$$(d\omega)|_U = \iota^* d\omega = d\iota^*\omega = d(\omega|_U).$$

**Example 5.3.5**
Let $U = (0, \infty) \times (0, 2\pi)$ in the $(r, \theta)$-plane $\mathbb{R}^2$. Define $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ by $F(r, \theta) = (r\cos\theta, r\sin\theta)$. Let $x, y$ be the standard coordaintes on the target $\mathbb{R}^2$. We compute $F^*(dx \wedge dy)$.

$$\begin{aligned}
F^*dx &= dF^*x \\
&= d(x \circ F) \\
&= d(r\cos\theta) \\
&= \cos(\theta)dr - r\sin\theta d\theta && \text{antiderivation} \\
F^*dy &= dF^*y \\
&= d(r\sin\theta) \\
&= (\sin\theta)dr + r\cos\theta d\theta.
\end{aligned}$$

Since the pullback commutes with the wedge product,

$$\begin{aligned}
F^*(dx \wedge dy) &= (F^*dx) \wedge (F^*dy) \\
&= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= (r\cos^2\theta + r\sin^2\theta) dr \wedge d\theta \\
&= r dr \wedge d\theta.
\end{aligned}$$

**Proposition 5.3.6**
If $F : N \to M$ is a smooth map of manifolds and $\omega$ is a smooth $k$-form on $M$, then $F^*\omega$ is a smooth $k$-form on $N$.

**Proof**
Fix $p \in N$. We show that there is an neighborhood about $p$ on which $F^*\omega$ is smooth. Choose a chart $(V, y^1, \ldots, y^m)$ on $M$ about $F(p)$. Let $F^i = y^i \circ F$ be the $i$-th coordinate of the map $F$ in this chart. By the continuity (smoothness) of $F$, there is a chart

$(U, x^1, \ldots, x^n)$ on $N$ about $p$ such that $F(U) \subseteq V$. Because $\omega$ is smooth on $V$,

$$\omega = \sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

for some $a_I \in C^\infty(V)$. By the properties of the pullback,

$$\begin{aligned} F^*\omega &= \sum_I (F^* a_I) F^*(dy^{i_1}) \wedge \cdots \wedge F^*(dy^{i_k}) \\ &= \sum_I (F^* a_I) dF^* y^{i_1} \wedge \cdots \wedge dF^* y^{i_k} \\ &= \sum_I (a_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k} \\ &= \sum_{I,J} (a_I \circ F) \frac{\partial(F^{i_1}, \ldots, F^{i_k})}{\partial(x^{j_1}, \ldots, x^{j_k})} dx^J. \end{aligned}$$

Since $a_I \circ F$ and $\partial(F^{i_1}, \ldots, F^{i_k})/\partial(x^{j_1}, \ldots, x^{j_k})$ are all smooth, we see that $F^*\omega$ is smooth as desired.

In summary, if $F : N \to M$ is a smooth map of manifolds, then the pullback map $F^*\Omega^*(M) \to \Omega^*(N)$ is a morphism of differential graded algebras, ie a degree-preserving algebra homomorphism that commutes with the differential.

### 5.3.6 Restriction of $k$-Forms to a Submanifold

The restriction of a $k$-form to an immersed submanifold is just like the restriction of a 1-form, but with $k$ arguments. Let $S$ be a regular submanifold of the manifold $M$. If $\omega$ is a $k$-form on $M$, then the *restriction* of $\omega$ to $S$ is the $k$-form $\omega|_S$ on $S$ given by

$$(\omega|_S)_p(v_1, \ldots, v_k) = \omega_p(v_1, \ldots, v_k)$$

for any $v_1, \ldots, v_k \in T_pS \subseteq T_pM$. Thus $(\omega|_S)_p$ is obtained from $\omega_p$ by restricting the domain of $\omega_p$ to $\times_{i=1}^k T_pS$. As before, the restriction of $k$-forms is the same as the pullback under the inclusion map $\iota : S \to M$.

We remark that a nonzero form on $M$ may restrict to the zero form on a submanifold $S$. For example, if $S$ is a smooth curve on $\mathbb{R}^2$ defined by the nonconstant function $f(x, y)$, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

is a nonzero 1-form on $\mathbb{R}^2$, but since $f$ is identically zero on $S$, $df \equiv 0$ on $S$.

Since pullbacks and exterior differentiation commute, we can write $df|_S$ to denote either one of $(df)|_S = d(f|_S)$.

### 5.3.7 A Nowhere-Vanishing 1-Form on the Circle

Recall that $-ydx+xdy$ is a nowhere-vanishing 1-form on the unit circle. As an application of the exterior derivative, we construct a different nowhere-vanishing 1-form on the circle. This construction generalizes to the construction of a nowhere-vanishing top form on a *smooth hypersurface* in $\mathbb{R}^{n+1}$, ie a regular level set of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$. As we will see, the existence of a nowhere-vanishing top form is intimately related to orientations on a manifold.

At $p = (1, 0)$, a basis for the tangent space $T_pS^1$ is $\partial/\partial y$. Although $dx$ is a nowhere-vanishing 1-form on $\mathbb{R}^2$, it vanishes at $(1, 0)$ when restricted to $S^1$ as

$$(dx)_p \left(\frac{\partial}{\partial y}\right) = 0.$$

In order to find a nowhere-vanishing 1-form on $S^1$, we take the exterior derivative of both sides of the equation

$$x^2 + y^2 = 1.$$

We get

$$2xdx + 2ydy = 0.$$

Note that this equation is valid only at a point in $S^1$. Define

$$\begin{aligned} U_x &:= \{(x, y) \in S^1 : x \neq 0\} \\ U_y &:= \{(x, y) \in S^1 : y \neq 0\}. \end{aligned}$$

By our calculations, on $U_x \cap U_y$,

$$\frac{dy}{x} = -\frac{dx}{y}.$$

Define a 1-form $\omega$ on $S^1$ by

$$\omega_p = \begin{cases} \frac{dy}{x}, & p \in U_x \\ -\frac{dx}{y}, & p \in U_y. \end{cases}$$

This is well-defined 1-form on $U_x \cup U_y$ by construction as the two 1-forms agree on $U_x \cap U_y$.

In order to show that $\omega$ is smooth and nowhere-vanishing, we need charts. Define

$$U_x^+ := \{(x, y) \in S^1 : x > 0\}$$

and similarly $U_x^-, U_y^+, U_y^-$. On $U_x^+$, $y$ is a local coordinate and so $dy$ is a basis for the cotangent space $T_p^*S^1$ at each $p \in U_x^+$. Since $\omega = dy/x$ on $U_x^+$, $\omega$ is smooth and nowhere zero on $U_x^+$. A similar argument applies to $dy/x$ on $U_x^-$ and $-dx/y$ on $U_y^+, U_y^-$. Hence $\omega$ is smooth and nowhere vanishing on $S^1$.

## 5.4 The Lie Derivative & Interior Multiplication

The exterior differentiation $d$ was first locally defined with respect to a chart. It turns out that $d$ is in fact global and intrinsic to the manifold. We seek to derive a global intrinsic formula for the exterior of a $k$-form such as the following:

$$(d\omega)(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

The proof uses the Lie derivative and interior multiplication, two other instrinsic operations on a manifold. The Lie derivative allows us to differentiate a vector field or a differential form on a manifold along another vector field. For any vector field $X$ on a manifold, the interior multiplication $\iota_X$ is an antiderivation of degree $-1$ on differential forms.

### 5.4.1 Families of Vector Fields and Differential Forms

a collection $\{X_t\}$ or $\{\omega_t\}$ of vector fields or diffferential forms on a manifold is said to be a 1-*parameter family* if the parameter $t$ runs over some subset of $\mathbb{R}$. Let $I \subseteq \mathbb{R}$ be an open interval and suppose $\{X_t\}$ is a 1-parameter family of vector fields on $M$ defined for all $t \in I \setminus \{t_0\}$ for some $t_0 \in I$. We say that the *limit*

$$\lim_{t\to t_0} X_t$$

exists if every point $p \in M$ has a coordinate neighborhood $(U, x^1, \dots, x^n)$ on which $X_t|_p = \sum a^i(t, p)\partial/\partial x^i|_p$ and $\lim_{t\to t_0} a^i(t, p)$ exists for all $i$. In this case, we define

$$\lim_{t\to t_0} X_t|_p := \sum_{i=1}^n \lim_{t\to t_0} a^i(t, p) \left.\frac{\partial}{\partial x^i}\right|_p .$$

It can be shown that this definition of the limit of $X_t$ as $t \to t_0$ is independent of the choice of the coordinate neighborhood $(U, x^1, \dots, x^n)$, as there is a smooth change of coordinates.

A 1-parameter family $\{X_t\}_{t\in I}$ of smooth vector fields on $M$ is said to *depend smoothly on* $t$ if every $p \in M$ has a coordinate neighborhood $(U, x^1, \dots, x^n)$ on which

$$(X_t)_p = \sum_i a^i(t, p) \left.\frac{\partial}{\partial x^i}\right|_p$$

for $(t, p) \in I \times U$ and smooth functions $a^i$ on $I \times U$. In this case we also say that $\{X_t\}_{t\in I}$ is a *smooth family of vector fields on* $M$.

For a smooth family of vector fields on $M$, one can define its derivative with respect to $t = t_0$ by

$$\left(\left.\frac{d}{dt}\right|_{t=t_0} X_t\right)_p = \sum_i \frac{\partial a^i}{\partial t}(t_0, p) \left.\frac{\partial}{\partial x^i}\right|_p$$

for $(t_0, p) \in I \times U$. It can be shown that this definition is independent of the chart $(U, x^1, \dots, x^n)$ containing $p$ by considering a smooth change of coordinates. Indeed, Let $(V, y^1, \dots, y^n)$ be another coordinate neighborhood of $p$ such that

$$X_t = \sum_j b^j(t, q) \frac{\partial}{\partial y^j}$$

on $V$. On the intersection $U \cap V$,

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

It follows that

$$b^j(t, p) = \sum_i a^i(t, p) \frac{\partial y^j}{\partial x^i}.$$

Differentiating both sides with respect to $t$ yields

$$\frac{\partial b^j}{\partial t} = \sum_i \frac{\partial a^i}{\partial t} \frac{\partial y^j}{\partial x^i}.$$

But then

$$\begin{aligned}\sum_j \frac{\partial b^j}{\partial t} \frac{\partial}{\partial y^j} &= \sum_{i,j} \frac{\partial a^i}{\partial t} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \\ &= \sum_i \frac{\partial a^i}{\partial t} \frac{\partial}{\partial x^i}\end{aligned}$$

as required. Observe that the derivative $d/dt|_{t=t_0} X_t$ is a smooth vector field on $M$.

Similarly, a 1-parameter family $\{\omega_t\}_{t \in i}$ of smooth $k$-forms on $M$ is said to *depend smoothly* on $t$ if every point of $M$ has a coordinate neighborhood $(U, x^1, \dots, x^n)$ on which

$$(\omega_t)_p = \sum_J b_J(t, p) dx^J|_p$$

for $(t, p) \in I \times U$ and some smooth functions $b_J$ on $I \times U$. We also call such a family $\{\omega_t\}_{t \in I}$ a *smooth family of $k$-forms on $M$* and define its derivative with respect to $t$ to be

$$\left( \frac{d}{dt}\bigg|_{t=t_0} \omega_t \right)_p = \sum_J \frac{\partial b_J}{\partial t}(t_0, p) dx^J|_p.$$

Similar to vector fields, this definition is independent of the chart and defines a smooth $k$-form $d/dt|_{t=t_0} \omega_t$ on $M$.

Note that we write $d/dt$ for the derivative of a smooth family of vector fields or differential forms, but $\partial/\partial t$ for the partial derivative of a function of several variables.

**Proposition 5.4.1 (Product Rule for $d/dt$)**
If $\{\omega_t\}, \omega\tau_t$ are smooth families of $k$-forms and $\ell$-forms respectively on a manifold $M$, then

$$\frac{d}{dt}(\omega_t \wedge \tau_t) = \left(\frac{d}{dt}\omega_t\right) \wedge \tau_t + \omega_t \wedge \left(\frac{d}{dt}\tau_t\right).$$

**Proof**
Written out in local coordinates, the statement reduces to the usual product rule in calculus.

**Proposition 5.4.2 (Commutation of $d/dt|_{t=t_0}$ with $d$)**
If $\{\omega_t\}$ is a smooth family of differential forms on a manifold $M$, then

$$\left.\frac{d}{dt}\right|_{t=t_0} d\omega_t = d\left(\frac{d}{dt}_{t=t_0}\omega_t\right).$$

**Proof**
We first check that

$$\frac{d}{dt}(d\omega_t) = d\left(\frac{d}{dt}\omega_t\right)$$

at an arbitrary point $p \in M$. Indeed, let $(U, x^1, \ldots, x^n)$ be a neighborhood of $p$ such that $\omega = \sum_J b_J dx^J$ for some smooth functions $b_J$ on $I \times U$. On $U$,

$$\begin{aligned}
&\frac{d}{dt}(d\omega_t)\\
&= \frac{d}{dt}\sum_{J,i}\frac{\partial b_J}{\partial x^i}dx^i \wedge dx^J\\
&= \sum_{i,J}\frac{\partial}{\partial x^i}\left(\frac{\partial b_J}{\partial t}\right)dx^i \wedge dx^J && \text{exchange order}\\
&= d\left(\sum_J \frac{\partial b_J}{\partial t}dx^J\right)\\
&= d\left(\frac{d}{dt}\omega_t\right).
\end{aligned}$$

Evaluating at $t = t_0$ on both sides of the equation commutes with $d$ as $d$ only involves partial derivatives with respect to the $x^i$ variables.

### 5.4.2 The Lie Derivative of a Vector Field

Recall the elementary calculus definition of the derivative of a real-valued function $f$ on $\mathbb{R}$ at a point $p \in \mathbb{R}$:

$$f'(p) := \lim_{t \to 0} \frac{f(p+t) - f(p)}{t}.$$

The issue in generalizing this to the derivative of a vector field $Y$ on a manifold $M$ is that at two nearby points $p, q \in M$, the tangent vectors $Y_p, Y_q$ are in different vector spaces $T_pM, T_qM$.so we cannot subtract them. One way around this is to use the local flow of another vector field $X$ to transport $Y_q$ to the tangent space $T_pM$ at $p$.

Recall that for any smooth vector field $X$ on $M$, there is a neighborhood $U$ of $p$ on which the vector field as a *local flow*: ie there is some $\varepsilon > 0$ and a map

$$\varphi : (-\varepsilon, \varepsilon) \times U \to M$$

such that if we write $\varphi_t(q) = \varphi(t, q)$, then

$$\frac{\partial}{\partial t}\varphi_t(q) = X_{\varphi_t(q)} \qquad \varphi_0(q) = q \qquad q \in U.$$

In other words, for each $q \in U$, the curve $\varphi_t(q)$ is an integral curve of $X$ with initial point $q$. By definition, $\varphi_0(q) = q$. The local flow also satisfies the property

$$\varphi_s \circ \varphi_t = \varphi_{s+t}$$

whenever both sides are defined. Thus for each $t$, the map $\varphi_t : U \to \varphi_t(U)$ is a diffeomorphism onto its image, with the smooth inverse $\varphi_{-t}$. Indeed,

$$\varphi_{-t} \circ \varphi_t = \varphi_0 = \mathrm{Id}, \qquad \varphi_t \circ \varphi_{-t} = \varphi_0 = \mathrm{Id}\,.$$

Let $Y$ be a smooth vector field on $M$. To compare the values of $Y$ at $\varphi_t(p)$ and at $p$, we use the diffeomorphism $\varphi_{-t} : \varphi_t(U) \to U$ to push $Y_{\varphi_t(p)}$ into $T_pM$.

**Definition 5.4.1 (Lie Derivative of a Vector Field)**
For $X, Y \in (M)$ and $p \in M$, let $\varphi : (-\varepsilon, \varepsilon) \times U \to M$ be a local flow of $X$ on a neighborhood $U$ of $p$. The *Lie derivative* $\mathcal{L}_X Y$ of $Y$ with respect to $X$ at $p$ is the tangent vector

$$\begin{aligned}
(\mathcal{L}_X Y)_p &:= \lim_{t \to 0} \frac{\varphi_{-t*}(Y_{\varphi_t(p)}) - Y_p}{t} \\
&= \lim_{t \to 0} \frac{(\varphi_{-t*}Y)_p - Y_p}{t} \\
&= \frac{d}{dt}\bigg|_{t=0} (\varphi_{-t*}Y)_p.
\end{aligned}$$

In the definition above, the limit is taken in the finite-dimensional vector space $T_pM$. For the derivative to exist, it suffices that $\{\varphi_{-t*}Y\}$ be a smooth family of vector fields on $M$.

To show the smoothness of the family, we write $\varphi_{-t*}Y$ in the local coordinates $x^1, \dots x^n$ in a chart. Let $\varphi_t^i$ and $\varphi^i$ be the $i$-th components of $\varphi_t, \varphi$ respectively. Then

$$(\varphi_t)^i(p) = \varphi^i(t,p) = (x^i \circ \varphi)(t,p).$$

Recall that relative to the frame $\{\partial/\partial x^j\}$, the differential $\varphi_{t*}$ at $p$ is represented by the Jacobian matrix

$$\left[\frac{\partial(\varphi_t)^i}{\partial x^j(p)}\right] = \left[\frac{\partial \varphi^i}{\partial x^j(t,p)}\right].$$

Hence

$$\varphi_{t*}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_i \frac{\partial \varphi^i}{\partial x^j}(t,p) \left.\frac{\partial}{\partial x^i}\right|_{\varphi_t(p)}.$$

Thus if $Y = \sum_j b^j \partial/\partial x^j$, then

$$\begin{aligned}\varphi_{-t*}(Y_{\varphi_t(p)}) &= \sum_j b^j(\varphi(t,p))\varphi_{-t*}\left(\left.\frac{\partial}{\partial x^j}\right|_{\varphi_t(p)}\right)\\ &= \sum_{i,j} b^j(\varphi(t,p))\frac{\partial \varphi^i}{\partial x^j}(-t,p) \left.\frac{\partial}{\partial x^i}\right|_p.\end{aligned}$$

When $X, Y$ are smooth vector fields on $M$, both $\varphi^i, b^j$ are smooth functions. Hence $\{\varphi_{-t*}Y\}$ is indeed a smooth family of vector fields on $M$. It follows that the Lie derivative $\mathcal{L}_X Y$ exists and is given in local coordinates by

$$\begin{aligned}(\mathcal{L}_X Y)_p &= \left.\frac{d}{dt}\right|_{t=0} \varphi_{-t*}(Y_{\varphi_t(p)})\\ &= \sum_{i,j} \left.\frac{\partial}{\partial t}\right|_{t=0} \left(b^j(\varphi(t,p))\frac{\partial \varphi^i}{\partial x^j}(-t,p)\right) \left.\frac{\partial}{\partial x^i}\right|_p.\end{aligned}$$

It turns out that we have already seen the Lie derivative.

**Theorem 5.4.3**
If $X, Y$ are smooth vector fields on a manifold $M$, then the Lie derivative $\mathcal{L}_X Y$ coincides with the Lie bracket $[X, Y]$.

Recall that the Lie brakcet $[X, Y]$ at $p$ is given by

$$[X,Y]_p f := (X_pY - Y_pX)f$$

for any germ $f$ of a smooth function at $p$. Here

$$Xf(p) := X_p f.$$

**Proof**

We check the equality $\mathcal{L}_X Y = [X, Y]$ at every point by expanding both sides in local coordinates. Suppose a local flow for $X$ is given by $\varphi : (-\varepsilon, \varepsilon) \times U \to M$, where $(U, x^1, \dots, x^n)$ is a coordinate chart. Let $X = \sum_i a^i \partial/\partial x^i$ and $Y = \sum_j b^j \partial/\partial x^j$ on $U$. Recall that a local flow $c(t)$ of $X$ satisfies

$$X_{c(t)} = \sum_i a^i(c(t)) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}$$
$$c'(t) = \sum_i \dot{c}^i(t) \left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

The condition that $\varphi_t(p)$ is an integral of $X$ translates into the ODE

$$\frac{\partial \varphi^i}{\partial t}(t, p) = a^i(\varphi(t, p)), \qquad i = 1, \dots, n \qquad (t, p) \in (-\varepsilon, \varepsilon) \times U.$$

The initial conditions state that at $t = 0$,

$$\frac{\partial \varphi^i}{\partial t}(0, p) = a^i(\varphi(0, p)) = a^i(p).$$

Recall that the Lie bracket in local coordinates is

$$[X, Y] = \sum_{i,k} \left( a^k \frac{\partial b^i}{\partial x^k} - b^k \frac{\partial a^i}{\partial x^k} \right) \frac{\partial}{\partial x^i}.$$

From our calculations above,

$$\begin{aligned}
&(\mathcal{L}_X Y)_p \\
&= \sum_{i,j} \left.\frac{\partial}{\partial t}\right|_{t=0} \left( b^j(\varphi(t,p)) \frac{\partial \varphi^i}{\partial x^j}(-t, p) \right) \left.\frac{\partial}{\partial x^i}\right|_p \\
&= \left[ \sum_{i,j,k} \left( \frac{\partial b^j}{\partial x^k}(\varphi(t,p)) \frac{\partial \varphi^k}{\partial t}(t,p) \frac{\partial \varphi^i}{\partial x^j}(-t,p) \right) \frac{\partial}{\partial x^i} \right. \\
&\quad \left. - \sum_{i,j} \left( b^j(\varphi(t,p)) \frac{\partial}{\partial x^j} \frac{\partial \varphi^i}{\partial t}(-t,p) \right) \frac{\partial}{\partial x^i} \right]_{t=0} && \text{exchange order} \\
&= \sum_{i,j,k} \left( \frac{\partial b^j}{\partial x^k}(p) a^k(p) \frac{\partial \varphi^i}{\partial x^j}(0,p) \right) - \sum_{i,j} \left( b^j(p) \frac{\partial a^i}{\partial x^j}(p) \right) \frac{\partial}{\partial x^i}.
\end{aligned}$$

But $\varphi(0, p) = p$ so $\varphi_0$ is the identity map and its Jacobian is the identity. In particular,

$$\frac{\partial \varphi^i}{\partial x^j}(0, p) = \delta^i_j$$

and the expression above simplifies above to

$$
\begin{aligned}
&(\mathcal{L}_X Y)_p \\
&= \sum_{i,j,k} \left( \frac{\partial b^j}{\partial x^k}(p) a^k(p) \frac{\partial \varphi^i}{\partial x^j}(0,p) \right) - \sum_{i,j} \left( b^j(p) \frac{\partial a^i}{\partial x^j}(p) \right) \frac{\partial}{\partial x^i} \\
&= \sum_{i,k} \left( a^k \frac{\partial b^i}{\partial x^k} - b^k \frac{\partial a^i}{\partial x^k} \right) \frac{\partial}{\partial x^i} \\
&= [X, Y].
\end{aligned}
$$

Although the Lie derivative of a vector fields does not give us anything new, it is a useful tool alongside the Lie derivative of differential forms.

### 5.4.3 The Lie Derivative of a Differential Form

Let $X$ be a smooth vector field and $\omega$ a smooth $k$-form on a manifold $M$. Fix $p \in M$ and let $\varphi_t : U \to M$ be a flow of $X$ in a neighborhood $U$ of $p$. The definition of the Lie derivative of a differential form is similar to that of the Lie derivative of a vector field. Instead of pushing a tangent vector at $\varphi_t(p)$ to $p$ via $(\varphi_{-t})_*$, we now pull the $k$-covector $\omega_{\varphi_t(p)}$ back to $p$ via $\varphi_t^*$.

**Definition 5.4.2 (Lie Derivative of a Differential Form)**
Let $X$ be a smooth vector field and $\omega$ a smooth $k$-form on a manifold $M$. The *Lie derivative* $\mathcal{L}_X\omega$ at $p \in M$ is

$$
\begin{aligned}
(\mathcal{L}_X \omega)_p &= \lim_{t \to 0} \frac{\varphi_t^*(\omega_{\varphi_t(p)}) - \omega_p}{t} \\
&= \lim_{t \to 0} \frac{(\varphi_t^*\omega)_p - \omega_p}{t} \\
&= \left. \frac{d}{dt} \right|_{t=0} (\varphi_t^*\omega)_p.
\end{aligned}
$$

By a similar argument to the case of vector fields, it can be shown that $\{\varphi_t^*\omega\}$ is a smooth family of $k$-forms on $M$ by expressing it in local coordiantes. The existence of $(\mathcal{L}_X\omega)_p$ is thus guaranteed.

**Proposition 5.4.4**
Let $f$ be a smooth function and $X$ be a smooth vector field on $M$. Then $\mathcal{L}_X f = Xf$.

**Proof**
Fix $p \in M$ and let $\varphi_t : U \to M$ be a local flow of $X$ as above. Since $\varphi_t(p)$ is a curve

through $p$ with initial vector $X_p$,

$$\begin{aligned}(\mathcal{L}_X f)_p &:= \frac{d}{dt}\bigg|_{t=0} (\varphi_t^* f)_p \\ &:= \frac{d}{dt}\bigg|_{t=0} (f \circ \varphi_t)(p) \\ &= X_p f.\end{aligned}$$

### 5.4.4 Interior Multiplication

We first define interior multiplication on a vector space.

**Definition 5.4.3 (Interior Multiplication)**
If $\beta$ is a $k$-covector on a vector space $V$ and $v \in V$, for $k \geq 2$ the *interior multiplication / contraction* of $\beta$ with $v$ is the $(k-1)$-covector $\iota_v \beta$ defined by

$$(\iota_v \beta)(v_2, \dots, v_k) = \beta(v, v_2, \dots, v_k)$$

for $v_2, \dots, v_k \in V$.

We define $\iota_v \beta = \beta(v) \in \mathbb{R}$ for a 1-covector $\beta$ on $V$ and $\iota_v \beta = 0$ for a 0-covector (constant) $\beta$ on $V$.

**Proposition 5.4.5**
For 1-covectors $\alpha^1, \dots, \alpha^k$ on a vector space $V$ and $v \in V$,

$$\iota_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k.$$

Here the hat over $\alpha^i$ indicates that $\alpha^i$ is omitted from the wedge product.

Recall that

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \alpha^1(v_{\sigma(1)}) \dots \alpha^k(v_{\sigma(k)}) = \det\big[\alpha^i(v_j)\big].$$

**Proof**
By computation,

$$\begin{aligned}
&(\iota_v)(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_2, \ldots, v_k) \\
&= (\alpha^1 \wedge \cdots \wedge \alpha^k)(v, v_2, \ldots, v_k) \\
&= \det\left[\alpha^i(v_j)\right] && v_1 = v \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v) \det\left[\alpha^\ell(v_j)\right]_{\ell \neq i, j \neq 2} && \text{expansion along 1st column} \\
&= \sum_{i=1}^{k} (-1)^{i+1} \alpha^i(v) \left(\alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k\right)(v_2, \ldots, v_k).
\end{aligned}$$

**Proposition 5.4.6**
Fix a vector $v$ in a vector space $V$. Let $\iota_v : \bigwedge^*(V^\vee) \to \bigwedge^{*-1}(V^\vee)$ be interior multiplication by $v$. Then

(i) $\iota_v \circ \iota_v = 0$

(ii) for $\beta \in \bigwedge^k(V^\vee)$ and $\gamma \in \bigwedge^\ell(V^\vee)$,

$$\iota_v(\beta \wedge \gamma) = (\iota_v \beta) \wedge \gamma + (-1)^k \beta \wedge \iota_v \gamma.$$

Thus $\iota_v$ is an antiderivation of degree $-1$ whose square is 0.

**Proof**
(i) Let $\beta \in \bigwedge^k(V^\vee)$. By the definition of interior multiplication,

$$\begin{aligned}
(\iota_v(\iota_v \beta))(v_3, \ldots, v_k) &= (\iota_v \beta)(v, v_3, \ldots, v_k) \\
&= \beta(v, v, v_3, \ldots, v_k) \\
&= 0.
\end{aligned}$$

The last equality follows as $\beta$ is alternating and there is a repeated argument $v$.

(ii) Recall that the wedge product and interior multiplication are both linear in its arguments. Hence it suffices to show the case there

$$\beta = \alpha^1 \wedge \cdots \wedge \alpha^k, \qquad \gamma = \alpha^{k+1} \wedge \cdots \wedge \alpha^{k+\ell},$$

where the $\alpha^i$'s are all 1-covectors. Then

$$\begin{aligned}
&\iota_v(\beta \wedge \gamma) \\
&= \iota_v(\alpha^1 \wedge \cdots \wedge \alpha^{k+\ell}) \\
&= \sum_{i=1}^{k+\ell} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^{k+\ell} \\
&= \left( \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k \right) \wedge \alpha^{k+1} \wedge \cdots \wedge \alpha^{k+\ell} \\
&\quad + (-1)^k \alpha^1 \wedge \cdots \wedge \alpha^k \wedge \sum_{i=1}^{k} (-1)^{i+1} \alpha^{k+i}(v) \alpha^{k+1} \wedge \cdots \wedge \widehat{\alpha^{k+i}} \wedge \cdots \wedge \alpha^{k+\ell} \\
&= (\iota_v \beta) \wedge \gamma + (-1)^k \beta \wedge \iota_v \gamma.
\end{aligned}$$

Interior multiplication on a manifold is defined pointwise. If $X$ is a smooth vector field on $M$ and $\omega \in \Omega^k(M)$, then $\iota_X \omega$ is the $(k-1)$-form defined by

$$(\iota_X \omega)_p := \iota_{X_p} \omega_p$$

for all $p \in M$. The form $\iota_X \omega$ on $M$ is smooth since for any smooth vector fields $X_2, \ldots, X_k$ on $M$,

$$(\iota_X \omega)(X_2, \ldots, X_k) = \omega(X, X_2, \ldots, X_k)$$

is a smooth function on $M$. In the case that $\omega$ is a 1-form, $\iota_X(\omega) = \omega(X)$. If $\omega = f$ is a -form (function) on $M$, then $\iota_X f = 0$. By the properties of interior multiplication at each point $p \in M$, the map $\iota_X : \Omega^*(M) \to \Omega^*(M)$ is an antiderivation of degree $-1$ such that $\iota_X \circ \iota_X = 0$.

Let $\mathcal{F}$ denote the ring $C^\infty(M)$ of smooth functions on the manifold $M$. As $\iota_X \omega$ is a point operator, ie its value at $p$ depends only on $X_p, \omega_p$, it is $\mathcal{F}$-linear in either argument. Thus $\iota_X \omega$ is additive in each argument.

**Proposition 5.4.7**
For any $f \in \mathcal{F}$,

(i) $\iota_{fX} \omega = f \iota_X \omega$

(ii) $\iota_X(f\omega) = f \iota_X \omega$

**Proof**
We omit the proof of (ii) as it is similar.

$$\begin{aligned}
(\iota_{fX} \omega)_p &= \iota_{f(p) X_p} \omega_p \\
&= f(p) \iota_{X_p} \omega_p \\
&= (f \iota_X \omega)_p.
\end{aligned}$$

**Example 5.4.8 (Interior Multiplication on $\mathbb{R}^2$)**
Let $X = x\partial/\partial x + y\partial/\partial y$ be the radial vector field and $\alpha = dx \wedge dy$ the area 2-form on the plane $\mathbb{R}^2$. We compute the contraction $\iota_X \alpha$.

Firstly,

$$\begin{aligned} \iota_X dx &= dx(X) = x \\ \iota_X dy &= y. \end{aligned}$$

By the antiderivation property of $\iota_X$,

$$\begin{aligned} \iota_X \alpha &= \iota_X(dx \wedge dy) \\ &= (\iota_X dx)dy - dx(\iota_X dy) \\ &= xdy - ydx. \end{aligned}$$

This restricts to the nowhere-vanishing 1-form $\omega$ on the circle $S^1$.

### 5.4.5 Properties of the Lie Derivative

We state and prove several basic properties of the Lie derivative, including its relation to exterior derivation and interior multiplication, two other intrinsic operations on a manifold.

**Theorem 5.4.9**
Let $X$ be a smooth vector field on a manifold $M$.

(i) The Lie derivative $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation, ie it is an $\mathbb{R}$-linear map such that for all $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$,

$$\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X \omega) \wedge \tau + \omega \wedge (\mathcal{L}_X \tau).$$

(ii) The Linear derivative $\mathcal{L}_X$ commutes with the exterior derivative $d$.

(iii) (Cartan homotopy formula) $\mathcal{L}_X = d\iota_X + \iota_X d$.

(iv) ("Product" formula) For all $\omega \in \Omega^k(M)$ and $Y_1, \dots, Y_k \in (M)$,

$$\mathcal{L}_X(\omega(Y_1, \dots, Y_k)) = (\mathcal{L}_X \omega)(Y_1, \dots, Y_k) + \sum_{i=1}^{k} \omega(Y_1, \dots, \mathcal{L}_X Y_i, \dots, Y_k).$$

Recall the product rule for smooth families $\{\omega_t\}, \{\tau_t\}$ of $k$-forms and $\ell$-forms:

$$\frac{d}{dt}(\omega_t \wedge \tau_t) = \left(\frac{d}{dt}\omega_t\right) \wedge \tau_t + \omega_t \wedge \frac{d}{dt}\tau_t.$$

**Proof (i)**
Let $p \in M$ and $\varphi_t : U \to M$ a local flow of $X$ in a neighborhood $U \ni p$.

The Lie derivative $\mathcal{L}_X$ is the $d/dt$ of a vector-valued function of $t$. Thus the derivation property is really just the product rule for smooth families of differential forms:

$$\begin{aligned}(\mathcal{L}_X(\omega \wedge \tau))_p &= \frac{d}{dt}\bigg|_{t=0} (\varphi_t^*(\omega \wedge \tau))_p \\ &= \frac{d}{dt}\bigg|_{t=0} (\varphi_t^*\omega)_p \wedge (\varphi_t^*\tau)_p \\ &= (\mathcal{L}_X\omega)_p \wedge \tau_p + \omega_p \wedge (\mathcal{L}_X\tau)_p.\end{aligned}$$

Recall that the exterior derivative commutes with the pullback by a smooth functions as well with $d/dt$ of a smooth family of differential forms.

**Proof (ii)**
Let $p \in M$ and $\varphi_t : U \to M$ a local flow of $X$ in a neighborhood $U \ni p$.

$$\begin{aligned}\mathcal{L}_X d\omega &:= \frac{d}{dt}\bigg|_{t=0} \varphi_t^* d\omega \\ &= \frac{d}{dt}\bigg|_{t=0} d\varphi_t^*\omega \\ &= d\left(\frac{d}{dt}\bigg|_{t=0} \varphi_t^*\omega\right) \\ &= d\mathcal{L}_X\omega.\end{aligned}$$

Recall that if $A, B$ are both superderivations of degree $m_1, m_2$, then $AB - (-1)^{m_1 m_2} BA$ is a superderivation of degree $m_1 + m_2$. In particular, if $A, B$ are antiderivations (superderivations of odd degree), then $AB + BA$ is a derivation of degree $m_1 + m_2$.

Also recall from an earlier proposition that

$$\mathcal{L}_X f = Xf$$

for any $f \in C^\infty(M)$ and $X \in \mathfrak{X}(X)$.

**Proof (iii)**
Let $p \in M$ and $\varphi_t : U \to M$ a local flow of $X$ in a neighborhood $U \ni p$. We claim that it suffices to check that

$$\mathcal{L}_X f = (d\iota_X + \iota_X d) f$$

for any $f \in C^\infty(U)$.

Indeed, for any $\omega \in \Omega^k(M)$, it suffices to check that at any $p \in M$, $\mathcal{L}_X \omega = (d\iota_X + \iota_X d)\omega$. By shrinking $U$ if necessary, we may assume we have a coordinate neighborhood

$(U, x^1, \dots, x^n)$. Moreover, by linearity, we may further assume that $\omega$ is a wedge product $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. Next, the LHS is a derivation by (i) and the RHS is also a derivation (superderivation of even degree). By (ii), the LHS commutes with exterior derivation and so does the RHS:

$$d(d\iota_X + \iota_X d) = d\iota_X d = (d\iota_X + \iota_X d)d.$$

thus both sides of the Cartan homotopy formula are derivations that commute with $d$.

Thus if the formula holds for two differential forms $\omega, \tau$, then it holds for the wedge product $\omega \wedge \tau$ and $d\omega$. It follows that the reduction above is justified.

We conclude the proof by verifying for $f \in C^\infty(U)$ that

$$\begin{aligned} (d\iota_X + \iota_X d) f &= \iota_X df & \iota_X f = 0 \\ &:= (df)(X) \\ &= Xf \\ &= \mathcal{L}_X f. \end{aligned}$$

Recall that $\{\varphi_{-t*} Y\}$ is a smooth family of vector fields for each $Y \in \mathfrak{X}(M)$.

Also recall that $\varphi_{-t*}(Y_{\varphi_t(p)})$ is smooth and hence continuous at a neighborhood of $(0, p)$.

**Proof (iv)**
Let $p \in M$ and $\varphi_t : U \to M$ a local flow of $X$ in a neighborhood $U \ni p$. The proof is similar to that of the standard product rule for the calculus derivative and we focus on the case of $k = 2$ for the sake of simplicity as the general case is similar but more tedious.

By the old trick of adding and subtracting terms,

$$\begin{aligned} &(\mathcal{L}_X(\omega(Y, Z)))_p \\ &= \lim_{t \to 0} \frac{(\varphi_t^*(\omega(Y, Z)))_p - (\omega(Y, Z))_p}{t} \\ &= \lim_{t \to 0} \frac{\omega_{\varphi_t(p)}(Y_{\varphi_t(p)}, Z_{\varphi_t(p)}) - \omega_p(Y_p, Z_p)}{t} \\ &= \lim_{t \to 0} \frac{\omega_{\varphi_t(p)}(Y_{\varphi_t(p)}, Z_{\varphi_t(p)}) - \omega_p(\varphi_{-t*}(Y_{\varphi_t(p)}), \varphi_{-t*}(Z_{\varphi_t(p)}))}{t} \\ &\quad + \lim_{t \to 0} \frac{\omega_p(\varphi_{-t*}(Y_{\varphi_t(p)}), \varphi_{-t*}(Z_{\varphi_t(p)})) - \omega_p(Y_p, \varphi_{-t*}(Z_{\varphi_t(p)}))}{t} \\ &\quad + \lim_{t \to 0} \frac{\omega_p(Y_p, \varphi_{-t*}(Z_{\varphi_t(p)})) - \omega_p(Y_p, Z_p)}{t}. \end{aligned}$$

We rewrite the first limit in the summation as

$$\begin{aligned}
&\frac{(\varphi_t^*\omega_{\varphi_t(p)})(\varphi_{-t*}(Y_{\varphi_t(p)}),\varphi_{-t*}(Z_{\varphi_t(p)}))-\omega_p(\varphi_{-t*}(Y_{\varphi_t(p)}),\varphi_{-t*}(Z_{\varphi_t(p)}))}{t}\\
&=\frac{\varphi_t^*(\omega_{\varphi_t(p)})-\omega_p}{t}(\varphi_{-t*}(Y_{\varphi_t(p)}),\varphi_{-t*}(Z_{\varphi_t(p)}))\\
&\to(\mathcal{L}_X\omega)_p(Y_p,Z_p). && t\to 0
\end{aligned}$$

Here the limit is justified as both the operator and arguments have limits.

By the bilinearity of $\omega_p$, the second term is

$$\begin{aligned}
&\lim_{t\to 0}\omega_p\left(\frac{\varphi_{-t*}(Y_{\varphi_t(p)})-Y_p}{t},\varphi_{-t*}(Z_{\varphi_t(p)})\right)\\
&=\omega_p((\mathcal{L}_XY)_p,Z_p).
\end{aligned}$$

Finally, by a similar calculation, the third term is given by $\omega_p(Y_p,(\mathcal{L}_XZ)_p)$.

**Remark 5.4.10** Unlike interior multiplication, the Lie derivative $\mathcal{L}_X$ is not $\mathcal{F}$-linear in either argument. By the derivation property,

$$\mathcal{L}_X(f\omega)=(\mathcal{L}_Xf)\omega+f\mathcal{L}_X\omega=(Xf)\omega+f\mathcal{L}_X\omega.$$

We note that the previous theorem can be used to compute the Lie derivative of a differential form.

**Example 5.4.11 (The Lie Derivative on a Circle)**
Let $\omega$ be the 1-form $-ydx+xdy$ and $X$ the tangent vector field $-y\partial/\partial x+x\partial/\partial y$ on the unit circle $S^1$. We have

$$\begin{aligned}
\mathcal{L}_X(-ydx)&=-(\mathcal{L}_Xy)dx-y\mathcal{L}_Xdx && \text{derivation}\\
&=-(Xy)dx-yd\mathcal{L}_Xx\\
&=-xdx-yd(Xx)\\
&=-xdx+ydy\\
\mathcal{L}_X(xdy)&=-ydy+xdx\\
\mathcal{L}_X\omega&=0.
\end{aligned}$$

### 5.4.6 Global Formulas for the Lie and Exterior Derivatives

The definition of the Lie derivative only makes sense in a neighborhood of a point as it is local. The product formula gives us access to a global formula for the Lie derivative.

**Theorem 5.4.12 (Global Formula for the Lie Derivative)**
For a smooth $k$-form $\omega$ and smooth vector fields $X, Y_1, \ldots, Y_k$ on a manifold $M$,

$$(\mathcal{L}_X\omega)(Y_1, \ldots, Y_k) = X(\omega(Y_1, \ldots, Y_k)) - \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k).$$

The definition of the exterior derivative $d$ is also local. Using the Lie derivative, we obtain a useful global formula for the exterior derivative. We begin with the case of a 1-form.

**Proposition 5.4.13**
If $\omega$ is a smooth 1-form and $X, Y$ are smooth vector fields on a manifold $M$, then

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

**Proof**
It suffices to check the formula in a chart $(U, x^1, \ldots, x^n)$, thus we assume without loss of generality that $\omega = \sum_i a_i dx^i$. Since both sides of the equation are $\mathbb{R}$-linear in $\omega$, we further assume that $\omega = f dg$ where $f, g \in C^\infty(U)$.

In this case,

$$d\omega = d(f dg) = df \wedge dg$$

and

$$d\omega(X, Y) = df(X)dg(Y) - df(Y)dg(X) = (Xf)Yg - (Yf)Xg.$$

On the other hand,

$$\begin{aligned} X\omega(Y) &= X(f dg(Y)) = X(fYg) = (Xf)Yg + fXYg, \\ Y\omega(X) &= Y(f dg(X)) = Y(fXg) = (Yf)Xg + fYXg, \\ \omega([X, Y]) &= f dg([X, Y]) = f(XY - YX)g. \end{aligned}$$

This concludes the proof.

**Theorem 5.4.14 (Global Formula for the Exterior Derivative)**
Fix $k \geq 1$. For a smooth $k$-form $\omega$ and smooth vector fields $Y_0, Y_1, \ldots, Y_k$ on a manifold $M$,

$$\begin{aligned} &(d\omega)(Y_0, \ldots, Y_k) \\ &= \sum_{i=0}^{k} (-1)^i Y_i \omega(Y_0, \ldots, \widehat{Y_i}, \ldots, Y_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([Y_i, Y_j], Y_0, \ldots, \widehat{Y_i}, \ldots, \widehat{Y_j}, \ldots, Y_k). \end{aligned}$$

We have already shown the case of $k = 1$. Assuming the formula for degrees $k - 1$, the case of degree $k$ can be shown by induction. Indeed, by the Cartan homotopy formula,

$$\begin{aligned}(d\omega)(Y_0, Y_1, \dots, Y_k) &= (\iota_{Y_0} d\omega)(Y_1, \dots, Y_k) \\ &= (\mathcal{L}_{Y_0}\omega)(Y_1, \dots, Y_k) - (d\iota_{Y_0}\omega)(Y_1, \dots, Y_k).\end{aligned}$$

The first term can be computed using the global formula for the Lie derivative $\mathcal{L}_{Y_0}\omega$ while the second term can be computed using the induction hypothesis.

# Chapter 6

# Integration

On a manifold, we integrate differential forms rather than functions. We focus on the integration of smooth forms over a submanifold. Note that it is nontheless possible to integrate noncontinuous forms over more general sets.

for integration over a manifold to be well-defined, the manifold must be oriented. We begin by discussing orientations on a manifold and enlarge the category of manifolds to include manifolds with boundary. Our treatment of integration culminates in Stokes' theorem for an $n$-dimensional manifold.

## 6.1 Orientations

Our goal is to define orientations for $n$-manifolds and to investigate various equivalent characterizations of orientations.

### 6.1.1 Orientations of a Vector Space

For this segment, we assume all vector spaces are finite-dimensional. Two ordered bases $u = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$ of a vector $V$ are *equivalent*, written $u \sim v$, if $u = vA$ for some $n \times n$ matrix with positive determinant.

**Definition 6.1.1 (Orientation)**
An orientation of a vector space $V$ is an equivalence class of ordered bases.

Note that any finite-dimensional vector space has exactly two orientations. If $\mu$ is an orientation of a finite-dimensional vector space $V$, we denote the other orientation by $-\mu$ and call it the *opposite* of the orientation $\mu$.

By convention, we define an orientation on the zero-dimensional vector space to be one of two signs $+, -$.

We typically write $v_1, \ldots, v_n$ for a basis in a vector space. We enclose the basis $(v_1, \ldots, v_n)$ if it is an ordered basis or alternatively we write it in matrix notation $\begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$. An orientation is denoted $[(v_1, \ldots, v_n)]$ where the square brackets now stand for an equivalence class.

### 6.1.2 Orientations & $n$-Covectors

Rather than an ordered basis, we can also use an $n$-covector to specify an orientation. This approach is based on the fact that the space $\bigwedge^n(V^\vee)$ of $n$-covectors on $V$ is one-dimensional.

**Lemma 6.1.1**
Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in a vector space $V$. Suppose

$$u_j = \sum_{i=1}^n a_j^i v_i$$

for some matrix $A = [a_j^i] \in \mathbb{R}^{n \times n}$. If $\beta$ is an $n$-covector on $V$, then

$$\beta(u_1, \ldots, u_n) = (\det A)\beta(v_1, \ldots, v_n).$$

**Proof**
By computation,

$$\begin{aligned}
\beta(u_1, \ldots, u_n) &= \beta\left(\sum_{i_1} v_{i_1} a_1^{i_1}, \ldots, \sum_{i_n} v_{i_n} a_n^{i_n}\right) \\
&= \sum_{i_1, \ldots, i_n} a_1^{i_1} \ldots a_n^{i_n} \beta(v_{i_1}, \ldots, v_{i_n}) \\
&= \sum_{\sigma \in S_n} a_1^{\sigma(1)} \ldots a_n^{\sigma(n)} \beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) && \beta \text{ is alternating} \\
&= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) a_1^{\sigma(1)} \ldots a_n^{\sigma(n)} \beta(v_1, \ldots, v_n) && \beta \text{ is alternating} \\
&= (\det A)\beta(v_1, \ldots, v_n).
\end{aligned}$$

It follows immediately that as ordered bases $(v_1, \ldots, v_n), (u_1, \ldots, u_n)$, $\beta(u_1, \ldots, u_n)$ and $\beta(v_1, \ldots, v_n)$ have the same sign if and only if $\det A > 0$.

We say that the $n$-covector $\beta$ *determines / specifies* the orientation $(v_1, \ldots, v_n)$ if $\beta(v_1, \ldots, v_n) >$

0. The previous lemma asserts that this is well-defined. Moreover, we see that two $n$-covectors $\beta, \beta'$ on $V$ determine the same orientation if and only if $\beta = a\beta'$ for some $a > 0$. We define an equivalence relation on the non-zero $n$-covectors on $V$ by setting $\beta \sim \beta'$ if they differ by a positive constant. Thus we alternatively describe an orientation of $V$ by an equivalence class of non-zero $n$-covectors.

A linear isomorphism $\bigwedge^n(V^\vee) \simeq \mathbb{R}$ identifies the set of non-zero $n$-covectors with $\mathbb{R} - \{0\}$ with two connected components, each of which determines an orientation of $V$.

> **Example 6.1.2**
> Let $e_1, e_2$ be the standard basis for $\mathbb{R}^2$ and $\alpha^1, \alpha^2$ its dual basis. Then the 2-covector $\alpha^1 \wedge \alpha^2$ determines the counterclockwise orientation since
>
> $$(\alpha^1 \wedge \alpha^2)(e_1, e_2) = 1 > 0.$$

> **Example 6.1.3**
> Let $\partial/\partial x|_p, \partial/\partial y|_p$ be the standard basis for $T_p\mathbb{R}^2$ and $(dx)_p, (dy)_p$ its dual basis. Then $(dx \wedge dy)_p$ determines the counterclockwise orientation of $T_p\mathbb{R}^2$.

### 6.1.3 Orientations on a Manifold

To orient a manifold $M$, we orient the tangent space at each point in $M$ in a "coherent" way.

Recall that a *frame* on an oepn set $U \subseteq M$ is an $n$-tuple of (possibly discontinuous) vector fields on $U$ such that at every $p \in U$, the $n$-tuple $(X_{1,p}, \dots, X_{n,p})$ is an ordered basis for $T_pM$. A *global frame* is a frame defined on all of $M$, while a *local frame* about $p \in M$ is a frame defined on some neighborhood of $p$.

We introduce an equivalence relation for frames on $U$: $(X_1, \dots, X_n), (Y_1, \dots, Y_n)$ are equivalent if and only if the unique change of basis matrix has positive determinant at every $p \in U$.

> **Definition 6.1.2 (Pointwise Orientation)**
> A *pointwise orientation* on a manifold $M$ assigns to each $p \in M$ an orientation $\mu_p$ of $T_pM$.

In terms of frames, a pointwise orientation on $M$ is an equivalence class of (possibly discontinuous) global frames on $M$.

**Definition 6.1.3 (Continuous Pointwise Orientation)**
We say that a pointwise orientation $\mu$ on $M$ is *continuous at* $p \in M$ if $p$ has a neighborhood $U$ on which $\mu$ is represented by a *continuous frame*, ie there exists continuous vector fields $Y_1, \dots, Y_n$ on $U$ such that $\mu_q = [(Y_{1,q}, \dots, Y_{n,q})]$ for all $q \in U$.

A continuous pointwise orientation is called an *orientation* on $M$. A manifold is said to be *orientable* if it has an orientation. A manifold together with an orientation is said to be *oriented*.

**Example 6.1.4**
$\mathbb{R}^n$ is oriented with orientation given by the continuous global frame

$$(\partial/\partial r^1, \dots, \partial/\partial r^n).$$

**Example 6.1.5 (Open Möbius Band)**
Let $R$ denote the rectangle

$$R := \{(x, y) \in \mathbb{R}^2 : x \in [0, 1], y \in (-1, 1)\}.$$

The open Möbius band $M$ is the quotient of the rectangle $R$ by the equivalence relation generated by $(0, y) \sim (1, -y)$. The interior of $R$ is the open rectangle

$$U := \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), y \in (-1, 1)\}.$$

Suppose towards a contradiction that $M$ is orientable. An orientation on $M$ restricts to an orientation on $U$. Without loss of generality, assume the orientation is given by $e_1, e_2$. By continuity, the orientations at $(0, 0), (1, 0)$ are both given by $e_1, e_2$. But under the identification, the ordered basis $e_1, e_2$ at $(1, 0)$ maps to $e_1, -e_2$ at $(0, 0)$, a contradiction.

**Proposition 6.1.6**
A connected orientable manifold $M$ has exactly two orientations.

Recall that a section of a tangent bundle is continuous (smooth) if and only if its coefficients with respect to a continuous (smooth) frame are continuous (smooth) functions on $U$.

**Proof**
Let $\mu, \nu$ be two orientations on $M$. At any $p \in M$, $\mu_p, \nu_p$ are orientations of $T_pM$. Thus they are either the same or are opposites. Define the function $f : M \to \{\pm 1\}$ by

$$f(p) := \begin{cases} 1, & \mu_p = \nu_p, \\ -1, & \mu_p = -\nu_p. \end{cases}$$

Fix $p \in M$. By continuity, there is a connected neighborhood $U \ni p$ on which $\mu = [(X_1, \dots, X_n)]$ and $\nu = [(Y_1, \dots, Y_n)]$ for some continuous vector fields $X_i, Y_j$ on $U$. Let $A = [a^i_j] : U \to \mathrm{GL}(n, \mathbb{R})$ be the change of basis matrix so that

$$Y_j = \sum_i a^i_j X_i.$$

The entries $a^i_j$ are continuous functions so that the determinant $\det A : U \to \mathbb{R}^\times$ is also continuous.

By the intermediate value theorem, the continuous nowhere-vanishing functions $\det A$ on the connected set $U$ is everywhere positive or everywhere negative, as $\mathbb{R}^\times$ has two connected components. Hence $\mu = \nu$ or $\mu = -v$ on $U$. Thus $f$ is locally constant. But a locally constant function on a connected set is constant, hence $\mu = \nu$ or $\mu = -\nu$ on all of $M$.

### 6.1.4 Orientations & Differential Forms

In practice, it is easier to manipulate the nowhere-vanishing top forms that specify a pointwise orientation. We aim to show that the continuity condition on a pointwise orientation translates to a smooth condition on nowhere-vanishing top forms.

**Lemma 6.1.7**
A pointwise orientation $\mu = [(X_1, \dots, X_n)]$ on a manifold $M$ is continuous if and only if each $p \in M$ has a coordinate neighborhood $(U, x^1, \dots, x^n)$ on which the function $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \dots, X_n)$ is everywhere positive.

**Proof**
$(\implies)$ Suppose the pointwise orientation $\mu$ is continuous. By definition, every $p \in M$ has a neighborhood $W$ on which $\mu$ is represented by a continuous frame $(Y_1, \dots, Y_n)$.

Choose a connected coordinate neighborhood $(U, x^1, \dots, x^n)$ of $p$ contained in $W$ and for simplicity write $\partial_i := \partial/\partial x^i$. Then $Y_j = \sum_i b^i_j \partial_i$ for a continuous matrix-valued function $[b^i_j] : U \to \mathrm{GL}(n, \mathbb{R})$. By a previous lemma,

$$(dx^1 \wedge \cdots \wedge dx^n)(Y_1, \dots, Y_n) = (\det\big[b^i_j\big])(dx^1 \wedge \cdots \wedge dx^n)(\partial_1, \dots, \partial_n) = \det\big[b^i_j\big] \neq 0.$$

As a continuous nowhere-vanishing real-valued function on a connected set, $(dx^1 \wedge \cdots \wedge dx^n)(Y_1, \dots, Y_n)$ is everywhere positive or everywhere negative on $U$. By consider $\tilde{x}^1 = -x^1$ if necessary, we assume it is everywhere positive on $U$.

Since $\mu = [(X_1, \dots, X_n)] = [(Y_1, \dots, Y_n)]$ on $U$, the change of basis matrix $C = [c^i_j]$ such that $X_j = \sum_i c^i_j Y_i$ has positive determinant. Applying the previous lemma once more yields that on $U$,

$$(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) = (\det C)(dx^1 \wedge \cdots \wedge dx^n)(Y_1, \ldots, Y_n) > 0.$$

$(\Longleftarrow)$ Fix $p \in M$ and suppose that on its neighborhood chart $(U, x^1, \ldots, x^n)$, the function $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) > 0$ over all $U$.

By shrinking $U$ if necessary, we have a local representation $X_j = \sum_j a^i_j \partial_i$. Thus

$$0 < (dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) = (\det[a^i_j])(dx^1 \wedge \cdots \wedge dx^n)(\partial_1, \ldots, \partial_n) = \det[a^i_j].$$

Thus on $U$, $[(X_1, \ldots, X_n)] = [(\partial_1, \ldots, \partial_n)]$ by definition and the pointwise orientation $\mu$ is continuous at $p$.

**Theorem 6.1.8**
An $n$-manifold $M$ is orientable if and only if there exists a smooth nowhere-vanishing $n$-form on $M$.

**Proof**
$(\Longrightarrow)$ Let $[(X_1, \ldots, X_n)]$ be an orientation on $M$. The previous lemma assures that each $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which

$$(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) > 0.$$

Let $\{(U_\alpha, x^1_\alpha, \ldots, x^n_\alpha)\}_\alpha$ be a collection of these open charts voering $M$, and $\{\rho_\alpha\}_\alpha$ a smooth partition of unity subordinate to the open cover $\{U_\alpha\}$. Being a locally finite sum, the $n$-form $\omega = \sum_\alpha \rho_\alpha dx^1_\alpha \wedge \ldots dx^n_\alpha$ is well defined and smooth on $M$. For any $p \in M$, since $\rho_\alpha(p) \geq 0$ for all $\alpha$ and there is at least one $\alpha$ for which it is positive,

$$\omega_p(X_{1,p}, \ldots, X_{n,p}) = \sum_\alpha \rho_\alpha(p)(dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha)_p(X_{1,p}, \ldots, X_{n,p}) > 0.$$

Thus $\omega$ is a smooth nowhere-vanishing $n$-form on $M$.

$(\Longleftarrow)$ Suppose $\omega$ is a nowhere-vanishing $n$-form on $M$. At each $p \in M$, choose an ordered basis $(X_{1,p}, \ldots, X_{n,p})$ for $T_pM$ such that $\omega_p(X_{1,p}, \ldots, X_{n,p}) > 0$. We show that at every point, there is a coordinate neighborhood on which $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n)$ is everywhere positive. The previous lemma concludes the proof.

Fix $p \in M$ and let $(U, x^1, \ldots, x^n)$ be a connected coordinate neighborhood of $p$. Then on $U$, $\omega = f dx^1 \wedge \cdots \wedge dx^n$ for a smooth nowhere-vanishing function $f$. Being continuous and nowhere vanishing on a connected set, $f$ is everywhere positive or negative on $U$. By taking $\tilde{x}^1 = -x^1$ if necessary, we may assume $f > 0$ on $U$. Then on $U$, $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) > 0$ as desired.

It can be shown that the unit sphere $S^2 \subseteq \mathbb{R}^3$ is orientation. A classical theorem from algebraic topology states that a continuous vector field on an even-dimensional sphere must vanish somewhere. Thus although the sphere $S^2$ has a continuous pointwise orientation, any global frame that represents the orientation is necessarily discontinuous.

If $\omega, \omega'$ are nowhere-vanishing smooth $n$-forms on an $n$-manifold, then $\omega = f\omega'$ for some nowhere-vanishing function $f$ on $M$. Locally on a chart $(U, x^1, \ldots, x^n)$, $\omega = h dx^1 \wedge \cdots \wedge dx^n$ and $\omega' = g dx^1 \wedge \cdots \wedge dx^n$, where $h, g$ are smooth nowhere-vanishing functions on $U$. Thus $f = h/g$ is also a smooth nowhere vanishing function on $U$. Since $U$ is an arbitrary chart, $f$ is smooth and nowhere vanishing function on $M$. On a *connected* manifold $M$, such a function is either everywhere positive or everywhere negative. Thus the nowhere-vanishing smooth $n$-forms on a connected orientable manifold $M$ are partitioned into two equivalence classes by the equivalence relation

$$\omega \sim \omega' \iff \omega = f\omega'$$

with $f > 0$.

To each orientation $\mu = [(X_1, \ldots, X_n)]$ on a connected orientable manifold $M$, we associate the equivalence class of smooth nowhere-vanishing $n$-forms $\omega$ on $M$ such that $\omega(X_1, \ldots, X_n) > 0$. Such an $\omega$ exists by theorem above. If $\mu \mapsto [\omega]$, then $-\mu \mapsto [-\omega]$. On a connected orientable manifold, this yields a bijective correspondance

$$\{\text{orientations on } M\} \leftrightarrow \{\text{equivalence classes of smooth nowhere-vanishing } n\text{-forms on } M\},$$

where each side is a set of two elements. By considering one conected component at a time, we see that the bijection still holds for an arbitrary orientable manifold, with each component having two possible orientations and two equivalence classes of smooth nowhere-vanishing $n$-forms. If $\omega$ is a smooth nowhere-vanishing $n$-form such that $\omega(X_1, \ldots, X_n) > 0$, we say that $\omega$ *determines* of *specifies* the orientation $[(X_1, \ldots, X_n)]$ and we call $\omega$ an *orientation form* on $M$. An oriented manifold can be described by a pair $(M, [\omega])$, where $[\omega]$ is the equivalence class of an orientation form on $M$. However, we typically just write $M$ if the orientation is clear from context. For example, $\mathbb{R}^n$ is oriented by $dx^1 \wedge \cdots \wedge dx^n$ unless otherwise specified.

**Remark 6.1.9 (Orientations on Zero-Dimensional Manifolds)** A connected manifold of dimension 0 is a point. The equivalence class of nowhere-vanishing 0-forms on a point is either $[-1]$ or $[+1]$. Hence a connected zero-dimensional manifold is always orientable with its two orientations specified by $\pm 1$.

a general zero-dimensional manifold $M$ is a countable discrete set of points and an orientatble is given by a function that assigns to each point either 1 or $-1$.

A diffeomorphism $F : (N, [\omega_N]) \to (M, \omega_M)$ of oriented manifolds is said to be *orientation-preserving* if $[F^*\omega_M] = [\omega_N]$. It is *orientation-reversing* if $[F^*\omega_M] = [-\omega_N]$.

**Proposition 6.1.10**
Let $U, V \subseteq \mathbb{R}^n$ be open, both with the standard orientation inherited from $\mathbb{R}^n$. A diffeomorphism $F : U \to V$ is orientation-preserving if and only if the Jacobian determinant $\det[\partial F^i/\partial x^j]$ is everywhere positive on $U$.

**Proof**
Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ be the standard coordinates on $U, V \subseteq \mathbb{R}^n$. By computation,

$$\begin{aligned} F^*(dy^1 \wedge \ldots dy^n) &= d(F^*y^1) \wedge \cdots \wedge d(F^*y^n) \\ &= d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\ &= dF^1 \wedge \cdots \wedge dF^n \\ &= \det\left[\frac{\partial F^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus $F$ is orientation-preserving if and only if $\det[\partial F^i/\partial x^j]$ is everywhere positive on $U$.

### 6.1.5 Orientations & Atlases

Using the characterization of an orientation-preserving diffoemorphism by the sign of its Jacobian determinant, we can describe the orientability of manifolds in terms of atlases.

**Definition 6.1.4 (Oriented Atlas)**
An atlas on $M$ is said to be *oriented* if for any two overlapping charts $(U, x^1, \ldots, x^n), (V, y^1, \ldots, y^n)$ of the atlas, the Jacobian determinant $\det[\partial y^i/\partial x^j]$ is everywhere positive on $U \cap V$.

**Theorem 6.1.11**
A manifold $M$ is orientable if and only if it has an oriented atlas.

**Proof**
($\Longrightarrow$) Let $\mu = [(X_1, \ldots, X_n)]$ be an orientation on the manifold $M$. By a prior lemma, each $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which

$$(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) > 0.$$

We claim that the collection $\mathfrak{U}$ of these charts is an oriented atlas.

If $(U, x^1, \ldots, x^n), (V, y^1, \ldots, y^n)$ are two overlapping charts from $\mathfrak{U}$, then on $U \cap V$,

$$(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n), (dy^1 \wedge \cdots \wedge dy^n)(X_1, \ldots, X_n) > 0.$$

But

$$dy^1 \wedge \cdots \wedge dy^n = (\det\big[\partial y^i/\partial x^j\big])dx^1 \wedge \cdots \wedge dx^n,$$

hence $\det[\partial y^i/\partial x^j] > 0$ on $U \cap V$.

$(\Longleftarrow)$ Suppose $\{(U, x^1, \ldots, x^n)\}$ is an oriented atlas. For each $p \in (U, x^1, \ldots, x^n)$, define $\mu_p$ to be the equivalence class of ordered bases $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ for $T_pM$. If two charts $(U, x^1, \ldots, x^n), (V, y_1, \ldots, y^n)$ in the oriented atlas contains $p$, then by the orientability of the atlas, $\det[\partial y^i/\partial x^j] > 0$, so that $(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)$ is equivalent to $(\partial/\partial y^1|_p, \ldots, \partial/\partial y^n|_p)$. Thus $\mu$ is a well-defined pointwise orientation on $M$. Moreover, it is continuous as every point has a coordinate neighborhood on which $\mu$ is represented by a continuous frame.

We say two oriented atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ on a manifold $M$ are *equivalent* if the transition functions

$$\phi_\alpha \circ \psi_\beta^{-1} : \psi_\beta(U_\alpha \cap V_\beta) \to \phi_\alpha(U_\alpha \cap V_\beta)$$

have positive Jacobian determinant for all $\alpha, \beta$.

It is not difficult to show that this is an equivalence relation on the set of oriented atlases on a manifold $M$. In the proof of the theorem above, an oriented atlas $\{(U, x^1, \ldots, x^n)\}$ on a manifold $M$ determines an orientation $p \mapsto [(\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p)]$ on $M$. Conversely, an orientation $[(X_1, \ldots, X_n)]$ on $M$ gives rise to an oriented atlas $\{(U, x^1, \ldots, x^n)\}$ on $M$ such that $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) > 0$ on $U$. It can be shown that the two induced maps

$$\{\text{equivalence classes of oriented atlases on } M\} \leftrightarrow \{\text{orientations on } M\}$$

are well-defined and inverse to each other. Thus we can also specify an orientation on an orientable manifold by an equivalence class of oriented atlases.

For an oriented manifold $M$, we denote by $-M$ the same manifold with the opposite orientation. If $\{(U, \phi)\} = \{(U, x^1, \ldots, x^n)\}$ is an oriented atlas specifying the orientation of $M$, the an oriented atlas specifying the orientation of $-M$ is $\{(U, \tilde{\phi})\} = \{(U, -x^1, x^2, \ldots, x^n)\}$.

## 6.2 Manifolds with Boundary

The prototype of a *manifold with boundary* is the *closed upper half-space*

$$\mathcal{H}^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\},$$

with the subspace topology inherited from $\mathbb{R}^n$. The points $x \in \mathcal{H}^n$ with $x^n > 0$ are called the *interior points* of $\mathcal{H}^n$, and the points with $x^n = 0$ are called the *boundary points* of $\mathcal{H}^n$. The sets of these two points are denoted $(\mathcal{H}^n)^\circ$ and $\partial(\mathcal{H}^n)$, respectively.

If $M$ is a manifold with boundary, then its boundary $\partial M$ turns out to be a manifold of dimension $n-1$ where $n = \dim(M^\circ)$. Moreover, an orientation on $M$ induces an orientation on $\partial M$.

### 6.2.1 Smooth Invariance of Domain in $\mathbb{R}^n$

In order to discuss smooth functions on a manifold with boundary, we need to extend the notion of a smooth function to allow for nonopen domains.

**Definition 6.2.1 (Smooth on an Arbitrary Set)**
Let $\subseteq \mathbb{R}^n$ be an arbitrary subset. A function $f : S \to \mathbb{R}^m$ os *smooth at* $p \in S$ if there exists a neighborhood $U \ni p$ and a smooth function $\tilde{f} : U \to \mathbb{R}^m$ such that $\tilde{f} = f$ on $U \cap S$. We say $f$ is *smooth on* $S$ if it is smooth at each point of $S$.

This definition allows us to make sense of an arbitrary subset $S \subseteq \mathbb{R}^n$ being diffeomorphic to an arbitrary subset $T \subseteq \mathbb{R}^m$.

**Proposition 6.2.1 (Smooth Functions on an Arbitrary Set)**
A function $f : S \to \mathbb{R}^m$ is smooth on $S \subseteq \mathbb{R}^n$ if and only if there is an open set $U \supseteq S$ and a smooth function $\tilde{f} : U \to \mathbb{R}^m$ such that $f = \tilde{f}|_S$.

**Proof**
Suppose $f : S \to \mathbb{R}^m$ is indeed smooth on $S \subseteq \mathbb{R}^n$. For each $p \in S$, let $U_p \ni p, \tilde{f}_p : U_p \to \mathbb{R}^m$ be the neighborhood of $p$ and smooth function on $U_p$ that restricts to $\tilde{f}$ on $U_p \cap S$. Then $\{U_p\}_{p \in S}$ is an open cover of the open submanifold

$$U := \bigcup_{p \in S} U_p.$$

Thus there is a smooth partition of unity $\{\varphi_p\}$ subordinate to $\{U_p\}$. The function $\tilde{f} : U \to \mathbb{R}$ given by

$$\tilde{f} := \sum_{p \in S} \varphi_p \tilde{f}_p$$

is the desired function.

The conversely clearly holds, concluding the proof.

The following theorem is the smooth analogue of a classical theorem from algebraic topology in the continuous category. We use it to show that interior points and boundary points are invariant under diffoemorphism of open subsets of $\mathcal{H}^n$.

**Theorem 6.2.2 (Smooth Invariance of Domain)**
Let $U \subseteq \mathbb{R}^n$ be an open subset, $S \subseteq \mathbb{R}^n$ an arbitrary subset, and $f : U \to S$ a diffeomorphism. Then $S$ is open in $\mathbb{R}^n$.

We remark that the theorem is non-trivial since a priori, we only know that $f : U \to S$ takes an open subset of $U$ (which is open in $\mathbb{R}^n$) to an open subset of $S$. Hence $f(U) \subseteq S$ is open in $S$, but not necessarily in $\mathbb{R}^n$.

**Proof**
Fix $p \in U$. Our goal is to find a neighborhood $V_{f(p)} \ni f(p)$ that is open in $\mathbb{R}^n$ and contained in $S$.

Since $f : U \to S$ is a diffeomorphism, there is an open set $V \subseteq \mathbb{R}^n$ containing $S$ and a smooth map $g : V \to \mathbb{R}^n$ such that $g|_S = f^{-1}$. Thus $g \circ f = \mathrm{Id}_U : U \to U$ is the identity map on $U$. By the chain rule,

$$g_{*,f(p)} \circ f_{*,p} = \mathrm{Id}_{T_pU} : T_pU \to T_pU$$

is the identity map on the tangent space $T_pU$. In particular, $f_{*,p}$ is necessarily injective. Since $U, V$ have the same dimension, it follows that $f_{*,p} : T_pU \to T_{f(p)}V$ is invertible. By the inverse function theorem, $f$ is locally invertible at $p$, meaning there are open neighborhoods $U_p \ni p$ in $U$ and $V_{f(p)} \ni f(p)$ in $V$ such that $f : U_p \to V_{f(p)}$ is a diffeomorphism. But

$$V_{f(p)} = f(U_p) \subseteq f(U) = S$$

with $V \subseteq \mathbb{R}^n$ open in $\mathbb{R}^n$ and $V_{f(p)} \subseteq V$ open in $V$, hence $V_{f(p)}$ is open in $\mathbb{R}^n$ as desired.

**Proposition 6.2.3**
Let $U, V$ be open subsets of the upper half-space $\mathcal{H}^n$ and $f : U \to V$ a diffeomorphism. Then $f$ maps interior points to interior points and boundary points to boundary points.

Note here we refer to openness in the relative topology on $\mathcal{H}^n$.

**Proof**
Let $p \in U$ be an interior point of $\mathcal{H}^n$. Then $p$ is contained in an open ball $B$, which is open in $\mathbb{R}^n$. By the smooth invariance of domain, $f(B)$ is open in $\mathbb{R}^n$ as well. Thus we necessarily have $f(B) \subseteq (\mathcal{H}^n)^\circ$. But then $f(p) \in f(B)$ is an interior point of $\mathcal{H}^n$.

If $p$ is a boundary point in $U \cap \partial H^n$, then $f^{-1}(f(p)) = p$ is a boundary point. But $f^{-1} : V \to U$ is a diffeomorphism, and by the contrapositive of what we just proved, $f(p)$ cannot be an interior point.

**Remark 6.2.4** Replacing Euclidean spaces by manifolds throughout this section, the identical proof steps yields the smooth invariance of domain for manifolds: If there is a diffeomorphism between an open subset $U$ of an $n$-manifold $N$ and an arbitrary subset $S$ of another $n$-manifold $M$, then $S$ must be open in $M$.

### 6.2.2 Manifolds with Boundary

In the upper half-space $\mathcal{H}^n$, we can distinguish open sets by those disjoint from the boundary, or those that intersect the boundary. Charts on a manifold are homeomorphim to only the first kind of open sets. A manifold with boundry generalizes the definition of a manifold by allowing both kinds of open sets. we say that a topological space $M$ is *locally* $\mathcal{H}^n$ if every point $p \in M$ has a neighborhood $U$ homeomorphim to an open subset of $\mathcal{H}^n$.

> **Definition 6.2.2 ($n$-Manifold with Boundary)**
> A *topological n-manifold with boundary* is a second countable, Hausdorff topological space that is locally $\mathcal{H}^n$.

Let $M$ be a topological $n$-manifold with boundary. For $n \geq 2$, a *chart* on $M$ is defined to be a pair $(U, \phi)$ consisting of an open set $U \subseteq M$ and a homeomorphism of $U$ with an open subset $\varphi(U) \subseteq \mathcal{H}^n$.

In the case of $n = 1$, a slight modification is necessary. We need to allow two local models, the *right half-line* $\mathcal{H}^1$ and the *left half-line*

$$\mathcal{L}^1 := \{x \in \mathbb{R} : x \leq 0\}.$$

A chart $(U, \varphi)$ in dimension 1 consists of an open set $U$ in $M$ and a homeomorphism $\phi$ of $U$ ewith an open subset of $\mathcal{H}^1$ or $\mathcal{L}^1$. Under this convention, if $(U, x^1, \ldots, x^n)$ is a chart of an $n$-manifold with boundary, then so is $(U, -x^1, x^2, \ldots, x^n)$ for any $n \geq 1$. A manifold with boundary has dimension at least 1, since a mnaifold of dimension 0, being a discrete set of points, necessarily has empty boundary.

A collection $\{(U, \phi)\}$ of charts is a *smooth atlas* if for any two charts $(U, \phi)$ and $(V, \psi)$, the transition map

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subseteq \mathcal{H}^n$$

is s diffeomorphism. A *smooth manifold with boundary* is a topological manifold with boundary together with a maximal smooth atlas.

A point $p \in M$ is an *interior point* if in some chart $(U, \phi)$, the point $\phi(p)$ is an interior point of $\mathcal{H}^n$. Similarly, $p$ is a boundary point of $M$ if $\phi(p)$ is a boundary point of $\mathcal{H}^n$. These concepts are well-defined independent of the choice of charts by the smooth invariance of domain. Indeed, consider any other chart $(V, \psi)$. Then $\psi \circ \phi^{-1}$ sends $\phi(p) \mapsto \psi(p)$, and $\phi(p), \psi(p)$ are either both interior points or both boundary points. The set of boundary points of $M$ is denoted $\partial M$.

Most of the concepts introduced for a manifold extend word for word to a manifold with boundary, with the only difference being that a chart can be either of two types. For example, a function $f : M \to \mathbb{R}$ is smooth at a boundary point $p \in \partial M$ if there is a chart $(U, \phi)$ about $p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p) \in \mathcal{H}^n$. This in turn translates to $f \circ \phi^{-1}$ having a smooth extension to a neighborhood of $\phi(p) \in \mathbb{R}^n$.

We may be used to other notions of interior and boundary from point-set topology, defined for a subset $A$ of a topological space $S$. A point $p \in S$ is said to be an *interior point of $A$* if there is an open subset $U \subseteq S$ such that

$$p \in U \subseteq A.$$

the point $p \in S$ is an *exterior point of $A$* if there is an open subset $U$ of $S$ such that

$$p \in U \subseteq S - A.$$

Finally, $p \in S$ is a *boundary point of $A$* if every neighborhood of $p$ contains both a point in $A$ and a point not in $A$. We denote by $\text{int}(A), \text{ext}(A), \text{bd}(A)$ the sets of interior, exterior, ad boundary points respectively of $A$ in $S$. Clearly the topological space $S$ is the disjoint union

$$S = \text{int}(A) \sqcup \text{ext}(A) \sqcup \text{bd}(A).$$

In the case the subset $A \subseteq S$ is a manifold with boundary, we call $\text{int}(A)$ the *topological interior* and $\text{bd}(A)$ the *topological boundary*, to distinguish them from the *manifold interior* $A^\circ$ and the *manifold boundary* $\partial A$. Nnote that the topological interior and the topological boundary of a set depends on an ambient space, while the manifold interior and manifold boundary are intrinsic.

> **Example 6.2.5 (Toplogical vs Manifold Boundary)**
> Let $A$ be the open unit disc in $\mathbb{R}^n$. Then $\text{bd}(A) = S^{n-1}$ but $\partial A = \varnothing$. If $B$ is the closed unit ball in $\mathbb{R}^n$, then $\text{bd}(B) = \partial B = S^{n-1}$.

> **Example 6.2.6 (Topological vs Manifold Interior)**
> Let $S$ be the upper half-plane $\mathcal{H}^2$ and $D$ the subset
>
> $$D := \{(x, y) \in \mathcal{H}^2 : y \leq 1\}.$$
>
> The topological interior of $D$ is the set
>
> $$\text{int}(D) = \{(x, y) \in \mathcal{H}^2 : y \in [0, 1)\},$$
>
> while the manifold interior of $D$ is the set
>
> $$D^\circ = \{(x, y) \in \mathcal{H}^2 : y \in (0, 1)\}.$$

To indicate the dependence of the topological interior of a set $A$ on its ambient space $S$, we may denote it by $\text{int}_S(A)$. In the example above,

$$\text{int}_{\mathcal{H}^2}(D) \neq \text{int}_{\mathbb{R}^2}(D) = D^\circ.$$

### 6.2.3 The Boundary of a Manifold with Boundary

Let $M$ be a manifold of dimension $n$ with boundary $\partial M$. If $(U, \phi)$ is a chart on $M$, we denote by $\phi' = \phi|_{U \cap \partial M}$ the restriction of the coordinate map $\phi$ to the boundary. Since $\phi$ maps boundary points to boundary points,

$$\phi' : U \cap \partial M \to \partial \mathcal{H}^n = \mathbb{R}^{n-1}.$$

Moreover, if $(U, \phi)$ and $(V, \Psi)$ are two charts on $M$, then

$$\psi' \circ (\phi')^{-1} : \phi'(U \cap V \cap \partial M) \to \psi'(U \cap V \cap \partial M)$$

is smooth. Thus an atlas $\{(U_\alpha, \phi_\alpha)\}_\alpha$ for $M$ induces an atlas $\{(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M})\}$ for $\partial M$, making $\partial M$ into a manifold of dimension $n - 1$ without boundary.

### 6.2.4 Tangent Vectors, Differential Forms, and Orientations

Let $M$ be a manifold with boundary and $p \in \partial M$. As before, two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on neighborhoods $U, V \ni p$ in $M$ are said to be *equivalent* if they agree on some neighborhood $W \in p$ contained in $U \cap V$. A *germ* of smooth functions at $p$ is an equivalence class of such functions. Along with the usual addition, multiplicaiton, and scalar multiplication of terms, the set $C_p^\infty(M)$ of germs of smooth functions at $p$ is an $\mathbb{R}$-algebra. The *tangent space* $T_pM$ at $p$ is then defined to be the vector space of all point-derivations on the algebra $C_p^\infty(M)$.

for instance, for $p \in \partial \mathcal{H}^2$, $\partial/\partial x|_p$ and $\partial/\partial y|_p$ are both derivations on $C_p^\infty(\mathcal{H}^2)$. The tangent space $T_p(\mathcal{H}^2)$ is represented by a 2-dimensional vector space with the origin at $p$. Since $\partial/\partial y|_p$ is a tangent vector to $\mathcal{H}^2$ at $p$, its negative $-\partial/\partial y|_p$ is also a tangent vector at $p$, although there is not curve through $p$ in $\mathcal{H}^2$ with initial velocity $-\partial/\partial y|_p$.

As before, the *cotangent space* $T_p^*M$ is defined to be the dual of the tangent space

$$T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R}).$$

*Differential k-forms* are also as before, as sections of the vector bundle $\bigwedge^k T^*M$. A differential $k$-form is smooth if it is smooth as a section of the vector bundle $\bigwedge^k T^*M$. For example, $dx \wedge dy$ is a smooth 2-form on $\mathcal{H}^2$.

An *orientation* on an $n$-manifold $M$ with boundary is again a continuous pointwise orientation on $M$. The previous discussion on orientations goes through for manifolds with boundary. Thus the orientability of a manifold with boundary is equivalent to the existence of a smooth nowhere-vanishing top form and to the existence of an oriented atlas. In one of the proofs in the discussion about orientations, it was necessarily to replace the chart $(U, x^1, \ldots, x^n)$ with the chart $(U, -x^1, x^2, \ldots, x^n)$. This is not possible for $n = 1$ if we did not allow the left half-line $\mathcal{L}^1$ as a local model in the definition of a chart on a 1-dimensional manifold with boundary.

**Example 6.2.7**
The closed interval $[0,1]$ is a smooth manifold with boundary. With $d/dx$ as a continuous pointwise orientation, $[0,1]$ is an oriented manifold with boundary. It has an oriented atlas with two charts $(U_1,\phi_1),(U_2,\phi_2)$ where $U_1=[0,1),\phi_1(x)=x$ and $U_2=(0,1],\phi_2(x)=x-1$. Note that $\phi_2$ maps to $\mathcal{L}^1$.

### 6.2.5 Outward-Pointing Vector Fields

**Definition 6.2.3 (Inward-Pointing)**
Let $M$ be a manifold with boundary and $p\in\partial M$. We say that a tangent vector $X_p\in T_pM$ is *inward-pointing* if $X_p\notin T_p(\partial M)$ and there is some $\varepsilon>0$ and a curve $c:[0,\varepsilon)\to M$ such that $c(0)=p$, $c(0,\varepsilon)\subseteq M^\circ$, and $c'(0)=X_p$.

A tangent vector $T_p\in T_pM$ is *outward-pointing* if $-X_p$ is inward-pointing.

**Example 6.2.8**
On the upper half-plane $\mathcal{H}^2$, the vector $\partial/\partial y|_p$ is inward-pointing and the vector $-\partial/\partial y|_p$ is outward-pointing at any $p$ on the $x$-axis.

**Definition 6.2.4 (Vector Field Along the Boundary)**
A vector field *along* $\partial M$ is a function $X$ that assigns to each $p\in\partial M$ a vector $X_p\in T_pM$ (as opposed to $T_p(\partial M)$.

In a coordinate neighborhood $(U,x^1,\ldots,x^n)$ of $p\in M$, any such vector field $X$ can be written as a linear combination

$$X_q=\sum_i a^i(q)\,\frac{\partial}{\partial x^i}\bigg|_q$$

for $q\in\partial M$. A vector field $X$ along $\partial M$ is said to be *smooth at* $p\in M$ if there is a coordinate neighborhood of $p$ for which the functions $a^i$ on $\partial M$ are smooth at $p$. Furthermore, $X$ is said to be *smooth* if it is smooth at every $p$. In terms of local coordinates, it can be shown that a tangent vector $X_p$ is outward-pointing if and only if $a^n(p)<0$.

**Proposition 6.2.9**
On any manifold $M$ with boundary $\partial M$, there is a smooth outward-pointing vector field along $\partial M$.

**Proof (Sketch)**
Cover $\partial M$ with coordinate open sets $(U_\alpha,x^1_\alpha,\ldots,x^n_\alpha)$ in $M$. On each $U_\alpha$, the vector field $X_\alpha=-\partial/\partial x^n_\alpha$ along $U_\alpha\cap\partial M$ is smooth and outward-pointing. Choose a partition of

unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha \cap \partial M\}_{\alpha\in A}$. Then one can check that $X := \sum_\alpha \rho_\alpha X_\alpha$ is a smooth outward-pointing vector field along $\partial M$.

### 6.2.6 Boundary Orientation

Our goal now is to show that the boundary of an orientable manifold $M$ with boundary is an orientable manifold without boundary. We will designate one of the orientations on the boundary as the boundary orientation. It is easily described in terms of an orientation form or of a pointwise orientation on $\partial M$.

Recall that the contraction $\iota_X\omega$ of the $k$-form by $X$ is the $(k-1)$-form given by

$$(\iota_X\omega)_p(v_2, \ldots, v_k) := \iota_{X_p}\omega_p(v_2, \ldots, v_k) := \omega_p(X_p, v_2, \ldots, v_k).$$

**Proposition 6.2.10**
Let $M$ be an oriented $n$-manifold with boundary. If $\omega$ is an orientation form on $M$ and $X$ is a smooth outward-pointing vector field on $\partial M$, then $\iota_X\omega$ is a smooth nowhere-vanishing $(n-1)$-form on $\partial M$. Hence, $\partial M$ is orientable.

**Proof**
Since $\omega$ and $X$ are both smooth on $\partial M$, so is the contraction $\iota_X\omega$. We argue by contradiction that $\iota_X\omega$ must be nowhere-vanishing on $\partial M$.

Suppose $\iota_X\omega$ vanishes at some $p \in \partial M$. This means that $(\iota_X\omega)_p(v_1, \ldots, v_{n-1}) = 0$ for all $v_1, \ldots, v_{n-1} \in T_p(\partial M)$. Let $e_1, \ldots, e_{n-1}$ be a basis for $T_p(\partial M)$. Then $X_p, e_1, \ldots, e_{n-1}$ is a basis for $T_pM$ as $X_p \notin \operatorname{span}\{e_1, \ldots, e_{n-1}\}$ and so

$$\omega_p(X_p, e_1, \ldots, e_{n-1}) = (\iota_X\omega)_p(e_1, \ldots, e_{n-1}) = 0.$$

This implies $\omega_p \equiv 0$ on $T_pM$, which is the desired contradiction.

In the notation of preceding proposition, we define the *boundary orientation* on $\partial M$ to be the orientation with orientation form $\iota_X\omega$. It can be checked that this is independent of the choice of the orientation form $\omega$ and of the outward-pointing vector field $X$.

**Proposition 6.2.11**
Suppose $M$ is an oriented $n$-manifold with boundary. Let $p$ be a point of the boundary $\partial M$ and $X_p$ an outward-pointing tangent vector in $T_pM$. An ordered basis $(v_1, \ldots, v_{n-1})$ for $T_p(\partial M)$ represents the boundary orientation at $p$ if and only if the ordered basis $(X_p, v_1, \ldots, v_{n-1})$ for $T_pM$ represents the orientation on $M$ at $p$.

**Proof**
The proof consists of unwrapping the definitions. For $p \in \partial M$, let $(v_1, \ldots, v_{n-1})$ be an

ordered basis for the tangent space $T_p(\partial M)$. Then

$$\begin{aligned}
&(v_1, \ldots, v_{n-1}) \text{ represents the boundary orientation on } \partial M \text{ at } p \\
\iff &(\iota_{X_p}\omega_p)(v_1, \ldots, v_{n-1}) > 0 \\
\iff &\omega_p(X_p, v_1, \ldots, v_{n-1}) > 0 \\
\iff &(X_p, v_1, \ldots, v_{n-1}) \text{ represents the orientation on } M \text{ at } p.
\end{aligned}$$

**Example 6.2.12 (The Boundary Orientation on $\partial\mathcal{H}^n$)**
An orientation form for the standard orientation on the upper half-space $\mathcal{H}^n$ is $\omega = dx^1 \wedge \cdots \wedge dx^n$. A smooth outward-pointing vector field on $\partial H^n$ is $-\partial/\partial x^n$. By definition, an orientation form for the boundary orientation on $\partial H^n$ is given by the contraction

$$\begin{aligned}
\iota_{-\partial/\partial x^n}\omega &= -\iota_{\partial/\partial x^n}(dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dx^n) \\
&= -(-1)^{n-1} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge \iota_{\partial/\partial x^n}(dx^n) \\
&= (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}.
\end{aligned}$$

Thus the boundary orientation on $\partial\mathcal{H}^1 = \{0\}$ is given by $-1$, the boundary orientation on $\partial\mathcal{H}^2$ is given by $dx^1$, and the boundary orientation on $\partial\mathcal{H}^3$ is given by $-dx^1 \wedge dx^2$.

**Example 6.2.13**
The closed interval $[a, b] \subseteq \mathbb{R}$ with coordinate $x$ has a standard orientation given by the vector field $d/dx$, with orientation form $dx$. At the right endpoint $b$, an outward vector is $d/dx$. Hence the boundary orientation at $b$ is given by $\iota_{d/dx}(dx) = 1$. Similarly, the boundary orientation at the left endpoint $a$ is given by $\iota_{-d/dx}(dx) = -1$.

**Example 6.2.14**
Suppose $c : [a, b] \to M$ is a smooth immersion whose image is a 1-dimensional manifold $C$ with boundary. An orientation on $[a, b]$ induces an orientation on $C$ via the differential $c_{*,p} : T_p[a, b] \to T_{c(p)}C$ at each $p \in [a, b]$. In a situation like this, we give $C$ the orientation induced from the standard orientation on $[a, b]$. Thus the boundary orientation on the boundary of $C$ is given by $+1$ at the endpoint $c(b)$ and $-1$ at the initial point $c(a)$.

## 6.3 Integration on Manifolds

We first recall Riemann integration for a function over a closed rectangle in Euclidean space. By Lebesgue's theorem, this theory can be extended to integrals over bounded subsets of $\mathbb{R}^n$ whose boundary has measure zero.

The integral of an $n$-form with compact support in an open set of $\mathbb{R}^n$ is defined to be the Riemann integral of the coefficient function. Using a partition of unity, we define the integral of an $n$-form with compact support on a manifold by writing the form as a sum of forms, each with compact support in a coordinate chart. We then prove the general Stokes theorem for an oriented manifold and show how it generalizes the fundamental theorem for line integrals as well as Green's theorem from calculus.

### 6.3.1 The Riemann Integral of a Function on $\mathbb{R}^n$

We assume familiarity with Riemann integration in $\mathbb{R}^n$ and only briefly summarize the Riemann integral of a bounded function over a bounded set in $\mathbb{R}^n$.

A *closed rectangle in* $\mathbb{R}^n$ is a Cartesian product $R = [a^1, b^1] \times \cdots \times [a^n, b^n]$ of closed intervals in $\mathbb{R}$. The *volume* $\mathrm{vol}(R)$ of the closed rectangle $R$ is defined to be

$$\mathrm{vol}(R) := \prod_{i=1}^{n}(b_i - a_i).$$

A *partition* of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \ldots, p_n\}$ such that

$$a = p_0 < p_1 < \cdots < p_n = b.$$

A *partition of the rectangle* $R$ is a collection $P = \{P_1, \ldots, P_n\}$ where each $P_i$ is a partition of $[a^i, b_i]$. The partition $p$ divides the rectangel $R$ into closed subrectangles, which we denote by $R_j$.

Let $f : R \to \mathbb{R}$ be a bounded function defined on a closed rectangle $R$. We define the *lower sum* and *upper sum* of $f$ with respect to the partition $p$ to be

$$L(f, P) := \sum_j (\inf_{R_j} f)\,\mathrm{vol}(R_j), \qquad U(f, P) := \sum_j (\sup_{R_j} f)\,\mathrm{vol}(R_j),$$

where each sum runs over all subrectangles of $P$. For any partition $P$, clearly $L(f, P) \leq U(f, P)$. In fact, we will soon see that for any two partitions $P, P'$ of the rectangle $R$,

$$L(f, P) \leq U(f, P').$$

A partition $P' = \{P'_1, \ldots, P'_n\}$ is a *refinement* of the partition $P = \{P_1, \ldots, P_n\}$ if $P_i \subseteq P'_i$ for all $i \in [n]$. If $P'$ is a refinement of $P$, then each subrectangle $R_j$ of $P$ is subdivided into subrectangles $R'_{jk}$ of $P'$. and it can be seen that

$$L(f, P) \leq L(f, P').$$

This is because if $R'_{jk} \subseteq R_j$, then $\inf_{R_j} f \leq \inf_{R'_{jk}} f$. Similarly, if $P'$ is a refinement of $P$, then

$$U(f, P') \leq U(f, P).$$

Any two partitions $P, P'$ of the rectangle $R$ have a common refinement $Q = \{Q_1, \ldots, Q_n\}$ with $Q_i := P_i \cup P'_i$. It follows that

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P').$$

It follows that the supremum of the lower sum $L(f, P)$ over all partitions $P$ of $R$ is less than or equal to the infimum of the upper sum $U(f, P)$ over all partitions $P$ of $R$. We define these two numbers to the *lower integral* $\underline{\int}_R f$ and the *upper integral* $\overline{\int}_R f$, respectively:

$$\underline{\int}_R f := \sup_P L(f, P), \qquad \overline{\int}_R f := \inf_P L(f, P).$$

**Definition 6.3.1**
Let $R$ be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be *Riemann integrable* if

$$\underline{\int}_R f = \overline{\int}_R f.$$

In this case, the Riemann integral of $f$ is this common value, denoted

$$\int_R f(x) dx^1 \ldots dx^n$$

where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$.

**Remark 6.3.1** When we speak of a rectangle in $\mathbb{R}^n$, we have implicitly chosen $n$ coordinate axes, with coordinates $x^1, \ldots, x^n$. Thus the definition of a Riemann integral depends on the coordinates $x^1, \ldots, x^n$.

If $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$, then the *extension of $f$ by zero* is the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ obtained from $f$ by setting it to be zero outside of $A$. Suppose $f : A \to \mathbb{R}$ is a bounded function on a bounded set $A$ in $\mathbb{R}^n$. We can enclose $A$ in a closed rectangle $R$ and define the Riemann integral of $f$ over $A$ to be

$$\int_A f(x) dx^1 \ldots dx^n = \int_R \tilde{f}(x) dx^1 \ldots dx^n$$

if the RHS exists. In this way, we can deal with the integral of a bounded function whose domain is an arbitrary bounded set in $\mathbb{R}^n$.

The *volume* $\mathrm{vol}(A)$ of a subset $A \subseteq \mathbb{R}^n$ is defined to be the integral $\int_A 1 dx^1 \ldots dx^n$ provided the integral exists. This concept generalizes the volume of a closed rectangle we previously defined.

### 6.3.2 Integrability Conditions

In this section, we describe some conditions under which a function defined on an open subset $\mathbb{R}^n$ is Riemann integrable.

Recall that a set $A \subseteq \mathbb{R}^n$ is said to have *(Lebesgue) measure zero* if for every $\varepsilon > 0$, there is a countable cover $\{R_i\}_{i\geq 1}$ of $A$ by closed rectangles $R_i$ such that $\sum_{i\geq 1} \text{vol}(R_i) < \varepsilon$.

The most useful criterion of Riemann integrability is due to Lebesgue:

**Theorem 6.3.2 (Lebesgue's Criterion)**
A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subseteq \mathbb{R}^n$ is Riemann integrable if and only if the set set of discontinuities of the extended function $\tilde{f}$ has measure zero.

**Proposition 6.3.3**
If a continuous function $f : U \to \mathbb{R}$ defined on an open subset $U \subseteq \mathbb{R}^n$ has compact support, then $f$ is Riemann integrable on $U$.

**Proof**
Being continuous on a compact set, the function $f$ is bounded. Being compact, the set $\operatorname{supp} f$ is closed and bounded in $\mathbb{R}^n$. We claim that the extension $\tilde{f}$ is continuous.

Since $\tilde{f}$ agrees with $f$ on $U$, the extended function $\tilde{f}$ is continuous on $U$. It remains to show $\tilde{f}$ is continuous on the complement. If $p \notin U$, then $p \notin \operatorname{supp} f$. Since $\operatorname{supp} f$ is a closed subset of $\mathbb{R}^n$, there is an open ball $B \ni p$ disjoint from $\operatorname{supp} f$. $\tilde{\equiv} f$ on $B$ implying that $\tilde{f}$ is continuous at $p \notin U$. By Lebesgue's theorem, $f$ is Riemann integrable on $U$.

**Remark 6.3.4** The support of a real-valued function is the closure *in its domain* of the subset where the function is not zero.

**Definition 6.3.2 (Domain of Integration)**
A subset $A \subseteq \mathbb{R}^n$ is called a *domain of integration* if it is bounded and its topological boundary $\text{bd}(A)$ is a set of measure zero.

Familiar figures such as triangles, rectangles, and circular disks are all domains of integration in $\mathbb{R}^2$.

**Proposition 6.3.5**
Every bounded continuous function $f$ defined on a domain of integration $A \subseteq \mathbb{R}^n$ is Riemann integrable over $A$.

**Proof**
Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of $f$ by zero. Since $f$ is continuous on $A$, the extension

is necessarily continuous at all interior points of $A$. But every exterior point has a neighborhood disjoint from $A$ on which $\tilde{f} \equiv 0$. But then the set of discontinuities is contained in $\operatorname{bd}(A)$, which has measure zero. We conclude the proof by Lebesgue's theorem.

### 6.3.3 The Integral of an $n$-Form on $\mathbb{R}^n$

Once a set of coordinates $x^1, \dots, x^n$ has been fixed on $\mathbb{R}^n$, $n$-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every $n$-form on $\mathbb{R}^n$ can be written as $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ for a unique function $f(x)$ on $\mathbb{R}^n$. In this way, the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to $n$-forms on $\mathbb{R}^n$.

**Definition 6.3.3 (Integral of $n$-Form)**
Let $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ be a smooth $n$-form on an open subset $U \subseteq \mathbb{R}^n$, with standard coordinates $x^1, \dots, x^n$. Its *integral* over a subset $A \subseteq U$ is defined to be the Riemann integral of $f(x)$:

$$\int_A \omega = \int_A f(x)dx^1 \wedge \cdots \wedge dx^n := \int_A f(x)dx^1 \dots dx^n,$$

assuming the Riemann integral exists.

Note that in this definition, we require the $n$-form to be written in the order $dx^1 \wedge \cdots \wedge dx^n$. If it is in any other order, we would need to rearrange it by the alternating property.

**Example 6.3.6**
If $f$ is a bounded continuous function defined on a domain of integration $A \subseteq \mathbb{R}^n$, then the integral $\int_A f dx^1 \wedge \cdots \wedge dx^n$ exists.

Let us see how the integral of an $n$-form $\omega = f dx^1 \wedge \cdots \wedge dx^n$ on an open subset $U \subseteq \mathbb{R}^n$ transofrm under a change of variables. A change of variables on $U$ is given by a diffeomorphism $T : V \subseteq \mathbb{R}^n \to U \subseteq \mathbb{R}^n$. Let $x^1, \dots, x^n$ be the standard coordinates on $U$ and $y^1, \dots, y^n$ the standard coordinates on $V$. Then $T^i := x^i \circ T = T^*(x_i)$ is the $i$-th component of $T$. We assume that $U, V$ are connected, and write $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$. Furthermore, denote by $J(T)$ the Jacobian matrix $[\partial T^i / \partial y^j]$. By the local formula for a wedge of differentials,

$$dT^1 \wedge \cdots \wedge dT^n = \det(J(T))dy^1 \wedge \cdots \wedge dy^n.$$

Recall also that the wedge product commutes with the wedge product:

$$F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau.$$

Hence

$$\begin{aligned}
\int_V T^*\omega &= \int_V (T^*f)T^*dx^1 \wedge \cdots \wedge T^*dx^n \\
&= \int_V (f\circ T)dT^1 \wedge \cdots \wedge dT^n && T^*d = dT^* \\
&= \int_V (f\circ T)\det(J(T))dy^1 \wedge \cdots \wedge dy^n \\
&= \int_V (f\circ T)\det(J(T))dy^1 \dots dy^n.
\end{aligned}$$

Recall that the change of variables formula from calculus (whose most intuitive proof is obtained from a measure-theoretic argument regarding the Lebesgue integral) gives

$$\int_U f dx^1 \dots dx^n = \int_V (f\circ T)|\det J(T)|dy^1 \dots dy^n$$

Hence puting the two equations above together yields

$$\int_V T^*\omega = \pm \int_U \omega,$$

depending on the sign on the Jacobian determinant.

Recall that a diffeomorphism of oriented manifolds $T: V \subseteq \mathbb{R}^n \to U \subseteq \mathbb{R}^n$ is *orientation-preserving* if $[F^*\omega_V] = [\omega_U]$, where $\omega_V, \omega_U$ are the orientation forms for $V, U$, respectively. We deduced that $T$ is oriented-preserving if and only if its Jacobian determinant $\det J(T)$ is everywhere positive on $V$. Our work above shows that the integral of a differential form is not invariant under all diffeomoprhisms of $V$ with $U$, but only under orientation-preserving diffeomorphisms.

### 6.3.4 Integral of a Differential Form over a Manifold

Integration of an $n$-form on $\mathbb{R}^n$ is not so different from integration of a function. Our approach to integration has several distinguishing features:

(i) The manifold must be oriented.

(ii) On an $n$-manifold, we can only integrate $n$-forms, not functions.

(iii) The $n$-forms must have compact support.

Let $M$ be an oriented manifold of dimension $n$, with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ giving the orientation of $M$. Denote by $\Omega_c^k(M)$ the vector space of smooth $k$-forms with compact support on $M$. Suppose $\{(U, \phi)\}$ is a chart in this atlas. If $\omega \in \Omega_c^n(U)$ is an $n$-form with compact support on $U$, then because $\phi: U \to \phi(U)$ is a diffeomorphism, $(\phi^{-1})^*\omega$ is an

$n$-form with compact support on the open subset $\phi(U) \subseteq \mathbb{R}^n$. We define the *integral* of $\omega$ on $U$ to be

$$\int_U \omega := \int_{\phi(U)} (\phi^{-1})^* \omega.$$

If $(U, \psi)$ is another chart in the oriented atlast with the same $U$, then $\phi \circ \psi^{-1} : \psi(U) \to \phi(U)$ is by definition an orientation-preserving diffeomorphism, and so

$$\int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\psi(U)} (\phi \circ \psi^{-1})^* (\phi^{-1})^* \omega = \int_{\psi(U)} (\psi^{-1})^* \omega.$$

Thus the integral $\int_U \omega$ on a chart $U$ of the atlas is well-defined, independent of the choice of coordinates on $U$. By the lienarity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega_c^n(U)$, then

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau.$$

Now let $\omega \in \Omega_c^n(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because $\omega$ has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha \omega$ are identically zero. In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a *finite* sum. Recall the elementary topological fact that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. this means that

$$\operatorname{supp}(\rho_\alpha \omega) \subseteq \operatorname{supp}(\rho_\alpha) \cap \operatorname{supp}(\omega).$$

In particular, $\operatorname{supp}(\rho_\alpha \omega)$ is a closed subset of the compact set $\operatorname{supp} \omega$ and is hence compact. Since $\rho_\alpha \omega$ is an $n$-form with compact support in the chart $U_\alpha$, its integral $\int_{U_\alpha} \rho_\alpha \omega$ is defined. Thus we can define the integral of $\omega$ over $M$ to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega.$$

For this integral to be well-defined, we must show that it is independent of the choices of oriented atlas and partition of unity. Let $\{V_\beta\}$ be another oriented atlas of $M$ specifying the same orientation and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then $\{(U_\alpha \cap V_\beta, \phi_\alpha|_{U_\alpha \cap V_\beta})\}$ and $\{(U_\alpha \cap B_\beta, \psi|_{U_\alpha \cap V_\beta})\}$ are two new atlases of $M$ specifying the orientation of $M$, and

$$\begin{aligned}
\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega &= \sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta \omega && \sum_\beta \chi_\beta = 1 \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha \chi_\beta \omega && \text{finite sums} \\
&= \sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega,
\end{aligned}$$

where the last line follows since $\text{supp}(\rho_\alpha \chi_\beta) \subseteq U_\alpha \cap V_\beta$. By symmetry, $\sum_\beta \int_{V_\beta} \chi_\beta \omega$ is equal to the same sum. Hence

$$\sum_\alpha \int_{U_\alpha} \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega,$$

as desired.

**Proposition 6.3.7**
Let $\omega$ be an $n$-form with compact support on an oriented $n$-manifold. If $-M$ denotes the same manifold with the oppposite orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$

**Proof**
By the definition of an integral, it suffices to show that for every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ and differential form $\tau \in \Omega_c^n(U)$, if $(U, \bar{\phi}) = (U, -x^1, x^2, \ldots, x^n)$ is the chart with the opposite orientation, then

$$\int_{\bar{\phi}(U)} (\bar{\phi}^{-1})^* \tau = -\int_{\phi(U)} (\phi^{-1})^* \tau.$$

Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then $x^i = r^i \circ \phi$ and $r^i = x^i \circ \phi^{-1}$. With $\bar{\phi}$, the only difference is that for $i = 1$,

$$-x^1 = r^1 \circ \bar{\phi}, \qquad r^1 = -x^1 \circ \bar{\phi}^{-1}.$$

Suppose $\tau = f dx^1 \wedge \cdots \wedge dx^n$ on $U$. Then

$$\begin{aligned}(\bar{\phi}^{-1})^* \tau &= (f \circ \bar{\phi}^{-1}) d(x^1 \circ \bar{\phi}^{-1}) \wedge d(x^1 \circ \bar{\phi}^{-1}) \wedge \cdots \wedge d(x^n \circ \bar{\phi}^{-1}) \\ &= -(f \circ \bar{\phi}^{-1}) dr^1 \wedge dr^2 \wedge \cdots \wedge dr^n.\end{aligned}$$

Similarly,

$$(\phi^{-1})^* \tau = (f \circ \phi^{-1}) dr^1 \wedge dr^2 \wedge \cdots \wedge dr^n.$$

Since $\phi \circ \bar{\phi}^{-1} : \bar{\phi}(U) \to \phi(U)$ is given by

$$(\phi \circ \bar{\phi}^{-1})(a^1, a^2, \ldots, a^n) = (-a^1, a^2, \ldots, a^n),$$

the absolute value of its Jacobian determinant is

$$|J(\phi \circ \bar{\phi}^{-1})| = |-1| = 1.$$

It follows that

$$\begin{aligned}
&\int_{\bar\phi(U)} (\bar\phi^{-1})^*\tau \\
&= -\int_{\bar\phi(U)} (f\circ\bar\phi^{-1})dr^1\dots dr^n && \text{calculations above}\\
&= -\int_{\bar\phi(U)} (f\circ\phi^{-1})\circ(\phi\circ\bar\phi^{-1})|J(\phi\circ\bar\phi^{-1})|dr^1\dots dr^n \\
&= -\int_{\phi(U)} (f\circ\phi^{-1})dr^1\dots dr^n && \text{change of variables in } \mathbb{R}^n\\
&= -\int_{\phi(U)} (\phi^{-1})^*\tau.
\end{aligned}$$

Our treatment of integration above can be extended nearly verbatim to oriented manifolds with boundary. It has the virtue of simplicity and utility in proving theorems. However, it is not practical for the actual computation of integrals. It is best to consider integrals over a parameterized set for explicit integral calculations.

**Definition 6.3.4 (Parameterized Set)**
A *parameterized set* in an oriented $n$-manifold $M$ is a subset $A \subseteq M$ together with a smooth map $F : D \to M$ from a compact domain of integration $D \subseteq \mathbb{R}^n$ to $M$ such that $A = F(D)$ and $F$ restricts to an orientation-preserving diffeomorphism from $\text{int}(D)$ to $F(\text{int}(D))$.

Note that by the smooth invariance of domain for manifolds, $F(\text{int}(D))$ is necessarily an open subset of $M$ The smooth map $F : D \to A$ is called a *parameterization* of $A$.

If $A$ is a parameterized set in $M$ with paramterization $F : D \to A$ and $\omega$ is a smooth $n$-form on $M$, not necessarily with compact support, then we define $\int_A \omega$ to be $\int_D F^*\omega$. It can be shown that the definition of $\int_A \omega$ is independent of the parameterization and in the case that $A$ is a manifold, it agrees with the earlier definition of integration over a manifold. Subdividing an oriented manifold into a union of parameterized sets can be an effective method of calculating an integral over the manifold.

**Example 6.3.8 (Integral over a Sphere)**
In spherical coordinates, $\rho$ is the distance $\sqrt{x^2+y^2+z^2}$ of the point $(x, y, z) \in \mathbb{R}^3$ to the origin, $\varphi$ is the angle that the vector $\langle x, y, z\rangle$ makes with the positive $z$-axis, and $\theta$ is the angle that the vector $\langle x, y\rangle$ in the $(x, y)$-plane makes with the positive $x$-axis. Let $\omega$ be

the 2-form on the unit sphere $S^2 \subseteq \mathbb{R}^3$ given by

$$\omega = \begin{cases} \frac{dy \wedge dz}{x}, & f \neq 0, \\ \frac{dz \wedge dx}{y}, & y \neq 0, \\ \frac{dx \wedge dy}{z}, & z \neq 0. \end{cases}$$

We with to compute $\int_{S^2} \omega$.

In Riemannian geometry, it can be shown that $\omega$ is the area form of the sphere $S^2$ with respect to the Euclidean metric. Therefore, the integral $\int_{S^2} \omega$ is the surface area of the sphere.

The sphere $S^2$ has a parametrization by spherical coordinates

$$F(\varphi, \theta) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$$

on $D := \{(\varphi, \theta) \in \mathbb{R}62 : \varphi \in [0, \pi], \theta \in [0, 2\pi]\}$. Since

$$\begin{aligned} F^*x &= \sin\varphi \cos\theta, \\ F^*y &= \sin\varphi \sin\theta, \\ F^*z &= \cos\varphi, \end{aligned}$$

we have

$$F^*dy = dF^*y = \cos\varphi \sin\theta d\varphi + \sin\varphi \cos\theta d\theta$$

and

$$F^*dz = -\sin\varphi d\varphi,$$

so for $x \neq 0$,

$$F^*\omega = \frac{F^*dy \wedge F^*dz}{F^*x} = \sin\varphi d\varphi \wedge d\theta.$$

For $y \neq 0$ and $z \neq 0$, similar calculations show that $F^*\omega$ is given by the same formula. Therefore, $F^*\omega = \sin\varphi d\varphi \wedge d\theta$ everywhere on $D$, and

$$\begin{aligned} \int_{S^2} \omega &= \int_D F^*\omega \\ &= \int_0^{2\pi} \int_0^{\pi} \sin\varphi d\varphi d\theta \\ &= 2\pi \left[-\cos\varphi\right]_0^{\pi} \\ &= 4\pi. \end{aligned}$$

### 6.3.5 Integration over a Zero-Dimensional Manifold

The discussion of integration so far implicitly assumes that the manifold $M$ has dimension $n \geq 1$. We now treat integration over a zero-dimensional manifold. A compact oriented 0-manifold $M$ is a finite collection of points, each point oriented by $+1, -1$. We write this as $M = \sum_i p_i - \sum_j q_j$. The integral of a 0-form $f : M \to \mathbb{R}$ is defined to be the sum

$$\int_M f := \sum_i f(p_i) - \sum_j f(q_j).$$

### 6.3.6 Stokes' Theorem

Let $M$ be an oriented manifold of dimension $n$ with boundary. we give its boundary $\partial M$ the boundary orientation and let $\iota : \partial M \to M$ be the inclusion map. If $\omega$ is an $(n-1)$-form on $M$, it is customary to write $\int_{\partial M} \omega$ instead of $\int_{\partial M} \iota^* \omega$.

**Theorem 6.3.9 (Stokes)**
For any smooth $(n-1)$-form $\omega$ with compact support on the oriented $n$-manifold $M$,

$$\int_M d\omega = \int_{\partial M} \omega.$$

**Proof**
Choose an atlas $\{(U_\alpha, \phi_\alpha)\}$ for $m$ in which each $U_\alpha$ is diffeormorphic to either $\mathbb{R}^n$ or $\mathcal{H}^n$ via an orientation-preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ and any half-disk containing its boundary diameter is diffeomorphic to $\mathcal{H}^n$. Let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. We showed in the preceding section that $\rho_\alpha \omega$ has compact support in $U_\alpha$.

Suppose Stokes' theorem holds for $\mathbb{R}^n$ and $\mathcal{H}^n$. Then it holds for all the charts in our atlas, which are diffeomorphic to $\mathbb{R}^n$ or $\mathcal{H}^n$. This is because we defined

$$\int_{U_\alpha} d\omega := \int_{\phi_\alpha(U)} (\phi_\alpha^{-1})^* \omega$$

with $\phi_\alpha(U) = \mathbb{R}^n$ or $\mathcal{H}^n$. Similarly,

$$\int_{\partial U_\alpha} \omega := \int_{\partial U_\alpha} \iota^* \omega := \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^* \omega$$

Note here that

$$(\partial M) \cap U_\alpha = \partial U_\alpha.$$

Therefore,

$$\begin{aligned}
\int_{\partial M}\omega &= \int_{\partial M}\sum_\alpha \rho_\alpha\omega && \sum_\alpha \rho_\alpha = 1\\
&= \sum_\alpha \int_{\partial M}\rho_\alpha\omega && \text{finite sum}\\
&= \sum_\alpha \int_{\partial U_\alpha}\rho_\alpha\omega && \operatorname{supp}(\rho_\alpha\omega)\subseteq U_\alpha\\
&= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha\omega) && \text{Stokes' for } U_\alpha\\
&= \sum_\alpha \int_{M} d(\rho_\alpha\omega) && \operatorname{supp} d(\rho_\alpha\omega)\subseteq \operatorname{supp}(\rho_\alpha\omega)\subseteq U_\alpha\\
&= \int_M d\left(\sum_\alpha \rho_\alpha\omega\right) && \text{finite sum}\\
&= \int_M d\omega.
\end{aligned}$$

Thus it suffices to prove Stokes' theorem for $\mathbb{R}^n$ and $\mathcal{H}^n$. We give a proof for $\mathcal{H}^2$ for the sake of simplicity.

Proof of Stokes' theorem in $\mathcal{H}^2$: Let $x, y$ be coordinates on $\mathcal{H}^2$. Then the standard orientation on $\mathcal{H}^2$ is given by $dx \wedge dy$, and the boundary orientation on $\partial\mathcal{H}^2$ is given by $\iota_{-\partial/\partial y}(dx \wedge dy) = dx$.

The form $\omega$ is a linear combination

$$\omega = f(x, y)dx + g(x, y)dy$$

for smooth functions $f, g$ with compact support in $\mathcal{H}^2$. Since the supports of $f, g$ are compact, we may choose a real number $a > 0$ sufficiently large so that the supports of $f, g$ are contained in the interior of the square $[-a, a] \times [0, a]$. We write $f_x, f_y$ to denote the partial derivatives of $f$ with respect to $x, y$, respectively. Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = (g_x - f_y)dx \wedge dy,$$

and

$$\begin{aligned}
\int_{\mathcal{H}^2} d\omega &= \int_{\mathcal{H}^2} g_x dxdy - \int_{\mathcal{H}^2} f_y dxdy\\
&= \int_0^a \int_{-a}^a g_x dxdy - \int_{-a}^a \int_0^a f_y dydx.
\end{aligned}$$

In the expression above,

$$\int_{-a}^{a} g_x(x,y)dx = g(x,y)\Big|_{x=-a}^{a} = 0$$

since $\text{supp}(g)$ lies in the interior of $[-a,a]\times[0,a]$. Similarly,

$$\int_{0}^{a} f_y(x,y)dy = f(x,y)\Big|_{y=0}^{a} = -f(x,0)$$

because $f(x,a)=0$. Thus the expression above reduces to

$$\int_{\mathcal{H}^2} d\omega = \int_{-a}^{a} f(x,0)dx.$$

On the other hand, $\partial\mathcal{H}^2$ is the $x$-axis and $dy=0$ on $\partial\mathcal{H}^2$. It follows from the definition of $\omega$ that

$$\int_{\partial\mathcal{H}^2} \omega = \int_{-a}^{a} f(x,0)dx.$$

### 6.3.7 Line Integrals & Green's Theorem

We now apply Stokes' theorem for manifolds to unify theorems of vector calculus on $\mathbb{R}^2, \mathbb{R}^3$. Recall the calculus notation $F\cdot dr = Pdx+Qdy+Rdz$ for a function vector field $F=\langle P,Q,R\rangle$ and coordinates $r=(x,y,z)$. As in calculus, we assume in this section that functions, vector fields, and regions of integration have sufficient smoothness or regularity properties so that all the integrals are defined.

**Theorem 6.3.10 (Fundamental Theorem for Line Integrals)**
Let $C$ be a curve in $\mathbb{R}^3$, parameterized by some $r(t)=(x(t),y(t),z(t)), t\in[a,b]$ and let $F$ be a vector field on $\mathbb{R}^3$. If $F=\text{grad}\, f$ for some scalar function $f$, then

$$\int_C F\cdot dr = f(r(b)) - f(r(a)).$$

Suppose in Stokes' theorem we take $M$ to be a curve with parameterization $r(t), t\in[a,b]$, and $\omega$ to be the function $f$ on $C$. Then

$$\int_C d\omega = \int_C df = \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \int_C \text{grad}\, f\cdot dr$$

and

$$\int_{\partial C} \omega = f\Big|_{r(a)}^{r(b)} = f(r(b)) - f(r(a)).$$

In this case Stokes' theorem specializes to the fundamental theorem for line integrals.

**Theorem 6.3.11 (Green)**
If $D$ is a plane region with boundary $\partial D$, and $P, Q$ are smooth functions on $D$, then

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

To obtain Green's theorem, let $M$ be a plane region $D$ with boundary $\partial D$ and let $\omega$ be the 1-form $Pdx + Qdy$ on $D$. Then

$$\int_{\partial D} \omega = \int_{\partial D} Pdx + Qdy$$

and

$$\begin{aligned}\int_D d\omega &= \int_D P_y dy \wedge dx + Q_x dy \wedge dy \\ &= \int_D (Q_x - P_y) dx \wedge dy \\ &= \int_D (Q_x - P_y) dxdy \\ &= \int_D (Q_x - P_y) dxdy.\end{aligned}$$

In this case, Stokes' theorem specializes to Green's theorem in the plane.