CO471: Semidefinite Optimization

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Chapter 1

Introduction

1.1 Positive Semidefinite Matrices

Definition 1.1.1 (Trace)
For $X \in \mathbb{R}^{n\times n}$, we let $\operatorname{Tr}(X)$ denote the trace of X

$$\operatorname{Tr}(X) := \sum_{i=1}^{n} X_{ii}.$$

Note that for $X, S \in \mathbb{R}^{n\times n}$, a commonly used *inner product* is

$$\begin{aligned}\langle X, S\rangle &:= \operatorname{Tr}\big(X^T S\big)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}s_{ij}\\ &= \operatorname{Tr}\big(SX^T\big).\end{aligned}$$

Note that this shows the trace is preserved under cyclic permutation.

Recall that given $X \in \mathbb{R}^{n\times n}$, the roots $\lambda_1, \dots, \lambda_n$ of the polynomial

$$\det(X - \lambda I) = 0$$

are the *eigenvalues* of X.

We denote the set of $n \times n$ symmetric matrices by

$$\mathbb{S}^n := \{X \in \mathbb{R}^{n\times n} : X^T = X\}.$$

For $X \in \mathbb{S}^n$, all eigenvalues of X are real and we usually order them

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X).$$

We can consider $\lambda : \mathbb{S}^n \to \mathbb{R}^n$ given by

$$\lambda(X)_i = \lambda_i(X).$$

For simplicty of notation we define the linear transformation Diag : $\mathbb{R}^n \to \mathbb{S}^n$ such that $\operatorname{Diag}(x)$ is the diagonal matrix whose entries are the entries of x. Moreover, we define

$$\operatorname{diag}(X) := \begin{bmatrix} x_{11} \\ x_{22} \\ \cdots \\ x_{nn} \end{bmatrix}.$$

Theorem 1.1.1 (Spectral Decomposition)
For $X \in \mathbb{S}^n$, there is some orthogonal $Q \in \mathbb{R}^{n \times n}$ such that

$$X = Q \operatorname{Diag}(\lambda(X)) Q^T.$$

Corollary 1.1.1.1
In the theorem above, the columns of Q are the eigenvectors of X.

Proof
We have

$$\begin{aligned} XQe_j &= Q \operatorname{Diag}(\lambda(X)) Q^T Q e_j \\ &= \lambda_j(X) Q e_j. \end{aligned}$$

Definition 1.1.2 (Positive Semidefinite)
$X \in \mathbb{S}^n$ is positive semidefinite if $h^T X h \geq 0$ for all $h \in \mathbb{R}^n$.

If $h^T X h > 0$ for each $0 \neq h \in \mathbb{R}^n$, we say that X is *positive definite.*

We denote the set of p.s.d. matrices by

$$\mathbb{S}^n_+.$$

Theorem 1.1.2 (Cholesky Decomposition)
Let $X \in \mathbb{S}^n$, then

(a) X is p.s.d. if and only if there is some lower triangular $B \in \mathbb{R}^{n\times n}$ such that $X = BB^T$.

(b) X is p.d. if and only if there is some non-singular lower triangular $B \in \mathbb{R}^{n\times n}$ for which $X = BB^T$.

Proposition 1.1.3
Let $X \in \mathbb{S}^n$. The following are equivalent:

(a) X is p.s.d.

(b) $\lambda(X) \geq 0$.

(c) There is some $\mu \in \mathbb{R}^n_+$ and $h^{(1)}, \ldots, h^{(n)} \in \mathbb{R}^n$ such that $X = \sum_{i=1}^n \mu_i h^{(i)}(h^{(i)})^T$.

(d) There is some $B \in \mathbb{R}^{n\times n}$ such that $X = BB^T$.

(e) For every nonempty $J \subseteq [n]$, $\det(X_j) \geq 0$, where $X_j := [x_{ij}]_{i,j\in J}$

(f) For every $S \in \mathbb{S}^n_+$, $\langle X, S\rangle \geq 0$.

1.2 Semidefinite Programming

Recall that linear programming (LP) is the problem of optimizing an affine function of finitely many real valued variables subject to finitely many linear contraints.

A "standard form" of an LP may look like the following:

$$\begin{aligned} \min c^T x \\ Ax = b \\ x \geq 0 \end{aligned}$$

Broadly speaking, semidefinite programming (SDP) is the problem of optimizing an affine function of finitely many matrix variables with real entries, subject to finitely linear constraints and some symmetry and positive semidefiniteness contraints on the matrix variables.

The power of SDP lies in the positive semidefiniteness constraints, as those can be highly nonlinear.

A possible form of an SDP is the following:

$$\begin{aligned} \inf \langle C, X\rangle & \\ \langle A_i, X\rangle = b_i & \qquad \forall i \in [m] \\ X \in \oplus_{j=1}^{k} \mathbb{S}_+^{n_j} & \end{aligned}$$

The last constraint says that X is a block diagonal matrix where each block is positive semidefinite.

Example 1.2.1
Suppose we have variables

$$v^{(i)} \in \mathbb{R}^n, i \in [n]$$

with objective functions and contraints which are affine functions of $\langle v^{(i)}, v^{(j)}\rangle$ for $i, j \in [n]$. Then we can express such a nonlinear, nonconvex optimization problem as an SDP.

Define a new matrix variable $X := VV^T \in \mathbb{S}_+^n$ where

$$V^T := \begin{bmatrix} v^{(1)} & v^{(2)} & \dots & v^{(n)} \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Then $X_{ij} = \langle v^{(i)}, v^{(j)}\rangle$ for each $i, j \in [n]$.

We can rewrite the original optimization problem using X as the variable within an SDP.

We denote the set of $n \times n$ positive definite matrices by

$$S_{++}^n.$$

Lemma 1.2.2 (Schur Complement)
Let $X \in \mathbb{S}^n$ and $T \in \mathbb{S}_{++}^m$.
Then

$$M := \begin{bmatrix} T & U^T \\ U & X \end{bmatrix} \in \mathbb{S}_+^{m+n}$$

if and only if

$$(X - UT^{-1}U^T) \in \mathbb{S}_+^n.$$

Moreover, $M \in \mathbb{S}_{++}^{m+n}$ if and only if $(X - UT^{-1}U^T) \in \mathbb{S}_{++}^n$.

Proof
Suppose $X \in \mathbb{S}^n$ and $T \in \mathbb{S}_{++}^m$. Then

$$M = \begin{bmatrix} T & U^T \\ U & X \end{bmatrix} = \begin{bmatrix} I & 0 \\ UT^{-1} & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & X - UT^{-1}U^T \end{bmatrix} \begin{bmatrix} I & T^{-1}U^T \\ 0I & \end{bmatrix}. = L \begin{bmatrix} T & 0 \\ 0 & X - UT^{-1}U^T \end{bmatrix} L^T.$$

> Notice that L is non-singular. Thus checking that the Rayleigh quotient of M is always positive (non-negative) is equivalent to checking the Rayleigh quotient of the block diagonal matrix
> $$\begin{bmatrix} T & 0 \\ 0 & X - UT^{-1}U^T \end{bmatrix}.$$
> But since we are given that $T \in \mathbb{S}^m_{++}$, this is equivalent to the condition that $X - UT^{-1}U^T$ is positive (semi)definite.

For $A, B \in \mathbb{S}^n$, we use the notation $A \succeq B$ to denote $A - B \in \mathbb{S}^n_+$ and $A \succ B$ to mean that $A - B \in \mathbb{S}^n_{++}$.

A special case of Lemma 1.2.2 with $m = 1, T = 1$ is that

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 \iff X - xx^T \succeq 0$$
$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succ 0 \iff X - xx^T \succ 0.$$

1.3 Duality

Given $C \in \mathbb{S}^n, b \in \mathbb{R}^m$ and a linear transformation $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, we define the SDP

$$\begin{aligned} \inf \langle C, X \rangle \qquad\qquad & (P) \\ \mathcal{A}(X) = b & \\ X \succeq 0 & \end{aligned}$$

as well as its dual

$$\begin{aligned} \sup b^T y \qquad\qquad & (D) \\ \mathcal{A}^*(y) + S = C & \\ S \succeq 0 & \end{aligned}$$

Recall from elementary linear algebra that $\mathcal{A}^* : \mathbb{R}^m \to \mathbb{S}^n$ is the adjoint of $\mathcal{A}$, which is the unique linear function satisfying

$$\forall X \in \mathbb{S}^n, \forall y \in \mathbb{R}^m, \langle \mathcal{A}^*(y), X \rangle = \langle y, A(X) \rangle.$$

Now, for every linear transformation $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$, there are $A_1, \dots, A_m \in \mathbb{S}^n$ such that

$$\forall i \in [m], [\mathcal{A}(X)]_i = \langle A_i, X \rangle.$$

Hence

$$\mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i.$$

We rewrite the primal and dual programs as

$$\begin{aligned} \inf \langle C, X\rangle & & (P) \\ \langle A_i, X\rangle = b_i & & \forall i \in [m] \\ X \succeq 0 & & \end{aligned}$$

$$\begin{aligned} \sup b^T y & & (D) \\ \sum_{i=1}^m y_i A_i + S = C & & \\ S \succeq 0 & & \end{aligned}$$

Theorem 1.3.1 (Weak Duality)
Let $\bar{X}$ be feasible in (P) and $(\bar{y}, \bar{S})$ be feasible in (D). Then

$$\langle C, \bar{X}\rangle - b^T\bar{y} = \langle \bar{X}, \bar{S}\rangle \geq 0.$$

Proof
We have

$$\begin{aligned} \langle C, \bar{X}\rangle - b^T\bar{y} &= \langle \mathcal{A}^*(\bar{y}) + \bar{S}, \bar{X}\rangle - b^T\bar{y} & & \mathcal{A}^*(\bar{y}) + \bar{S} = C \\ &= \langle \bar{S}, \bar{X}\rangle + \langle \mathcal{A}^*(\bar{y}), \bar{X}\rangle - b^T\bar{y} & & \\ &= \langle \bar{X}, \bar{S}\rangle + \bar{y}^T\mathcal{A}(\bar{X}) - b^T\bar{y} & & \\ &= \langle \bar{X}, \bar{S}\rangle & & \mathcal{A}(\bar{X}) = b \\ &\geq 0 & & \bar{X}, \bar{S} \succeq 0 \qquad \textit{Proposition } 1.1.3. \end{aligned}$$

Corollary 1.3.1.1

(a) If (P) is unbounded, then (D) is infeasible.

(b) If (D) is unbounded, then (P) is infeasible.

(c) If feasible solutions $\bar{X}, (\bar{y}, \bar{S})$ of (P), (D) satisfy $\langle \bar{X}, \bar{S}\rangle = 0$, then $\bar{X}$ is optimal in (P) and $(\bar{y}, \bar{S})$ is optimal in (D).

Note that the dual of (D) is "equivalent to (P), thus SDP duality is an involution. A rigorous proof and a rigorous definition of "equivalent" amounts to putting (D) in the form of (P), applying the definition of the dual and simplying.

Alternatively, let us assume both (P), (D) have feasible solutions $\hat{X}, (\hat{y}, \hat{S})$. Define

$$L := \mathrm{Null}(\mathcal{A}).$$

The feasible region of (P) can be written as

$$(L + \hat{X}) \cap \mathbb{S}^n_+.$$

Consider

$$L^\perp = \{S \in \mathbb{S}^n : \forall X \in L, \langle X, S\rangle = 0\}.$$

Notice that $\mathrm{Range}(\mathcal{A}^*) = L^\perp$. Thus $y \in \mathbb{R}^m, S \in \mathbb{S}^n$ satisfy $\mathcal{A}^*(y) + S = C$ if and only if $(S - \bar{S}) \in L^\perp$. In other words, $S \in \mathbb{S}^n$ is part of a feasible solution of (D) if and only if

$$S \in (L^\perp + \bar{S}) \cap \mathbb{S}^n_+.$$

The objective function of (P) for X satisfying $\mathcal{A}(S) = b$ can be written as

$$\langle C, X\rangle = \langle \mathcal{A}^*(y) + \bar{S}, X\rangle = b^T\bar{y} + \langle \hat{S}, X\rangle.$$

Since $b^T\bar{y}$ is a constant, (P) is really just

$$\inf\Big\{\langle \hat{S}, X\rangle : X \in (L + \hat{X}) \cap \mathbb{S}^n_+\Big\}.$$

Similarly, the objective function value of (D) for (y, S) satisfying $\mathcal{A}^*(y) + S = C$ is given by

$$b^T y = \mathcal{A}(\hat{X})^T y = \langle \hat{X}, \mathcal{A}^*(y)\rangle = \langle \hat{X}, C - S\rangle = \langle C, \hat{X}\rangle - \langle \hat{X}, S\rangle.$$

Again $\langle C, \hat{X}\rangle$ is constant, hence (D) is "equivalent" to

$$\inf\Big\{\langle \hat{X}, S\rangle, S \in (L^\perp + \hat{S}) \cap \mathbb{S}^n_+\Big\}.$$

From this, it is clear that the operations which yield (D) from (P) will yield (P) from (D).

Another attractive standard form for SDPs is given by a linear transformation $\mathcal{A} : \mathbb{S}^n \to \mathbb{S}^k, C \in \mathbb{S}^n$, and $B \in \mathbb{S}^k$.

$$\begin{aligned} \inf\langle C, X\rangle & \qquad\qquad (P) \\ \mathcal{A}(X) \succeq B & \\ X \succeq 0 & \end{aligned}$$

$$\begin{aligned} \sup\langle B, Y\rangle & \qquad\qquad (D) \\ \mathcal{A}^*(Y) \preceq C & \\ Y \succeq 0 & \end{aligned}$$

Recall that a PSD matrix $X = Q\,\mathrm{Diag}(\lambda(X))Q^T$ has a square root

$$X^{\frac{1}{2}} := Q[\mathrm{Diag}(\lambda(X))]^{\frac{1}{2}}Q^T.$$

Proposition 1.3.2
Let $X, S \in \mathbb{S}^n_+$. Then
$$\langle X, S\rangle = 0 \iff XS = 0.$$

Proof
$(\impliedby)$ We have
$$\langle X, S\rangle = \mathrm{Tr}(XS) = \mathrm{Tr}(0) = 0.$$

$(\implies)$ Suppose $X, S \in \mathbb{S}^n_+$ satisfy $\langle X, S\rangle = 0$. Then
$$0 = \mathrm{Tr}(XS) = \mathrm{Tr}\Big(X^{\frac{1}{2}}SX^{\frac{1}{2}}\Big).$$

But since X is symmetric, the Rayleigh quotient with respect to $X^{\frac{1}{2}}SX^{\frac{1}{2}}$ is always non-negative as $S \succeq 0$. This shows that $X^{\frac{1}{2}}SX^{\frac{1}{2}} \succeq 0$.

But then by Proposition 1.1.3, $\lambda(X^{\frac{1}{2}}SX^{\frac{1}{2}}) \geq 0$. So in fact, all eigenvalues are 0 which shows that $0 = X^{\frac{1}{2}}SX^{\frac{1}{2}}$.

It follows that
$$0 = \mathrm{Tr}\Big(X^{\frac{1}{2}}SX^{\frac{1}{2}}\Big) = \mathrm{Tr}\Big[(X^{\frac{1}{2}}S^{\frac{1}{2}})(X^{\frac{1}{2}}S^{\frac{1}{2}})^T\Big] = \|X^{\frac{1}{2}}S^{\frac{1}{2}}\|_2^2$$

and hence $X^{\frac{1}{2}}S^{\frac{1}{2}} = 0$. Finally,
$$XS = X^{\frac{1}{2}}(X^{\frac{1}{2}}S^{\frac{1}{2}})S^{\frac{1}{2}} = 0.$$

Proposition 1.3.3

(i) $\mathbb{S}^n_{++} = \operatorname{int}(\mathbb{S}^n_+)$

(ii) Let $X \in \mathbb{S}^n$. Then the following are equivalent:

  (a) X is positive definite.

  (b) $\lambda(X) > 0$.

  (c) There is some $\mu \in \mathbb{R}^n_{++}$ and linearly independent $h^{(1)}, \ldots, h^{(n)} \in \mathbb{R}^n$ such that $X = \sum_{i=1}^n \mu_i h^{(i)}(h^{(i)})^T$.

  (d) There is some nonsingular $b \in \mathbb{R}^{n \times n}$ such that $X = BB^T$.

  (e) For $k \in [n]$ and $J_k := [k]$, $\det(X_{J_k}) > 0$.

  (f) For any $S \in \mathbb{S}^n_+ \setminus \{0\}$, $\langle X, S \rangle > 0$.

  (g) $X \succeq 0$ and $\operatorname{rank}(X) = n$.

Chapter 2

Duality

2.1 Strong Duality

Using weak duality, we can generate concise certificates that our feasible solutions are optimal or near-optimal. We can also derive optimality conditions which help design efficient, robus algorithms.

2.1.1 Notions of Duality

Definition 2.1.1 (Dual Cone)
Given $K \subseteq \mathbb{R}^d$, the dual cone is given by

$$K^* := \{s \in \mathbb{R}^d : \forall x \in K, \langle x, s \rangle \geq 0\}.$$

Under the Euclidean inner product, $(\mathbb{R}^d_+)^* = \mathbb{R}^d_+$. Under the trace inner product, $(\mathbb{S}^n_+)^* = \mathbb{S}^n_+$.

Definition 2.1.2 (Polar Set)
Given $K \subseteq \mathbb{R}^d$,

$$K^o := \{s \in \mathbb{R}^d : \forall x \in K, \langle x, s \rangle \leq 1\}$$

is defined to be the polar set of K.

Observe that if K is a cone, then $K^o = -K^*$.

Definition 2.1.3 (Legendre-Fenchel Conjugate)
For a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, its convex conjugate is given by

$$f_*(s) := \sup_{x \in \mathbb{R}^d} -\langle s, x\rangle - f(x).$$

Example 2.1.1
Consider $f : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ given by

$$f(X) := \begin{cases} -\ln\det(X), & X \in \mathbb{S}^n_{++} \\ \infty, & \text{else} \end{cases}$$

Its conjugate is precisely

$$f_*(S) = \begin{cases} -\ln\det(S) - n, & S \in \mathbb{S}^n_{++} \\ \infty, & \text{else} \end{cases}$$

2.1.2 Preliminaries

Theorem 2.1.2 (Hyperplane Separation for Closed Convex Sets)
Let $\varnothing \neq G \subseteq \mathbb{R}^d \setminus \{0\}$ be closed and convex. There is some $a \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{R}_+$ sch that

$$G \subseteq \{x \in \mathbb{R}^d : a^T x \geq \alpha\}.$$

Corollary 2.1.2.1
Let $\varnothing \neq G_1, G_2 \subseteq \mathbb{R}^d$ be disjoint closed convex sets. If either G_1, G_2 is bounded, there exists some $a \in \mathbb{R}^d \setminus \{0\}$ such that

$$\inf\{a^T x : x \in G_1\} > \sup\{a^T x : x \in G_2\}.$$

If both sets are unbounded, we cannot guarantee the strictness of the inequality above.

Theorem 2.1.3
Let $\varnothing \neq G \subseteq \mathbb{R}^d$ be convex with $0 \notin G$. Then there is some $A \in \mathbb{R}^d \setminus \{0\}$ such that

$$G \subseteq \{x \in \mathbb{R}^d : a^T x \geq 0\}.$$

Corollary 2.1.3.1
Let $\varnothing \neq G_1, G_2 \subseteq \mathbb{R}^d$ be disjoint convex sets. There is some $a \in \mathbb{R}^d \setminus \{0\}$ such that

$$\inf\{a^T x : x \in G_1\} \geq \sup\{a^T x : x \in G_2\}.$$

Recall our primal and dual SDPs.

$$\begin{aligned} &\inf \langle C, X\rangle && (P) \\ \mathcal{A}(X) &= b \\ X &\succeq 0 \end{aligned}$$

$$\begin{aligned} &\sup b^T y && (D) \\ \mathcal{A}^*(y) + S &= C \\ S &\succeq 0 \end{aligned}$$

Definition 2.1.4 (Slater Point)
We say that (P) satisfies the Slater Condition, or (P) has a Slater point, if there is some $\bar{X} \in \mathbb{S}^n_{++}$ such that

$$\mathcal{A}(\bar{X}) = b.$$

Similarly, (D) satisfies the Slater Condition or has a Slater point if there are $\bar{y} \in \mathbb{R}^m$ and $\bar{S} \in \mathbb{S}^n_{++}$ for which

$$\mathcal{A}^*(\bar{y}) + \bar{S} = C.$$

2.1.3 Strong Duality

Theorem 2.1.4 (Strong Duality)
Suppose (D) has a Slater point and the objective value of (D) is bounded from above. Then (P) attains its optimal value and the optimum values of (P), (D) are the same.

Proof
Suppose there is some $\bar{y} \in \mathbb{R}^m, \bar{S} \in \mathbb{S}^n_{++}$ such that $\mathcal{A}^*(\bar{y}) + \bar{S} = C$. Further assume that there exists $\gamma \in \mathbb{R}$ such that $b^T y \leq \gamma$ for all feasible solutions (y, S) of (D).

Let

$$z^* := \sup\{b^T y : \mathcal{A}^*(y) + S = C, S \succeq 0\}.$$

We may as well assume that $b \neq 0$, or else by taking $\bar{X} := 0$ results in $\bar{X}, (\bar{y}, \bar{S})$ being optimal solutions to (P), (D) both with objective value 0.

Put

$$\begin{aligned} G_1 &:= \mathbb{S}^n_{++} \\ G_2 &:= \{S \in \mathbb{S}^n : \exists y \in \mathbb{R}^m, S = C - \mathcal{A}^*(y), b^T y \geq z^*\}. \end{aligned}$$

It is easy to see that G_1, G_2 are convex with $G_1 \neq \varnothing$. Now, by the definition of z^*, there must be some $(\hat{y}, \hat{S})$ such that $\mathbb{S}^n \supseteq \hat{S} \in G_2$ as linear functions attain their supremum over affine subspaces. Note that $\hat{S}$ is not necessarily positive semidefinite.

We claim that $G_1 \cap G_2 = \varnothing$. Otherwise, there is some $\tilde{y} \in \mathbb{R}^m$ such that $\mathcal{A}^*(\tilde{y}) \prec C$ and $b^T \tilde{y} \geq z^*$. But then by setting $\hat{y} := \tilde{y} + \epsilon b$ for sufficiently small $\epsilon > 0$, we see that $\mathcal{A}^*(\hat{y}) \prec C$ as well as

$$b^T \hat{y} = b^T \tilde{y} + \epsilon \|b\|_2^2 > z^*,$$

which is a contradiction.

By a previous corollary, there exists some $\tilde{X} \in \mathbb{S}^n \setminus \{0\}$ such that

$$\inf\{\langle \tilde{X}, S\rangle : S \in \mathbb{S}^n_{++}\} \geq \sup\{\langle \tilde{X}, S\rangle : S \in G_2\}.$$

But $\mathbb{S}^n_{++}$ is a cone, hence we must have that $\langle \tilde{X}, S\rangle \geq 0$ for each $S \in \mathbb{S}^n_{++}$, or else the infimum over G_1 is $-\infty$. This inequality holds over the closure, hence

$$\forall S \in \mathrm{cl}(\mathbb{S}^n_{++}) = \mathbb{S}^n_+, \langle \tilde{X}, S\rangle \geq 0.$$

By a previous proposition, it follows that $\tilde{X} \in \mathbb{S}^n_+$. By taking a sequence $S^{(k)} := \frac{1}{k} I \to 0$, we see that the LHS infimum must be 0. In other words,

$$0 \geq \langle \tilde{X}, C\rangle - \langle \tilde{X}, \mathcal{A}^*(y)\rangle$$

for every $y \in \mathbb{R}^m$ such that $b^T y \geq z^*$. This happens if and only if $\mathcal{A}(\tilde{X})^T y \geq \langle C, \tilde{X}\rangle$ for every $y \in \mathbb{R}^m$ such that $b^T y \geq z^*$.

Thus, $\mathcal{A}(\tilde{X})^T y$ is bounded below on the set $\{y \in \mathbb{R}^m : b^T y \geq z^*\}$. But since the latter is a closed half-space, it must be that $\mathcal{A}(\tilde{X}) = \alpha b$ for some $\alpha \geq 0$. Note that $\alpha \neq 0$, or else $\mathcal{A}(\tilde{X}) = 0$ and

$$0 \geq \langle C, \tilde{X}\rangle = \langle \mathcal{A}^*(\bar{y}) + \bar{S}, \tilde{X}\rangle = \mathcal{A}(\tilde{X})^T \bar{y} + \langle \bar{S}, \tilde{X}\rangle > 0,$$

where the last inequality comes from our characterization of PD matrices. This is a contradiction. It follows that $\alpha > 0$.

Define

$$\hat{X} := \frac{1}{\alpha} \tilde{X} \in \mathbb{S}^n_+.$$

We have $\mathcal{A}(\hat{X}) = b$ and $\mathcal{A}(\hat{X})^T y \geq \langle C, \hat{X} \rangle$ for all $y \in \mathbb{R}^m$ such that $b^T y \geq z^*$. Therefore, $\langle C, \hat{X} \rangle \leq z^*$. By the Weak Duality theorem, $\hat{X}$ is optimal in (P) and the optimal objective values of (P), (D) are the same.

Corollary 2.1.4.1
If (D) has a feasible solution and (P) has a Slater point, then (D) attains its optimal objective value, and the optimal objective values of (P), (D), coincide.

Corollary 2.1.4.2
If (P), (D) both have Slater points, then both (P), (D), attain their optimal objective values and these objective values are the same.

Remark that the above theorem and its proof generalize to the conic convex optimization setting where we may simply pick our standard form as

$$\begin{aligned} &\inf \langle c, x \rangle && (P) \\ \mathcal{A}(x) &= b \\ \mathcal{B}(x) &\geq d \\ x &\in K \end{aligned}$$

$$\begin{aligned} &\sup \langle b, y \rangle + \langle d, u \rangle && (D) \\ \mathcal{A}^*(y) + \mathcal{B}^*(u) + s &= c \\ u &\geq 0 \\ s &\in K^* \end{aligned}$$

Here we let $c \in \mathbb{R}^n, b \in \mathbb{R}^{m_1}, d \in \mathbb{R}^{m_2}$, with $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^{m_1}, \mathcal{B} : \mathbb{R}^n \to \mathbb{R}^{m_2}$ being linear transformations, and $K \subseteq \mathbb{R}^n$ a closed convex cone.

In this more general setting, $\bar{x} \in \mathbb{R}^n$ is a *restricted Slater point* if it is feasible and satisfies $\bar{x} \in \operatorname{int}(K)$. Moreover, $(\bar{y}, \bar{u}, \bar{s})$ is a restricted Slater point for (D) if it is dual feasible with $\bar{s} \in \operatorname{int}(K^*)$.

2.1.4 Pitfalls

The existence of Slater points is crucial. Attainment of the optimal solution cannot be guaranteed without them. The primal dual optimality gap can also be made arbitrarily large when there are no Slater points.

In the special case of LP problems, if (P) and (D) both have feasible solutions, then they both have optimal solutions. Moreover, for every pair of optimal solutions $\bar{x}, (\bar{y}, \bar{s})$, complementarity holds: $\bar{x}_j \cdot \bar{s}_j = 0$ for each j. Furthermore, there is an optimal pair $\hat{x}, (\hat{y}, \hat{s})$ for which strict complementarity holds:

$$\hat{x} + \hat{s} > 0.$$

In the SDP generalization, we showed that with the existence of Slater points, both (P), (D) have optimal solutions (strong duality). In addition, every pair of optimal solutions $\bar{X}, (\bar{y}, \bar{S})$ satifies complementarity (weak duality):

$$\bar{X}\bar{S} = \bar{S}\bar{X} = 0.$$

However, as shown in A1Q4(c), strict complementarity may fail even in the presence of Slater points.

Fixed dimensions $n > m \geq 1$ and consider the set of data $(\mathcal{A}, b, c) \in \mathcal{L}(\mathbb{S}^n, \mathbb{R}^m) \oplus \mathbb{R}^m \oplus \mathbb{R}^n$ with feasible primal solutions. The set of instances without Slater points has measure zero. Morever, if we focus on the data for with feasible primal and dual solutions, the set of instances which do not have Slater points or do not satisfy strict complementarity has measure zero.

2.2 Certifying Infeasibility & Unboundedness

We seek a generalization of Farkas' lemma. Unfortunately, there is no direct generalization.

Definition 2.2.1 (Almost Feasible)
Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be a linear transformation and $C \in \mathbb{S}^n$. Then we say that $\mathcal{A}^*(y) \preceq C$ is almost feasible if for every $\epsilon > 0$, there is some $C' \in \mathbb{S}^n$ such that

$$\|C - C'\| < \epsilon$$

and $\mathcal{A}^*(y) \preceq C'$ is feasible.

Notice that feasibility implies almost feasibility.

Theorem 2.2.1
Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be linear and $C \in \mathbb{S}^n$. Then

(i) If $\exists D \in \mathbb{S}^n_+$ such that $\mathcal{A}(D) = 0$ with $\operatorname{Tr}(CD) < 0$, then the system $\mathcal{A}^*(y) \preceq C$ is not feasible.

(ii) If $\nexists D \in \mathbb{S}^n_+$ such that $\mathcal{A}(D) = 0$ with $\operatorname{Tr}(CD) < 0$, then $\mathcal{A}^*(y) \preceq 0$ is almost feasible.

Proof
(i): The proof is similar to weak duality. If there is a feasible $\bar{y}$,

$$\begin{aligned} 0 &\leq \langle C, D\rangle - \langle \mathcal{A}^*(\bar{y}), D\rangle \\ &= \langle C, D\rangle - \bar{y}^T \mathcal{A}(D) \\ &= \langle C, D\rangle \\ &< 0. \end{aligned}$$

This is a contradiction.

(ii): We set up an SDP and apply strong duality. Consider

$$\begin{aligned} \sup \eta & \qquad\qquad (D_1) \\ \mathcal{A}^*(y) + \eta I \preceq C & \\ \eta \leq 0 & \end{aligned}$$

$$\begin{aligned} \inf \langle C, X\rangle & \qquad\qquad (P_1) \\ \mathcal{A}(X) = 0 & \\ \mathrm{Tr}(X) \leq 1 & \\ X \succeq 0. & \end{aligned}$$

The point $(\bar{y}, \eta) : +(0, -\|C\|_2 - 1)$ is a Slater point by assumption and $\bar{X} = 0$ is primal feasible. Thus the strong duality theorem applies.

If there is no such $D \succeq 0$ such that $\mathcal{A}(D) = 0$ and $\mathrm{Tr}(CD) < 0$, then the optimal objective value of (P_1) is 0. This is also the optimal objective value of (D_1) and there is a sequence of dual feasible solutions approaching 0.

$$\mathcal{A}^*(y^{(k)}) + \eta_k I \preceq C, \eta_k \to 0^-.$$

By definition, $\mathcal{A}^*(y) \preceq C$ is almost feasible.

Theorem 2.2.2
There exists $D \in \mathbb{S}^n_+$ such that $\mathcal{A}(D) = 0$ with $\mathrm{Tr}(CD) < 0$, if and only if $\mathcal{A}^*(y) \preceq 0$ is NOT almost feasible.

Proof
$(\Longleftarrow)$ Previous theorem.

($\implies$) Suppose there exists such D. We may assume $\operatorname{Tr}(CD) = -1$ by scaling if necessary. Then for every $C' \in \mathbb{S}^n$ such that

$$\|C - C'\|_F < \frac{1}{\|D\|_F},$$

$\mathcal{A}^*(y) \preceq C'$ is infeasible. Indeed, if $0 \preceq C' - \mathcal{A}^*(y)$, then $\langle D, \mathcal{A}^*(y)\rangle \leq \operatorname{Tr}(C'D)$ by our characterization of positive semidefinite matrices.

$$\begin{aligned}
0 &= \langle y, 0\rangle \\
&= \langle y, \mathcal{A}(D)\rangle \\
&= \langle \mathcal{A}^*(y), D\rangle \\
&\leq \operatorname{Tr}(CD) - \operatorname{Tr}[(C - C')D] \\
&\leq -1 + \|C - C'\|_F \cdot \|D\|_F && \text{Cauchy-Schwartz} \\
&< 0.
\end{aligned}$$

2.3 Slater Condition, Facial Reduction, & Extended Langrange-Slater Dual

Another way to deal with duality gaps, dual attainment issues, as well as infeasibility and unboundedness theorems is to change the definition of the dual.

Facial Reduction

Definition 2.3.1 (Face)
Let $k \subseteq \mathbb{R}^d$ be a closed convex cone. A convex cone $G \subseteq K$ is a face of K if for every $u, v \in K$ such that $u + v \in G$, we have $u \in G, v \in G$.

A face G of K is *exposed*, if there is some $a \in \mathbb{R}^d \setminus \{0\}$ such that

$$G = \{x \in K : \langle a, x\rangle = 0\}$$

and

$$K \subseteq \{x \in \mathbb{R}^d : \langle a, x\rangle \leq 0\}.$$

So G is the intersection of K with one of its supporting hyperplanes.

Note that $a \in -K^*$.

A face G of K is *proper* if $\{0\} \subsetneq G \subsetneq K$.

Theorem 2.3.1

(a) Every face $\varnothing \neq G \subseteq \mathbb{S}^n_+$ is characterized by a unique subspace $L \subseteq \mathbb{R}^n$ such that $G = \{X \in \mathbb{S}^n_+ : \mathrm{Null}(X) \supseteq L\}$ and $\mathrm{relint}(G) = \{X \in \mathbb{S}^n_+ : \mathrm{Null}(X) = L\}$.

(b) Every proper face of $\mathbb{S}^n_+$ is exposed.

(c) $\mathbb{S}^n_+$ is *projectionally exposed*. That is, every nonempty face G can be expressed as $G = (I - Q)\mathbb{S}^n_+(I - Q)$, where $Q \in \mathbb{S}^n$ is the projection onto the unique subspace L defining G.

The above theorem implies that every proper face of $\mathbb{S}^n_+$ is linearly isomorphic to $\mathbb{S}^k_+$ for some $k \in [n-1]$. In other words G is a proper face of $\mathbb{S}^n_+$ if and only if there is some $k \in [n-1]$ and $Q \in \mathbb{R}^{n\times n}$ orthogonal such that

$$G = \left\{ Q \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} Q^T : X \in \mathbb{S}^k_+ \right\}.$$

Given $(\mathcal{A}, b, c)$, suppose we find the inclusion-wise minimal face $\bar{G} \subseteq \mathbb{S}^n_+$ which contains the feasible region of (P). Our problem is equivalent to

$$\begin{aligned} \inf \mathrm{Tr}(CX) & & (\tilde{P}) \\ \mathcal{A}(X) = b & & \\ X \in \bar{G} & & \end{aligned}$$

Let $Q \in \mathbb{R}^{n\times n}$ be orthogonal such that

$$\bar{G} = \left\{ Q \begin{bmatrix} \bar{X} & 0 \\ 0 & 0 \end{bmatrix} Q^T : \bar{X} \in \mathbb{S}^k_+ \right\}.$$

By composing $C, \mathcal{A}$ with the linear function $Q(\cdot)Q^T$ to obtain $\bar{C}, \bar{\mathcal{A}}$, we see that $(\tilde{P})$ is equivalent to

$$\begin{aligned} \inf \mathrm{Tr}\big(\bar{C}\bar{X}\big) & & (\bar{P}) \\ \bar{\mathcal{A}}(\bar{X}) = b & & \\ \bar{X} \in \mathbb{S}^k_+ & & \end{aligned}$$

Now since $\bar{G}$ was inclusion-wise minimal, the maximal rank of a feasible solution must be precisely k, or else we can find a smaller face which contains the feasible region. But then $(\bar{P})$ necessarily satisfies the Slater condition!

Unfortunately, finding the minimal face of $\mathbb{S}^n_+$ containing the feasible region of (P) is no easier than finding a solution to (P) in the worst case. However, the following result is useful.

Lemma 2.3.2
Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be linear and $b \in \mathbb{R}^m$. Then exactly one of the following two systems has a solution.
(I) $\mathcal{A}(X) = b, X \in \mathbb{S}^n_{++}$
(II) $\mathcal{A}^*(y) \in \mathbb{S}^n_+ \setminus \{0\}, b^T y = 0$

Thus either (P) has a Slater point, or we can find some $\bar{y}$ for system (II) such that

$$\forall X, \mathcal{A}(X) = b \implies \langle X, \mathcal{A}^*(\bar{y}) \rangle = \langle \mathcal{A}(X), \bar{y} \rangle = b^T \bar{y} = 0.$$

This is a supporting hyperplane of $\mathbb{S}^n_+$ containing the feasible region of (P). Note that $X \in \mathbb{S}^n_+, \langle X, \mathcal{A}^*(\bar{y}) \rangle = 0$ defines a proper face of $\mathbb{S}^n_+$. Thus we can repeatedly apply this lemma, and create SDPs over $\mathbb{S}^k_+, k < n$ at most n times to arrive at some SDP with a Slater point.

This process is sometimes called *facial reduction*.

Extended Lagrange-Slater Dual

Suppose we wish to solve the SDP

$$\begin{aligned} \sup b^T y \\ \mathcal{A}^*(y) \preceq C \end{aligned} \qquad (D)$$

Define its *Extended Lagrange-Slater Dual* as

$$\begin{aligned} \inf \operatorname{Tr}(C(U + W)) \\ \mathcal{A}(U + W) = b \\ W \in \mathcal{W}_n \\ U \succeq 0 \end{aligned} \qquad \text{(ELSD)}$$

Here $\mathcal{W}_n \subseteq \mathbb{S}^n$ is a linear subspace defined using $\mathcal{A}, C$ through $n(m+1)$ linear equations and n PSD matrix inequalities on $2n \times 2n$ matrices.

Theorem 2.3.3
If (D) has a finite optimal value, then the optimal values of (D) and (ELSD) are the same and (ELSD) attains its optimal value.

Theorem 2.3.4
Let $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ be linear and $C \in \mathbb{S}^n$. Then exactly one of the following systems has a solution.

(I) $\mathcal{A}^*(y) \preceq C$

(II) $\mathcal{A}(U+W) = 0, W \in \mathcal{W}_n, U \succeq 0, \mathrm{Tr}(C(U+W)) = -1$

Theorem 2.3.5
In the real number computation model, the problem of deciding SDP feasibility is in NP and co-NP.

The following are open problems:

1. Does there exist a more efficient representation of $\mathcal{W}_n$?
2. Is SDP feasibility in NP in the Turing machine model?

2.4 Slater Condition in SDP Relaxations

We usually employ SDPs as a relaxation of a much harder problem of optimizing $c^T x$ over a difficult nonconvex set F.

Definition 2.4.1 (Homogeneous Equality Form)
Suppose there is some $\mathcal{A} : \mathbb{S}^{n+1} \to \mathbb{R}^m$ linear such that

$$F = \left\{ x \in \mathbb{R}^n : \mathcal{A} \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} = 0 \right\}.$$

Then F can be represented in Homogeneous Equality Form.

Which sets F can be represented in Homogeneous Equality Form?

Proposition 2.4.1
Every system of finitely many multivariate polynomial equations and inequalities can be put into Homogeneous Equality Form.

We omit the proof but the ideas are elementary. For instance, a quadratic inequality given by $Q \in \mathbb{S}^n, q \in \mathbb{R}^n, \gamma \in \mathbb{R}$ is

$$x^T Q x + 2q^T x + \gamma \le 0.$$

We can introduce the slack variable $\tilde{s}^2$. So that the above happens if and only if

$$\operatorname{Tr}\begin{bmatrix}\gamma & q^T & 0\\ q & Q & 0\\ 0 & 0^T & 1\end{bmatrix}\begin{bmatrix}1 & x^T & \tilde{s}\\ x & xx^T & \tilde{s}x\\ \tilde{s} & \tilde{s}x^T & \tilde{s}^2\end{bmatrix} = 0.$$

Note that the lower diagonal block of the RHS matrix is hh^T where $h = (1, x, \tilde{s})^T$.

We can also reduce higher degree polynomials, say $x^4 = b$ as

$$x^2 - x' = 0, (x')^2 = b.$$

An SDP relaxation is very natural.

$$\begin{aligned}\inf \operatorname{Tr}\begin{bmatrix}0 & \frac{1}{2}c^T\\ \frac{1}{2}c & 0\end{bmatrix}\begin{bmatrix}1 & x^T\\ x & X\end{bmatrix} & \qquad (\hat{P})\\ \mathcal{A}\begin{bmatrix}1 & x^T\\ x & X\end{bmatrix} = 0 &\\ \begin{bmatrix}1 & x^T\\ x & X\end{bmatrix} \succeq 0 &\end{aligned}$$

Theorem 2.4.2
If $\operatorname{conv} F$ is full dimensional, then the Slater condition holds for the SDP relaxation above.

Proof
Suppose $\operatorname{conv} F$ is full-dimensional. By definition, there are $v^{(i)} \in F, i \in [n+1]$ affinely independent vectors. In other words,

$$h^{(i)} := \begin{bmatrix}1\\ v^{(i)}\end{bmatrix}, i \in [n+1]$$

are linearly independent.

Consider

$$\bar{X} := \frac{1}{n+1}\sum_{i=1}^{n+1} h^{(i)}(h^{(i)})^T.$$

By the characterization of positive definite matrices, $\bar{X} \in \mathbb{S}^n_{++}$. Moreover,

$$\bar{X} \in \operatorname{conv}\left\{\begin{bmatrix}1 & x^T\\ x & xx^T\end{bmatrix} \in \mathbb{S}^{n+1} : x \in F\right\} \subseteq \hat{P}.$$

Thus $\bar{X}$ is a Slater point for the SDP relaxation.

In the case where $\dim \operatorname{conv}(F) = d \leq n-1$, we can determine the affine hull of F. Thus we find $L \in \mathbb{R}^{d\times n}$ and $\ell \in \mathbb{R}^n$ with $\operatorname{rank} L = d$ and

$$x \in F \implies \exists y \in \mathbb{R}^d, x = \ell + L^T y.$$

Define the linear transformation $\mathcal{L} : \mathbb{S}^{n+1} \to \mathbb{S}^{d+1}$ given by

$$\mathcal{L}(Z) := \begin{bmatrix} 1 & \ell^T \\ 0 & L \end{bmatrix} Z \begin{bmatrix} 1 & 0^T \\ \ell & L^T \end{bmatrix}.$$

Its adjoint is $\mathcal{L}^* : \mathbb{S}^{d+1} \to \mathbb{S}^{n+1}$ given by

$$\mathcal{L}^*(W) = \begin{bmatrix} 1 & 0^T \\ \ell & L^T \end{bmatrix} W \begin{bmatrix} 1 & \ell^T \\ 0 & L \end{bmatrix}.$$

Note that for $x \in F$, there is some y such that $x = L^T y + \ell$ and

$$\mathcal{L}^* \begin{bmatrix} 1 & y^T \\ y & yy^T \end{bmatrix} = \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix}.$$

Furthermore, define $\bar{\mathcal{A}} : \mathbb{S}^{d+1} \to \mathbb{R}^m$ by

$$\bar{\mathcal{A}}(W) := \mathcal{A}(\mathcal{L}^*(W)).$$

We can then express

$$F = \left\{ \ell + L^T y : \bar{A} \begin{bmatrix} 1 & y^T \\ y & yy^T \end{bmatrix} = 0, y \in \mathbb{R}^d \right\}$$

which leads to the SDP relaxation

$$\hat{P}_L := \left\{ \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \in \mathbb{S}^{d+1} : \bar{\mathcal{A}} \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} = 0, \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \succeq 0 \right\}.$$

Theorem 2.4.3
$\hat{P}_L \cap \mathbb{S}^{d+1}_{++} \neq \varnothing$.

Thus we can always guarantee the Slater condition holds in a wide clas of SDP relaxations, provided we can identity the affine hull of F. Moreover, in many cases we decrease the size of the SDP.

Chapter 3

Solving SDPs

3.1 Ellipsoids & the Ellipsoid Method

Definition 3.1.1 (Ellipsoid)
$E \subseteq \mathbb{R}^d$ is an ellipsoid if there is some $c \in \mathbb{R}^d$ (center) and $A \in \mathbb{S}^d_{++}$ (shape & size) such that

$$E = \{x \in R^d : (x-c)^T A^{-1}(x-c) \leq 1\} =: E(A, c).$$

Note that

$$\begin{aligned} E(A,c) &= \left\{ x \in \mathbb{R}^d : \left\| A^{-\frac{1}{2}}(x-c) \right\|_2^2 \leq 1 \right\} \\ &= \left\{ A^{\frac{1}{2}} z + c : \|z\|_2^2 \leq 1, z \in \mathbb{R}^d \right\} \\ &= c + A^{\frac{1}{2}} B_d(0,1). \end{aligned}$$

Since ellipsoids are simply affine images of the unit ball, many of their attributes are easy to handle. For instance,

$$\operatorname{vol}(E(A,c)) = \sqrt{\det A} \cdot \operatorname{vol}(B_d(0,1)).$$

The longest and shortest axis of $E(A,c)$ each correpond to an eigenvector of A determining $\lambda_1(A), \lambda_d(A)$, respectively.

Theorem 3.1.1 (Löwner-John)
For every compact convex set in $\mathbb{R}^d$ with nonempty interior, there exists a unique minimal volume ellipsoid containing that set. Moreover, shrinking that ellipsoid around its center by a factor of at most d gives an ellipsoid contained in the convex set.

The Ellipsoid method does not require an explicit description of the feasible region. It suffices to have a *weak separation oracle.*

Definition 3.1.2 (δ-Relaxation)
Let $G \subseteq \mathbb{R}^d$ be convex. Given $\delta > 0$, the δ-relaxation of G is

$$\mathrm{relax}(G, \delta) := \left\{u \in \mathbb{R}^d : \exists x \in G, \|u - x\|_2 \leq \delta\right\}.$$

Note that $\mathrm{relax}(G, \delta)$ is convex by definition.

Definition 3.1.3 (Weak Separation Oracle)
This oracle takes as input $\bar{x} \in \mathbb{Q}^d$ and $\delta \in \mathbb{Q}_{++}$ and outputs either $\bar{x} \in \mathrm{relax}(G, \delta)$ or $a \in \mathbb{Q}^d$ such that $\|a\|_\infty = 1$ and

$$\forall x \in \mathrm{relax}(G, \delta), \langle a, \bar{x}\rangle \geq \langle a, x\rangle - \delta.$$

3.1.1 Feasibility Algorithm

Our input is $A \in \mathbb{S}^d_{++}, c \in \mathbb{R}^d$ such that $E(A, c) \supseteq G$ and $\epsilon \in \mathbb{Q}_{++}$.

1) Using the separation oracle, check if $c \in G$. If $c \in$, STOP. Otherwise, retrieve $a \in \mathbb{Q}^d, \|a\|_\infty = 1$ separating c from G.
2) If $\mathrm{vol}(E(A, c)) < \epsilon$, STOP. Else consider $\tilde{E} := \{x \in E(A, c) : \langle a, x\rangle \leq \langle a, c\rangle\}$.
3) Compute the minimum volume ellipsoid $E(A, c)$ (update A, c) containing $\tilde{E}$ and goto the first step.

Lemma 3.1.2
Let $A_+ \in \mathbb{S}^d_{++}, c_+ \in \mathbb{R}^d$ such that $E(A_+, c_+)$ is the minimum volume ellipsoid containing
$$\tilde{E} := \{x \in E(A, c) : \langle a, x\rangle \leq \langle a, c\rangle\}.$$
Then
$$c_+ = c - \frac{1}{(d+1)\sqrt{a^T A a}} A a$$
$$A_+ = \frac{d^2}{d^2 - 1}\left[A - \frac{2}{(d+1)a^T A a} A a a^T A\right].$$
Moreover,
$$\ln\left(\frac{\operatorname{vol}(E(A_+, c_+))}{\operatorname{vol}(E(A, c))}\right) \leq -\frac{1}{2d}.$$

Proof
Let $G \subseteq \mathbb{R}^d$ be convex such that we have access to a weak separation oracle for G, and $G \subseteq B_d(0, R)$ for a given $R \in \mathbb{Q}_{++}$.

Then for any $\epsilon \in \mathbb{Q}_{++}$,
$$O\left(d^2 \ln\left(\frac{R}{\epsilon}\right)\right)$$
iterations of the feasibility ellipsoid method suffices to compute either $\bar{x} \in \operatorname{relax}(G, \epsilon)$ or prove that $\operatorname{vol}(G) \leq \epsilon$.

3.1.2 Optimization Algorithm

We can extend the feasibility algorithm to handle convex optimization problem of the form
$$\inf_{x \in G} f(x)$$
where $f : \mathbb{R}^d \to \mathbb{R}$ is convex.

A *subgradient oracle for f* takes as input $\bar{x} \in \mathbb{R}^d$ and returns $f(\bar{x})$ as well as
$$h \in \partial f(x) := \{h \in \mathbb{R}^d : \forall x \in \mathbb{R}^d, f(x) \geq f(\bar{x}) + h^T(x - \bar{x})\}.$$

Theorem 3.1.3
Let $G \subseteq \mathbb{R}^n$ be convex and $f : \mathbb{R}^d \to \mathbb{R}$ a convex function such that

(i) There is a weak separation oracle for G.

(ii) There is a subgradient oracle for f.

(iii) $r, R \in \mathbb{Q}_{++}$ are given such that $B_d(\tilde{x}, r) \subseteq G \subseteq B_d(0, R)$ for some $\tilde{x} \in \mathbb{R}^d$.

Then after

$$O\left(d^2\left[\ln\left(\frac{R}{r}\right) + \ln\left(\frac{\mu_0}{\epsilon}\right)\right]\right)$$

iterations of an ellipsoid method, we obtain $\bar{x} \in G$ such that

$$f(\bar{x}) \leq \inf_{x \in G} f(x) + \epsilon.$$

Here

$$\mu_0 := \epsilon + \sup_{x \in B_d(0,R)} f(x) - \inf_{x \in b_d(0,R)} f(x).$$

This algorithm applies to all convex optimization problems, including SDPs! Suppose (P), (D) have Slater points $\bar{X}, (\bar{y}, \bar{S})$ respectively. Then we can replace (P) by $(\tilde{P})$.

$$\begin{aligned} \inf \langle C, X \rangle & \qquad\qquad\qquad (\tilde{P}) \\ \mathcal{A}(X) = b & \\ \langle \bar{S}, X \rangle \leq 2\langle \bar{X}, \bar{S} \rangle & \\ X \succeq 0 & \end{aligned}$$

Theorem 3.1.4

(a) $(P), (\tilde{P})$ have optimal solutions.

(b) The optimal solution sets of $(P), (\tilde{P})$ are the same.

(c) Let $G \subseteq \mathbb{S}^n_+$ denote the feasible solution set for $(\tilde{P})$. Then G is compact and convex. Moreover, if B_G denotes the Euclidean ball in $\operatorname{aff}(G)$,

$$b_G(\bar{X}, \lambda_n(\bar{X})) \subseteq G \subseteq B_G\left(0, \frac{2\langle \bar{X}, \bar{S} \rangle}{\lambda_n(\bar{S})}\right).$$

(d) $\max_{X \in G}\langle C, X \rangle - \min_{X \in G}\langle C, X \rangle \leq \frac{4n\|C\|_2 \langle \bar{X}, \bar{S} \rangle}{\lambda_n(\bar{S})}$.

This theorem allows us to use the ellipsoid method to solve SDPs.

3.2 Primal-Dual Interior-Point Methods

Consider algorithms which start with $X^{(o)} \succ 0, y^{(0)} \in \mathbb{R}^{\cdot} S^{(0)} \succ 0$ and generate

$$\left\{(X^{(k)}, y^{(k)}, S^{(k)}) : k \geq 1\right\}$$

such that $X^{(k)} \succ 0, S^{(k)} \succ 0$ for each k.

Assuming (P), (D) have optimal solutions with the same optimal values, we wish to maintain positive definitess and

$$\begin{aligned} \|\mathcal{A}(X^{(k)}) - b\| &\to 0 && \text{primal feasibility} \\ \|\mathcal{A}^*(y^{(k)}) + S^{(k)} - C\| &\to 0 && \text{dual feasibility} \\ \langle X^{(k)}, X^{(k)} \rangle &\to 0 && \text{complementary slackness} \end{aligned}$$

For simplicity of presentation, we will assume $\mathcal{A}(X^{(0)}) = b, \mathcal{A}^*(y^{(0)}) + S^{(0)} = C$ and that A is surjective. Define $f : \mathbb{S}^n \to \mathbb{R} \cup \{\infty\}$ by

$$f(X) := \begin{cases} -\ln\det(X) & X \in \mathbb{S}^n_{++} \\ \infty, & \text{else} \end{cases}$$

Note that for any sequence of positive definite matrices converging to the boundary of the SDP cone, $f(X^{(k)}) \to \infty$.

Proposition 3.2.1
f above is strictly convex on $\mathbb{S}^n_{++}$. Moreover, for any $X \in \mathbb{S}^n_{++}, H \in \mathbb{S}^n$,

1. $\langle f'(X), H \rangle = -\operatorname{Tr}(X^{-1}H)$
2. $\langle f''(X)H, H \rangle = \operatorname{Tr}(X^{-1}HX^{-1}H) = \operatorname{Tr}\Big(X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\Big)^2$
3. $f'''(X)[H, H, H] = -2\operatorname{Tr}\Big(X^{-\frac{1}{2}}HX^{-\frac{1}{2}}\Big)^3$

3.2.1 Central Path

For each $\mu > 0$, define

$$\begin{aligned} \inf \frac{1}{\mu}\langle C, X \rangle + f(x) \qquad\qquad (P_\mu) \\ \mathcal{A}(X) = b \end{aligned}$$

Necessary and sufficient optimality conditions for (P_μ) under the Slater point assumption for (P) and (D) are

(i) $\mathcal{A}(X) = b$

(ii) $X \succ 0$

(iii) $-\mathcal{A}^*(y) - X^{-1} + \frac{1}{\mu}C = 0$

If we let $\mu y \to y$ and $S := \mu X^{-1}$. Then these conditions translate to

(i) $\mathcal{A}(X) = b$

(ii) $X \succ 0$

(iii) $\mathcal{A}^*(y) + S = C$

(iv) $S = \mu X^{-1} (\succ 0)$

For each $\mu > 0$, the unique solution $(X(\mu), y(\mu), S(\mu))$ defines the primal-dual central path:

$$\{X(\mu), y(\mu), X(\mu) \in \mathbb{S}^n \oplus \mathbb{R}^m \oplus \mathbb{S}^n : \mu > 0\}.$$

Theorem 3.2.2
Suppose (P), (D) have Slater points and $\mathcal{A}$ is surjective. Then for every $\mu > 0$, (P_μ) has a unique optimal solution $X(\mu)$. Moreover, the following system

$$\begin{aligned} \mathcal{A}(X) &= b \\ X &\succ 0 \\ \mathcal{A}^*(y) + S &= C \\ S &= \mu X^{-1} \end{aligned}$$

has a unique solution $(X(\mu), y(\mu), S(\mu))$.

The system above also characterizes the unique optimal solution of

$$\begin{aligned} &\sup \frac{1}{\mu} b^T y + f(S) \qquad (D_\mu) \\ &\mathcal{A}^*(y) + S = C \end{aligned}$$

Consider the solutions $(X(\mu), y(\mu), S(\mu))$ for $\mu > 0$ and focus on the condition

$$S(\mu) = \mu[X(\mu)]^{-1}.$$

This implies that

$$\begin{aligned} \langle X(\mu), S(\mu) \rangle &= \langle X(\mu), \mu[X(\mu)]^{-1} \rangle \\ &= \mu \operatorname{Tr}(I) \\ &= n\mu. \end{aligned}$$

So as $\mu \to 0$, the duality gap approaches 0.

3.2.2 Path-Following Algorithms

While the central path is theoretically elegant, it is not efficient to follow it exactly. This class of algorithms follow the central path approximately as $\mu \to 0$.

To derive these algorithms, one can use Newton's Method and its variants locally on the system of nonlinear equations.

$$\begin{aligned} \mathcal{A}(X) &= b \\ \mathcal{A}^*(y) + S &= C \\ S &= \mu X^{-1} \end{aligned}$$

or

$$\begin{aligned} \mathcal{A}(X) &= b \\ \mathcal{A}^*(y) + S &= C \\ X &= \mu S^{-1} \end{aligned}$$

or some equivalent systems. We enforce $X^{(k)}, S^{(k)} \succ 0$ by starting a PSD point and careful step size selection rules.

Given a pair of Slater points X, S for (P) and (D) respectively, we can easily measure how close (X, S) is to the central path. One example would be to take $\mu(X, S) := \frac{\mathrm{Tr}(XS)}{n}$ and consider

$$\|S - \mu(X, S)X^{-1}\|.$$

Another more direct way relating to f is

$$\psi(X, S) := n \ln\left(\frac{\mathrm{Tr}(XS)}{n}\right) + f(X) + f(S).$$

Theorem 3.2.3
For every $(X, S) \in \mathbb{S}^n_{++} \oplus \mathbb{S}^n_{++}$, $\psi(X, S) \geq 0$. Moreover, the equality holds if and only if

$$S = \mu X^{-1}, \mu = \frac{\langle X, S\rangle}{n}.$$

Let $\lambda := \lambda(S^{\frac{1}{2}}XS^{\frac{1}{2}})$. Then

$$\psi(X, S) = n \ln\left[\frac{\mathrm{Tr}\left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right)}{n}\right] - \ln\det\left(S^{\frac{1}{2}}XS^{\frac{1}{2}}\right).$$

This is precisely the ratio between the arithmetic mean and geometric mean of entries of λ. Hence it is always nonnegative.

3.2.3 Primal-Dual Potential Function

We use two attributes for judging how good a pair of Slater points (X, S) is:

(i) We want a small duality gap $\langle X, S\rangle$.

(ii) We want to be close to the central path (small $\psi(X, S)$)

For $\rho > 0$, define

$$\phi_\rho(X, S) := \rho \ln\langle X, S\rangle + \psi(X, S)$$

where $\rho > 0$.

Theorem 3.2.4
Suppose $X^{(0)}, S^{(0)} \in \mathbb{S}^n_{++}$ are Slater points for (P), (D) respectively and they satisfy

$$\psi(X^{(0)}, S^{(0)}) \leq \sqrt{n} \ln\left(\frac{1}{\epsilon}\right)$$

for some $\epsilon \in (0, 1)$.
Generate a sequence $\{(X^{(k)}, S^{(k)})\}$ of feasible solutions for (P), (D) respectively such that

$$\phi_{\sqrt{n}}(X^{(k)}, S^{(k)}) \leq \phi_{\sqrt{n}}(X^{(k-1)}, S^{(k-1)}) - \delta$$

for every $k \geq 1$, where $\delta > 0$ is an absolute constant. Then for some $\bar{k} = O\left(\sqrt{n}\ln\left(\frac{1}{\epsilon}\right)\right)$, we have

$$\langle X^{(k)}, S^{(k)}\rangle \leq \epsilon\langle X^{(0)}, X^{(0)}\rangle$$

for all $k \geq \bar{k}$.

We will design an algorithm with the property required in the previous theorem as well as primam-dual symmetry and scale-invariance.

Given the current iteration $(X^{(k)}, S^{(k)})$, we will find a pair of search directions D_X, D_S such that for all $\alpha \geq 0$ (step size), $(X^{(k)} + \alpha D_X)$ and $(S^{(k)} + \alpha D_S)$ satisfy

$$\exists y \in \mathbb{R}^m, \mathcal{A}(X) = b, \mathcal{A}^*(y) + S = C \iff \exists d_y \in \mathbb{R}^m, \mathcal{A}(D_X) = 0, \mathcal{A}^*(d_y) + D_S = 0.$$

To achieve primal-dual symmetry and scale-invarience, for every pair $X, S \in \mathbb{S}^n_{++}$, we find some $T : \mathbb{S}^n \to \mathbb{S}^n$ such that

(i) $T \in \mathrm{Aut}(\mathbb{S}^n_+)$

(ii) $T(S) = T^{-1}(X) =: V$

(iii) $T(X^{-1}) = T^{-1}(S^{-1}) = V^{-1}$

Then, we transform the X-space via T^{-1} and the S-space via T. (X, S) is mapped to (V, V).

$$\begin{aligned}
\bar{\mathcal{A}}(\cdot) &:= \mathcal{A}(T(\cdot)) \\
\bar{C} &:= T(C) \\
\bar{D}_X &:= T^{-1}(D_X) \\
\bar{D}_S &:= T(D_S)
\end{aligned}$$

Thus (P), (D) become

$$\begin{aligned}
&\inf\langle \bar{C}, X\rangle \\
\bar{\mathcal{A}}(X) &= b \\
X &\in T^{-1}(\mathbb{S}^n_+) = \mathbb{S}^n_+
\end{aligned} \tag{$\bar{P}$}$$

$$\begin{aligned}
&\sup b^T y \\
\bar{\mathcal{A}}^*(y) + S &= \bar{C} \\
S &\in T(\mathbb{S}^n_+) = \mathbb{S}^n_+
\end{aligned} \tag{$\bar{D}$}$$

Theorem 3.2.5
For every pair of $X, S \in \mathbb{S}^n_{++}$, there is some $T \in \mathrm{Aut}(\mathbb{S}^n_+)$ such that
(i) $T(S) = T^{-1}(X) =: V$
(ii) $T(X^{-1}) = T^{-1}(S^{-1}) = V^{-1}$

Proof
We find $W \in \mathbb{S}^n_{++}$ such that
$$T(Z) := WZW$$
satisfies the desired condition. Note that given such a W, for every $Z \in \mathbb{S}^n$, $T(Z) \in \mathbb{S}^n$ and $Z \in \mathbb{S}^n_+ \iff T(Z) \in \mathbb{S}^n_+$. Hence $T \in \mathrm{Aut}(\mathbb{S}^n_+$.

For our choice of T, the equation $T(S) = T^{-1}(X)$ is

$$\begin{aligned}
WSW = W^{-1}XW^{-1} &\iff W^2SW^2 = X \\
&\iff S^{\frac{1}{2}}W^2SW^2S^{\frac{1}{2}} = S^{\frac{1}{2}}XS^{\frac{1}{2}} \\
&\iff (S^{\frac{1}{2}}W^2S^{\frac{1}{2}})^2 = S^{\frac{1}{2}}XS^{\frac{1}{2}} \\
&\iff S^{\frac{1}{2}}W^2S^{\frac{1}{2}} = (S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}} \\
&\iff W^2 = S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}}.
\end{aligned}$$

Moreover, we have $W \in \mathbb{S}^n_{++}$ such that

$$\begin{aligned} &WSW = W^{-1}XW^{-1} =: V \\ \iff& W^{-1}S^{-1}W^{-1} = WX^{-1}W = V^{-1} \\ \iff& T^{-1}(S^{-1}) = T(X^{-1}). \end{aligned}$$

Observe that in the proof about $T \in \mathrm{Aut}(\mathbb{S}^n_+)$ is self adjoint, ie $T = T^*$.

3.2.4 Search Direction

Recall that we want a small duality gap as well as being close to the central path, ie small $\psi(X, S) = n \ln(\mathrm{Tr}(XS)/n) + f(x) + f(S)$.

Let $D_S, D_X \in \mathbb{S}^n$ denote the search directions. We define the family of "next" solutions

$$\begin{aligned} X(\alpha) &:= X + \alpha D_X \\ S(\alpha) &:= S + \alpha D_S. \end{aligned}$$

Recall that we require

$$\exists d_y \in \mathbb{R}^m, \mathcal{A}(D_X) = 0, \mathcal{A}^*(d_y) + D_S = 0.$$

Thus

$$\begin{aligned} \langle X(\alpha), S(\alpha)\rangle &= \langle X, S\rangle + \alpha\left[\langle X, D_S\rangle + \langle D_X, S\rangle\right] + \alpha^2 \langle D_X, D_S\rangle \\ &= \langle X, S\rangle + \alpha\left[\langle T^{-1}(X), T(D_S)\rangle + \langle T^{-1}(D_X), T(S)\rangle\right] \quad D_X \in \ker \mathcal{A}, D_S \in \operatorname{Im} \mathcal{A}^* \\ &= \langle X, S\rangle + \alpha \langle V, \bar{D}_X, +\bar{D}_S\rangle. \end{aligned}$$

It is clear the biggest decrease is by taking $\bar{D}_X + \bar{D}_S = -V$. Thus $\bar{D}_X$ is the orthogonal projection of $-V$ onto $\ker(\bar{\mathcal{A}})$ and $\bar{D}_S$ is the orthogonal projection of $-V$ onto $\operatorname{Im}(\bar{\mathcal{A}})$.

It remains to control the distance to the central path.

Lemma 3.2.6
Let $X \in \mathbb{S}^n_{++}$ and $D \in \mathbb{S}^n$ such that

$$\|D\|_X := \langle D, X^{-1}DX^{-1}\rangle^{\frac{1}{2}} \leq 1.$$

Then

$$f(X) + \langle f'(X), D\rangle \leq f(X + D) \leq f(X) + \langle f'(X), D\rangle + \frac{\|D\|_X^2}{2(1 - \|D\|_X)^2}.$$

Proof
Convexity as well as Taylor's theorem.

We can understand the condition of the lemma as

$$\begin{aligned} 1 &\geq \langle D, X^{-1}DX^{-1}\rangle^{\frac{1}{2}} \\ &= \left[\operatorname{Tr}\left(X^{-\frac{1}{2}}DX^{-\frac{1}{2}}\right)^2\right]^{\frac{1}{2}} \\ &= \left\|X^{-\frac{1}{2}}DX^{-\frac{1}{2}}\right\|_F \\ &\geq \lambda_1\left(X^{-\frac{1}{2}}DX^{-\frac{1}{2}}\right). \end{aligned}$$

By using the fact $X^{\frac{1}{2}} \cdot X^{\frac{1}{2}} \in \operatorname{Aut}(\mathbb{S}^n_+)$, this is equivalent to

$$-I \preceq X^{-\frac{1}{2}}DX^{-\frac{1}{2}} \preceq I \iff X \pm D \succeq 0.$$

Focusing on the first-order estimate from the previous lemma,

$$\begin{aligned} \langle f'(X), D_X\rangle + \langle f'(S), D_S\rangle &= \langle -X^{-1}, D_X\rangle + \langle -S^{-1}, D_S\rangle && \text{previous proposition} \\ &= -\langle T(X^{-1}), T^{-1}(D_X)\rangle - \langle T^{-1}(S^{-1}), T(D_S)\rangle \\ &= -\langle V^{-1}, \bar{D}_X + \bar{D}_S\rangle. \end{aligned}$$

This suggests setting

$$\bar{D}_X + \bar{D}_S = \kappa_1 V^{-1} - \kappa_2 V$$

for some $\kappa_1, \kappa_2 > 0$.

Setting $\kappa_1 := 1$ and $\kappa_2 := \frac{n+\sqrt{n}}{\langle X,S\rangle}$ with a suitable choice for step size such that $\alpha := \frac{\lambda_n(V)}{8}$ yields that

$$\phi_{\sqrt{n}}(X(\alpha), X(\alpha)) - \phi_{\sqrt{n}}(X, S) < -\frac{1}{12}$$

which is an absolute constant. Take

$$\begin{aligned} \tilde{U} &:= V^{-1} - \frac{n+\sqrt{n}}{\langle X, S\rangle}V \neq 0 \\ U &:= \frac{\tilde{U}}{\|\tilde{U}\|_F}. \end{aligned}$$

3.2.5 The Algorithm

We assume we are given inputs $X^{(0)}, S^{(0)} \in \mathbb{S}^n_{++}$ and $\epsilon \in (0,1)$ such that $X^{(0)}, S^{(0)}$ are feasible in (P), (D) respectively. Moreover, we assume that $\psi(X^{(0)}, S^{(0)}) \leq \sqrt{n}\ln\left(\frac{1}{\epsilon}\right)$.

Set $k := 0$. While $\langle X^{(k)}, S^{(k)} \rangle > \epsilon \langle X^{(0)}, S^{(0)} \rangle$:

1) $W^2 := (S^{(k)})^{-\frac{1}{2}} \left[(S^{(k)})^{\frac{1}{2}} X^{(k)} (S^{(k)})^{\frac{1}{2}} \right]^{\frac{1}{2}} (S^{(k)})^{\frac{1}{2}}$
2) $\bar{\mathcal{A}} := \mathcal{A}(W \cdot W)$
3) $V := W S^{(k)} W$
4) $\tilde{U} := V^{-1} - \frac{n+\sqrt{n}}{\langle X^{(k)}, S^{(k)} \rangle} V$
5) $U := \frac{\tilde{U}}{\|\tilde{U}\|_F}$
6) Solve the linear system of equations

$$\begin{aligned} \bar{\mathcal{A}}(\bar{D}_X) &= 0 \\ \bar{\mathcal{A}}^*(d_y) + \bar{D}_S &= 0 \\ \bar{D}_X + \bar{D}_S &= U \end{aligned}$$

7) Compute

$$\begin{aligned} \bar{\alpha} &:= \operatorname{argmin}\left\{ \phi_{\sqrt{n}}(X(\alpha), S(\alpha)) : \alpha > 0 \right\} \\ X^{(k+1)} &:= X^{(k)} + \bar{\alpha} W \bar{D}_X W \\ S^{(k+1)} &:= S^{(k)} + \bar{\alpha} W^{-1} \bar{D}_S W^{-1} \\ k &:= k + 1 \end{aligned}$$

Theorem 3.2.7
The above algorithm terminates in at most

$$24\sqrt{n} \ln \left(\frac{1}{\epsilon} \right)$$

iterations with $X^{(k)}, S^{(k)}$ feasible in (P), (D) respectively satisfying

$$\langle X^{(k)}, S^{(k)} \rangle \leq \epsilon \langle X^{(0)}, S^{(0)} \rangle.$$

3.2.6 Initial Solution

How can we obtain initial Slater points? Introduce an artificial variable $\xi \geq 0$ and construct the auxiliary SDP

$$\begin{aligned} \inf \xi & & (P_{\text{aux}}) \\ \mathcal{A}(X) + \xi(b - \mathcal{A}(I)) &= b & \\ \langle I, X\rangle &\leq M & \text{large constant} \\ \xi &\geq 0 & \\ X &\geq 0 & \end{aligned}$$

Then $(X^{(0)}, \xi_0) := (I, 1)$ is Slater point for (P_{aux}).

The dual is

$$\begin{aligned} \sup b^T y + M\eta & & (D_{\text{aux}}) \\ \mathcal{A}^*(y) + \eta I + S &= 0 & \\ b^T y - \operatorname{Tr}(\mathcal{A}^*(y)) &\leq 1 & \\ \eta &\leq 0 & \\ S &\succ 0 & \end{aligned}$$

Notice that $(y^{(0)}, S^{(0)}, \eta_0) := (0, I, -1)$ is a Slater point for the dual auxiliary program.

For this starting pair,

$$\psi(\dots, \dots) = (n+1)\ln\left(\frac{M+1}{n+2}\right) - \ln(M-n).$$

In order to actually find a Slater point, we need to further introduce a small constant γ and the additional constraint that

$$X \succeq 0 \mapsto X \succeq \gamma I.$$

If the optimal value fo ξ is positive, we can only say that the system

$$\{X \in \mathbb{S}^n : X \succeq \gamma I \wedge \operatorname{Tr}(X) \leq M\}$$

does not contain any feasible solutions of (P). But since we can control γ, M, we simply increase M, decrease γ, and try again.

3.2.7 Remarks

In the case of LP problems with rational data, we can pick $\gamma \approx 2^{-L}$ and $M \approx 2^L$ where L is the number of bits required to express the data (A, b, c). However, we can construct instances of SDP with data containing only 0, 1, 2 where

$$\gamma \approx 2^{2^{-L}} \vee M \approx 2^{2^L}.$$

Another computational difficulty is that given a feasible solution of an LP whose objective value is within 2^{-2L} of the optimum, then EVERY extreme point whose objective is at least as good as the feasible solution is optimal. Thus we can compute an exact optimal sollution very efficiently.

However, given a SDP solution, we can compute "in practice" an extreme point solution whose objective value is at least as good, BUT there may be infinitely many extreme point solutions of (P) that are strictly better.

Furthermore, SDPs may have a unique optimal solution that is irrational.

Problem 1 (SDP Feasibility)
Given $A_1, \dots, A_m \in \mathbb{S}^n \cap \mathbb{Z}^{n\times n}$ and $b \in \mathbb{Z}^m$, does there exist $\bar{X} \in \mathbb{S}^n_+$ such that

$$\langle A_i, \bar{X} \rangle = b_i$$

for each $i \in [m]$.

It is an open problem whether SDP feasibility lies in P.

In theoretical applications, the ellipsoid method is very powerful. Interior-point algorithms have better complexity bounds and in applications requiring high accuracy, if we can perform one iteration in a reasonable time, they are hard to beat.

However, when we cannot even perform a single interation of an interior-point algorithm (the instance is huge and there is no easily exploitable structure), we resort to first-order algorithms (not ellipsoid method).

Chapter 4

Approximation Algorithms Based on SDP

4.1 Maximum Cut

Problem 2
Given an undirected graph $G = (V, E)$ and $w \in \mathbb{R}^E_+$, find $U \subseteq V$ maximizing

$$\sum_{ij \in \delta(U)} w_{ij}.$$

4.1.1 Nonconvex Optimization Formulation

Let $n := |V|$. We represent each cut by a vector $u \in \{-1, 1\}^n$. such that $u_i = 1$ if and only if $i \in U$. Also, extend w to be 0 on all non edge pairs $i, j \in V$. Then the problem is

$$\begin{aligned} \max \frac{1}{4} \sum_{i \in V} \sum_{j \in V} & w_{ij}(1 - u_i u_j) \qquad (P) \\ u \in \{-1, 1\}^n & \end{aligned}$$

Let $W \in \mathbb{S}^n$ be such that $W_{ij} = w_{ij}$. Put $\bar{e}$ as the all-ones vector.

An equivalent nonconvex formulation is the following:

$$\begin{aligned} \max \frac{1}{4}\langle W, \bar{e}\bar{e}^T\rangle - \frac{1}{4}\langle W, X\rangle & \qquad (P) \\ \operatorname{diag}(X) = \bar{e} & \\ X \succeq 0 & \\ \operatorname{rank} X = 1 & \end{aligned}$$

Here we identity $X \iff uu^T$.

4.1.2 SDP Relaxation

We simply drop the rank 1 constraint.

$$\max -\frac{1}{4}\operatorname{Tr}(WX) + \frac{1}{4}\bar{e}^T W \bar{e} \qquad \text{(SDP)}$$

$$\begin{aligned} \operatorname{diag}(X) &= \bar{e} \\ X &\succeq 0 \end{aligned}$$

$$\min \bar{e}^T y + \frac{1}{4}\bar{e}^T W \bar{e} \qquad \text{(SDD)}$$

$$\begin{aligned} \operatorname{Diag}(y) - S &= -\frac{1}{4}W \\ S &\succeq 0 \end{aligned}$$

Now,

$$\begin{aligned} \bar{X} &:= I \\ \bar{y} &:= \left(\frac{1}{4}\bar{e}^T W \bar{e} + 1\right)\bar{e} \end{aligned}$$

yield Slater points for (SDP) and (SDD).

4.1.3 Goemans-Williamson Algorithm

If we find an exact optimal solution $\bar{X}$ usch that $\operatorname{rank} \bar{X} = 1$, then we are done. Otherwise, let

$$\begin{aligned} \bar{X} &= BB^T \\ B^T &=: \begin{bmatrix} v^{(1)} & \dots & v^{(n)} \end{bmatrix} \\ v^{(i)} &\in \mathbb{R}^d \\ d &\leq n \end{aligned}$$

Thus $\hat{X}_{ij} = \langle v^{(i)}, v^{(j)} \rangle$ for all $i, j \in [n]$. Since each diagonal entry is 1,

$$1 = \hat{X}_{ii} = \langle v^{(i)}, v^{(i)} \rangle = \|v^{(i)}\|^2$$

for each $i \in [n]$.

Random Hyperplane Technique

We generate $r \in \mathbb{R}^d$ on the unit hypersphere randomly. Then, set

$$U := \{i \in V : r^T v^{(i)} \geq 0\}$$

to be a cut in G.

Lemma 4.1.1
Let $v^{(i)}$ and r be as above. Then

$$P\left(\operatorname{sign}(r^T v^{(i)}) \neq \operatorname{sign}(r^T v^{(j)})\right) = \frac{\theta}{\pi},$$

where

$$\theta := \arccos\langle v^{(i)}, v^{(j)} \rangle.$$

For $v \in \mathbb{R}^d$, we write $\operatorname{sign}(v) \in \{-1, 1\}^d$ to denote the vector where

$$[\operatorname{sign}(v)]_j := \begin{cases} 1, & v_j \geq 0 \\ -1, & v_j < 0 \end{cases}$$

Lemma 4.1.2
For every $u \in [-1, 1]$, we have

$$\begin{aligned} \frac{1}{\pi} \arccos(u) &\geq \frac{\rho}{2}(1 - u) \\ 1 - \frac{1}{\pi} \arccos(u) &\geq \frac{\rho}{2}(1 + u) \end{aligned}$$

where $\rho := 0.87856$.

Theorem 4.1.3
The expected weight of the cut generated by the Random Hyperplane Technique based on $\hat{X}$ is at least

$$\frac{\rho}{4} \sum_{i,j \in V} w_{ij}(1 - \langle v^{(i)}, v^{(j)} \rangle) = \rho \, \mathrm{OPT}(P).$$

Proof
Let $\hat{X} \in \mathbb{S}_+^n$ be an optimal solution of (SDP). Then

$$\begin{aligned} E[\text{RHT-Cut}] &= \sum_{i,j} w_{ij} \frac{\arccos\big(\langle v^{(i)}, v^{(j)} \rangle\big)}{2\pi} && \text{previous lemma} \\ &\geq \frac{\rho}{4} \sum_{i,j} w_{ij}(1 - \langle v^{(i)}, v^{(j)} \rangle) && \text{previous lemma} \\ &= \rho \cdot \mathrm{OPT}(\mathrm{SDP}). \end{aligned}$$

Theorem 4.1.4
Let $G = (V, E)$ with $w \in \mathbb{Q}_+^E$ be given. Then a cut of value at least $\rho \cdot \mathrm{OPT}$ can be generated in polynomial time.

Note that we do not need a exact optimal solution of (SDP) as an approximate optimal solution $\tilde{X}$ suffices. Moreover, the algorithm can be "derandomized". This approximation ratio is the best possible unless the "Unique Games Conjecture" is false.

4.2 Maximum Satisfiability

Problem 3 (Satisfiability)
Given a boolean formual in CNF, decide whether there is an assignment of values to the variables so that the formula evaluates to "True"

If the given formula is $C_1 \wedge \cdots \wedge C_m$, and consider the following integer programming formulation.

$$\begin{aligned} \sum_{j: x_j \in C_i} x_j + \sum_{j: \bar{x}_i \in C_i} (1 - x_j) &\geq 1 && \forall i \in [m] \\ x &\in \{0,1\}^n \end{aligned}$$

Problem 4 (MaxSat)
Given a boolean formula $C_1 \wedge \cdots \wedge C_m$ and weights on the clauses $w_i \in \mathbb{R}_+$, find an assignment of values to the variables which maximizes the total weight of the satisfied clauses.

We can consider either k-SAT or Max k-SAT, where every clause has at most k literals.

Theorem 4.2.1
For every $k \geq 3$, k-SAT is NP-complete.
For every $k \geq 2$, Max k-SAT is NP-hard.

Max 2-SAT is closely related to MaxCut. Let G be an instance of MaxCut where every edge has weight one. Make a variable x_v for all $v \in V$. Create a clause $(x_u \vee x_v)$ for each $uv \in E$ with weight $w_{uv} = 2$. Finally, make a clause $(\bar{x}_v)$ for every $v \in V$ with weight $w_v := |\delta(v)|$. Then

$$\mathrm{OPT}(\text{Max 2-SAT}) = \mathrm{OPT}(\text{MaxCut}) + 2|E|.$$

Approximation results for MaxCut extend to Max 2-SAT. Can we extend them to more general nonconvex optimization problem?

4.3 Quadratic Optimization over Sign Vectors

Let $W \in \mathbb{S}^n$. Consider the following quadratic programs.

$$\begin{aligned} \bar{f}(W) := \max_{x \in \{-1,1\}^n} x^T W x = \max \operatorname{Tr}(WX) & \\ \operatorname{diag}(X) &= \bar{e} \\ X &\succeq 0 \\ \operatorname{rank} X &= 1 \end{aligned}$$

$$\begin{aligned} \underline{f}(W) := \min_{x \in \{-1,1\}^n} x^T W x = \min \operatorname{Tr}(WX) & \\ \operatorname{diag}(X) &= \bar{e} \\ X &\succeq 0 \\ \operatorname{rank}(X) &= 1 \end{aligned}$$

Their SDP relaxations are

$$
\begin{aligned}
\bar{F}(W) &:= \max \operatorname{Tr}(WX) \\
\operatorname{diag}(X) &= \bar{e} \\
X &\succeq 0
\end{aligned}
$$

$$
\begin{aligned}
\underline{F}(W) &:= \min \operatorname{Tr}(WX) \\
\operatorname{diag}(X) &= \bar{e} \\
X &\succeq 0
\end{aligned}
$$

Moreover, the SDP duals are

$$
\begin{aligned}
&\min \bar{e}^T y \\
\operatorname{Diag}(y) &\succeq W
\end{aligned}
$$

$$
\begin{aligned}
&\max \bar{e}^T y \\
\operatorname{Diag}(y) &\preceq w
\end{aligned}
$$

Note that both the primal and dual programs have Slater points as constructed in the MaxCut relaxation.

Proposition 4.3.1
For every $W \in \mathbb{S}^n$, we have

(i) $\underline{f}(W) = -\bar{f}(-W)$

(ii) $\underline{F}(W) = -\bar{F}(-W)$

(iii) $\underline{F}(W) \leq \underline{f}(W) \leq \bar{f}(W) \leq \bar{F}(W)$

We can apply the random hyperplane technique here!

Lemma 4.3.2
Let $W \in \mathbb{S}^n$. Then $\bar{f}(W)$ is equal to the two optimization problems

$$\begin{aligned} \max \xi^T W \xi & & (\hat{P}) \\ \xi = \operatorname{sign}(Br) & & \\ \|B^T e_i\|_2 = 1 & & \forall i \\ \|r\|_2 = 1 & & \\ B \in \mathbb{R}^{n\times n} & & \\ r \in \mathbb{R}^n & & \end{aligned}$$

and

$$\begin{aligned} \max E_r[\xi^T W \xi] & & (\tilde{P}) \\ \xi = \operatorname{sign}(Br) & & \\ \|B^T e_i\|_2 = 1 & & \forall i \\ \|r\|_2 = 1 & & \\ B \in \mathbb{R}^{n\times n} & & \\ r \in \mathbb{R}^n & & \end{aligned}$$

Proof
It is clear that $\mathrm{OPT}(\hat{P}) \geq \mathrm{OPT}(\tilde{P})$. Moreover, the constraints from $(\hat{P})$ ensure that ξ is always a sign vector. Hence it must be that $\bar{f}(W) \geq \mathrm{OPT}(\hat{P})$.

$\underline{\bar{f}(W) \leq \mathrm{OPT}(\bar{P})}$: Now suppose that $\bar{f}(W) = \bar{w}^T W \bar{x}$. Pick any $r \in \mathbb{R}^n$ with $\|r\|_2 = 1$. Define $B \in \mathbb{R}^{n\times m}$ by

$$B^T e_i = \begin{cases} r, & \bar{x}_i = 1 \\ -r, & \bar{x}_i = -1 \end{cases}$$

Then $\xi = \bar{x}$.

$\underline{\bar{f}(W) \leq \mathrm{OPT}(\tilde{P})}$: Again, suppose that $\bar{f}(W) = \bar{w}^T W \bar{x}$. Define $B \in \mathbb{R}^{n\times n}$ by

$$B^T e_i := \begin{cases} \frac{1}{\sqrt{n}}\bar{x}, & \bar{x}_i = 1 \\ -\frac{1}{\sqrt{n}}\bar{x}, & \bar{x}_i = -1 \end{cases}$$

Then

$$\begin{aligned} E_r\left[\operatorname{sign}(r^T B^T e_i)\operatorname{sign}(r^T B^T e_j)\right] &= \begin{cases} 1, & \bar{x}_i = \bar{x}_j \\ -1, & \bar{x}_i \neq \bar{x}_j \end{cases} \\ &= \bar{x}_i \bar{x}_j & \forall i, j \end{aligned}$$

Notice that we used the fact that $r \in B(0,1)$ restricted to being orthogonal to both $B^T e_i, B^T E_j$ has zero $(n-1)$-th dimensional measure.

But then

$$\begin{aligned} E_r[\xi^T W \xi] &= \sum_{i,j} W_{ij} E_r[\xi_i \xi_j] \\ &= \sum_{i,j} W_{ij} \bar{x}_i \bar{x}_j \\ &= \bar{x}^T W \bar{x} \\ &= \bar{f}(W). \end{aligned}$$

Lemma 4.3.3
For every $W \in \mathbb{S}^n$, $\bar{f}(W)$ is equal to

$$\begin{aligned} \max \frac{2}{\pi} \langle W, \arcsin(X) \rangle \qquad & \qquad (P') \\ \operatorname{diag}(X) = \bar{e} & \\ X \succeq 0 & \end{aligned}$$

Assuming this lemma, we can make the difficult $\operatorname{rank}(X) = 1$ constraint and convert it difficulty into the objective function. So that two following problems are equivalent

$$\begin{aligned} \max \operatorname{Tr}(WX) & \\ \operatorname{diag}(X) &= \bar{e} \\ X &\succeq 0 \\ \operatorname{rank} X &= 1 \end{aligned}$$

$$\begin{aligned} \max \frac{2}{\pi} \operatorname{Tr}(W \arcsin(X)) & \\ \operatorname{diag}(X) &= \bar{e} \\ X &\succeq 0 \end{aligned}$$

Proof
Since $X \succeq 0$, $\operatorname{diag}(X) = \bar{e}$ implies $|X_{ij}| \leq 1$. So the problem is well-defined. The feasible region is nonempty and compact, with the objective function being continuous over the feasible region. Hence the maximum is finite and attained.

$\underline{\bar{f}(W) \leq \mathrm{OPT}(P')}$: Given W, apply the previous lemma to see that

$$\bar{f}(W) = \max\left\{E_r[\xi^T W \xi] : \xi = \mathrm{sign}(Br) \wedge \forall i, \|B^T e_i\|_2 = 1 \wedge \|r\|_2 = 1 \wedge B \in \mathbb{R}^{n\times n} \wedge r \in \mathbb{R}^n\right\}.$$

Let $\hat{B} \in \mathbb{R}^{n\times n}$ be an optimal solution of this last problem and write

$$\hat{B}^T =: \begin{bmatrix} v^{(1)} & \dots & v^{(n)} \end{bmatrix}.$$

Then

$$\begin{aligned}
&E_r\left[\mathrm{sign}(r^T\hat{B}^Te_i)\,\mathrm{sign}(r^T\hat{B}^Te_j)\right] \\
&= P\left(\mathrm{sign}(r^T\hat{B}^Te_i) = \mathrm{sign}(r^T\hat{B}^Te_j)\right) - P\left(\mathrm{sign}(r^T\hat{B}^Te_i)i \neq \mathrm{sign}(r^T\hat{B}^Te_j)\right) \\
&= 1 - 2P(\mathrm{sign}(r^T\hat{B}^Te_i) \neq \mathrm{sign}(r^T\hat{B}^Te_j)) \\
&= 1 - \frac{2}{\pi}\arccos\langle v^{(i)}, v^{(j)}\rangle \\
&\text{(Random Hyperplane Method Lemma)} \\
&= \frac{2}{\pi}\arcsin\langle v^{(i)}, v^{(j)}\rangle \qquad \forall i, j
\end{aligned}$$

Thus the objective value is precisely

$$E_r\left[\mathrm{sign}(\hat{B}r)^T W\,\mathrm{sign}(\hat{B}r)\right] = \frac{2}{\pi}\left\langle W, \arcsin\left(\hat{B}\hat{B}^T\right)\right\rangle.$$

$\underline{\bar{f}(W) \geq \mathrm{OPT}(P')}$: Let $X' \in \mathbb{S}^n_+$ be an optimal solution of

$$\max\{\mathrm{Tr}(W\arcsin(X)) : \mathrm{diag}(X) = \bar{e}, X \succeq 0\}.$$

Let $B' \in \mathbb{R}^{n\times n}$ be such that $X' = B'B'^T$. We have

$$\frac{2}{\pi}\mathrm{Tr}(W\arcsin(X')) = E_r\left[\mathrm{sign}(B'r)^T W\,\mathrm{sign}(B'r)\right].$$

An application of a previous lemma yields the desired inequality.

Lemma 4.3.4
For every $X \in \mathbb{S}^n_+$ such that $|X_{ij}| \leq 1$ for each i, j. We have

$$\arcsin(X) \succeq X.$$

Theorem 4.3.5
For every $W \in \mathbb{S}^n_+$, $\bar{f}(W)$ is at least

$$\begin{aligned} \max \frac{2}{\pi}\langle W, X\rangle \\ \operatorname{diag}(X) = \bar{e} \\ X \succeq 0 \end{aligned}$$

Thus for every $W \in \mathbb{S}^n_+$,

$$\frac{2}{\pi}\bar{F}(W) \le \bar{f}(W) \le \bar{F}(W).$$

The last theorem assumes $W \in \mathbb{S}^n_+$, which includes MaxCut instances as a special case: TAke $W \in \mathbb{S}^V$ to be the Laplacian of G with respect to weights w. W is *diagonall dominant*, hence by the Gershgorin Disk Theorem, $W \succeq 0$.

4.3.1 Arbitrary Weights

In SDP relaxations defining $\underline{F}(W)$ and $\bar{F}(W)$, we have the dual constraints $W - \operatorname{Diag}(y) \succeq 0$ and $\operatorname{Diag}(y) - W \succeq 0$. Moreover, we can make any $W \in \mathbb{S}^n$ PSD by adding a multiple of the identity. This motivates changes to $\underline{f}, \bar{f}, \underline{F}, \bar{F}$ under diagonal perturbations.

Fix $y \in \mathbb{R}^n$,

$$\begin{aligned} \underline{f}(W + \operatorname{Diag}(y)) &= \min_{x \in \{-1,1\}^n} x^T W x + x^T \operatorname{Diag}(y) x \\ &= \underline{f}(W) + \bar{e}^T y \\ \bar{f}(W + \operatorname{Diag}(y)) &= \bar{f}(W) + \bar{e}^T y. \end{aligned}$$

In similar fashion,

$$\begin{aligned} \underline{F}(W + \operatorname{Diag}(y)) &= \min_{X \succeq 0, \operatorname{diag}(X) = \bar{e}} \operatorname{Tr}(WX) + \langle \operatorname{Diag}(y), X\rangle \\ &= \underline{F}(W) + \bar{e}^T y \\ \bar{F}(W + \operatorname{Diag}(y)) &= \bar{F}(W) + \bar{e}^T y \end{aligned}$$

Theorem 4.3.6
For every $W \in \mathbb{S}^n$, we have

$$\begin{aligned}
\underline{F}(W) &\leq \underline{f}(W) \\
&\leq \frac{2}{\pi}\underline{F}(W) + \left(1 - \frac{2}{\pi}\right)\bar{F}(W) \\
&\leq \left(1 - \frac{2}{\pi}\right)\underline{F}(W) + \frac{2}{\pi}\bar{F}(W) \\
&\leq \bar{f}(W) \\
&\leq \bar{F}(W).
\end{aligned}$$

Proof
Let $\bar{y} \in \mathbb{R}^n$ be an optimal solution to the dual of the SDP relaxation determining $\bar{F}(W)$:

$$\begin{aligned}
\bar{F}(W) &= \max\{\mathrm{Tr}(WX) : \mathrm{diag}(X) = \bar{e}, X \in \mathbb{S}^n_+\} \\
&= \min\{\bar{e}^T y : \mathrm{Diag}(y) \succeq W\} \\
&= \bar{e}^T \bar{y}
\end{aligned}$$

such that $\mathrm{Diag}(\bar{y}) - W \succeq 0$.

We have

$$\begin{aligned}
\bar{F}(W) - \underline{f}(W) &= \bar{e}^T\bar{y} + \bar{f}(-W) && \text{previous proposition} \\
&= \bar{f}(\mathrm{Diag}(\bar{y}) - W) && \text{observation above} \\
&\geq \bar{F}(\mathrm{Diag}(\bar{y}) - W) && \mathrm{Diag}(\bar{y}) - W \succeq 0 \\
&= \frac{2}{\pi}\left[\bar{e}^T\bar{y} + \bar{F}(-W)\right] && \text{observation above} \\
&= \frac{2}{\pi}\left[\bar{e}^T\bar{y} - \underline{F}(W)\right] && \text{previous proposition} \\
&= \frac{2}{\pi}\left[\bar{F}(W) - \underline{F}(W)\right] && \text{strong duality} \\
\underline{f}(W) &\leq \frac{2}{\pi}\underline{F}(W) + \left(1 - \frac{2}{\pi}\right)\bar{F}(W).
\end{aligned}$$

The case for $\bar{f}$ is identical.

$$\bar{f}(W) \geq \left(1 - \frac{2}{\pi}\right)\underline{F}(W) + \frac{2}{\pi}\bar{F}(W).$$

Corollary 4.3.6.1
For every $W \in \mathbb{S}^n$, the value

$$v := \left(1 - \frac{2}{\pi}\right) \underline{F}(W) + \frac{2}{\pi}\bar{F}(W)$$

satisfies

$$\frac{\bar{f}(W) - v}{\bar{f}(W) - \underline{f}(W)} < \frac{4}{7}.$$

This also allows us to handle linear terms in the objective function. Suppose $W \in \mathbb{S}^n, w \in \mathbb{R}^n$ are given.

$$\begin{aligned} \max_{x \in \{-1,1\}^n} 2w^T x + x^T W x &= \max_{x \in \{-1,1\}^n, x_0 \in \{-1,1\}} 2x_0 w^T x + x^T W x \\ &= \max_{(x,x_0) \in \{-1,1\}^{n+1}} \begin{bmatrix} x_0 & x^T \end{bmatrix} \begin{bmatrix} 0 & w^T \\ w & W \end{bmatrix} \begin{bmatrix} x_0 \\ x \end{bmatrix}. \end{aligned}$$

4.4 Burer-Monteiro Approach

This is a first-order algorithm which has good practical performance in some large-scale instances. It involves a simple nonconvex reformulation.

$$\begin{aligned} \max \operatorname{Tr}(WX) \\ \operatorname{diag}(X) &= \bar{e} \\ X &\succeq 0 \end{aligned}$$

is equivalent to the system

$$\begin{aligned} \max \operatorname{Tr}\big(WLL^T\big) \\ \operatorname{diag}(LL^T) &= \bar{e} \\ L &\in \mathbb{T}^n \end{aligned} \qquad (P_\Delta)$$

where $\mathbb{T}^n$ is the space of $n \times n$ lower triangular matrices.

Note that any $L \in \mathbb{T}^n$ with no zero rows can be made feasible through scaling. This is because $(LL^T)_{ii} = \langle L_i, L_i \rangle$. Moreover, we can restrict L to $\mathbb{T}^{n,r}$, the lower triangular matrices that are $n \times r$ for some $r < n$. This way we can enforce that

$$\operatorname{rank}(LL^T) = \operatorname{rank}(L) \leq r.$$

Once we choose r, we can easily construct $L^{(0)} \in \mathbb{T}^{n,r}$ such that $\operatorname{diag}(L^{(0)}(L^{(0)})^T) = \bar{e}$. Then in each iteration k, we compute the gradient of the objective function at $L^{(k-1)}$ and project this gradient so that a linearization of the constraints is satisfied:

$$\operatorname{diag}\left[(L^{(k-1)} + d_L)(L^{(k-1)} + d_L)^T\right] = \bar{e}$$

ignoring the quadratic term in d_L yields

$$\operatorname{diag}[L^{(k-1)} d_L^T + d_L (L^{(k-1)})^T] = 0.$$

This projected gradient determines the search direction d_L. Then, choosing a step size $\alpha > 0$ (ie satisfying Armijo-Goldstein-Wolfe conditions, etc) for the objective function. Then take

$$L^{(k)} := L^{(k-1)} + \alpha d_L.$$

Again, scale the rows of $L^{(k)}$ so that every row has unit 2-norm.

There are many first-order algorihtms for solving the SDP relaxation of MaxCut problems as well as general SDPs. These include the bundle methods, multiplicative weights updated methods, proximal point algorithms, as well as software SDPNAL+.

Chapter 5

Geometric Representation of Graphs Based on SDP

5.1 Geometric Representation

Let $G = (V, E)$ be an undirected graph.

Definition 5.1.1 (Geometric Representation)
A map $u : V \to \mathbb{R}^d$ for some $d \geq 0$.

A geometric representation u of G is an *unit-distance representation* of G, if for all $ij \in E$,

$$\|u(i) - u(j)\|_2 = 1.$$

Theorem 5.1.1
Every graph $G = (V, E)$ admits a unit distance representation in $\mathbb{R}^{n-1}$ where $n := |V|$.

Proof
Embed K_n as the vertices of a simplex in $\mathbb{R}^{n-1}$ where every edge of the simplex is of unit length.

The geometric representations of graphs have an amazing range of applications.

We can define the dimension of a graph $\dim(G) := d \in \mathbb{Z}_+$ for which G admits a unit distance representation in $\mathbb{R}^d$.

Theorem 5.1.2
Deciding whether $\dim G \leq 2$ is NP-hard.

Consider instead computing a unit distance representation of G which is contained in an Euclidean ball with smallest possible radius. Let $T_b(G)$ denote the square of this minimum radius.

Theorem 5.1.3
For every graph $G = (V, E)$, $t_b(G)$ is the optimal solution to

$$\begin{aligned} \min t & \\ X_{ii} \leq t & \qquad \forall i \in V \\ X_{ii} + X_{jj} - 2X_{ij} = 1 & \qquad \forall ij \in E \\ X \in \mathbb{S}^V_+ & \end{aligned}$$

First we construct Slater points for both the primal and dual SDP to ensure the optimums are attained. Given an optimal solution

$$\hat{X} = BB^T$$

where $B \in \mathbb{R}^{n \times k}$ for some $k \leq n-1$, we have

$$B^T = \begin{bmatrix} u(1) & u(2) & \dots & u(n) \end{bmatrix}$$

so $u(i) \in \mathbb{R}^k$. Then $\langle u(i), u(j) \rangle = \hat{X}_{ii}$. Hence the t is the maximum squared Euclidean norm of the representation and $\|u(i) - (j)\|^2 = \hat{X}_{ii} + \hat{X}_{jj} - \hat{2}X_{ij}$ for all $ij \in E$.

The reverse direction is by putting the unit distance representation in B^T and considering $\tilde{X} = BB^T$.

Next, consider applying a unit distance representation of G contained in a hypersphere of minimum radius. Let $t_h(G)$ denote the square of this minimum radius.

Theorem 5.1.4
For every graph $G = (V, E)$, $t_h(G)$ is equal to the optimal value of

$$\begin{aligned} \min t & \\ \operatorname{diag}(X) = t\bar{e} & \\ X_{ii} + X_{jj} - 2X_{ij} = 1 & \qquad \forall ij \in E \\ X \in \mathbb{S}^V_+ & \end{aligned}$$

Moreover, $t_h(G) = t_b(G)$.

These SDPs provide EXACT mathematical models for their respective problems. Moreover, we can use these ideas for other applications.

Corollary 5.1.4.1
For every graph $G = (V, E)$,

$$t_b(G) \le t_h(G) \le \frac{1}{2} - \frac{1}{2|V|} < \frac{1}{2}.$$

Proof
For every graph G, every unit distance representation contained in a hypersphere of radius r is also contained in an Euclidean ball of radius r.

Note that

$$t_h(G) \le t_h(K_n).$$

For every $\epsilon > 0$ consider

$$\begin{aligned}
X(\epsilon) &:= \frac{1}{2} I - \epsilon \bar{e}\bar{e}^T \\
t(\epsilon) &:= \frac{1}{2} - \epsilon \\
[X(\epsilon)]_{ii} &= t(\epsilon) && \forall i \in V \\
&[X(\epsilon)]_{ii} + [X(\epsilon)]_{jj} - 2[X(\epsilon)]_{ij} \\
&= \frac{1}{2} - \epsilon + \frac{1}{2} - \epsilon + 2\epsilon \\
&= 1 && \forall i \neq j
\end{aligned}$$

Moreover, for every $h \in \mathbb{R}^n$ such that $\|h\| = 1$,

$$\begin{aligned}
h^T X(\epsilon) h &= \frac{1}{2}\|h\|_2^2 - \epsilon(\bar{e}^T h)^2 \\
&\ge \frac{1}{2} - n\epsilon \\
&\ge 0 && \forall \epsilon \le \frac{1}{2n}
\end{aligned}$$

Therefore $\left[X\left(\frac{1}{2n}\right), t\left(\frac{1}{2n}\right)\right]$ is a feasible solution to the SDP in Theorem 63.

5.2 Orthonormal Representation of Graphs

Definition 5.2.1 (Orthonormal Graph Representation)
Given a graph $G = (V, E)$, $v : V \to \mathbb{R}^d$ is an orthonormal representation of G if

(i) $\|v(i)\|_2 = 1$ for all $i \in V$

(ii) $\langle v(i), v(j) \rangle = 0$ for all $ij \in \bar{E}$

Consider a unit-distance hypersphere representation of G, say $u : V \to \mathbb{R}^d$ on a hypersphere of radius $\sqrt{t}$ with $t < \frac{1}{2}$. Let $v : V \to \mathbb{R}^{d+1}$ be obtained from u as follows

$$v(i) := \sqrt{2} \begin{bmatrix} \sqrt{\frac{1}{2} - t} \\ u(i) \end{bmatrix}$$

for every $i \in V$.

Indeed, for every $i \in V$,

$$\begin{aligned} \|v(i)\|_2^2 &= 2\left(\frac{1}{2} - t + \langle u(i), u(i) \rangle\right) \\ &= 2\left(\frac{1}{2} - t + t\right) \\ &= 1 \end{aligned}$$

and for all $ij \in E$,

$$\begin{aligned} \langle v(i), u(j) \rangle &= 2\left(\frac{1}{2} - t + \langle u(i), u(j) \rangle\right) \\ &= 2\left(\frac{1}{2} - t + \frac{1}{2}(X_{ii} + X_{jj} - 1)\right) \\ &= 2\left(\frac{1}{2} - t + \frac{1}{2}(2t - 1)\right) \\ &= 0 \end{aligned}$$

Thus $v : V \to \mathbb{R}^{d+1}$ is an orthonormal representation of $\bar{G}$.

Theorem 5.2.1
Every graph $G = (V, E)$ admits an orthonormal representation in $\mathbb{R}^n$, where $n := |V|$. Moreover, all orthonormal representations of G can be realized in $\mathbb{R}^n$.

Suppose now that we are given an orthonormal representation of G, say $v : V \to \mathbb{R}^d$. We claim that $u(i) := \frac{1}{\sqrt{2}} v(i)$ is a unit distance representation of $\bar{G}$ on a hypersphere.

Indeed, for each $ij \in \bar{E}$,

$$\begin{aligned} \|u(i) - u(j)\|_2^2 &= \frac{1}{2} + \frac{1}{2} - 0 \\ &= 1. \end{aligned}$$

For each $i \in V$,

$$\begin{aligned} \|u(i)\|_2^2 &= \frac{1}{2}\|v(i)\|_2^2 \\ &= \frac{1}{2}. \end{aligned}$$

5.2.1 Lovász Theta Body

An important application of orthonormal representations of graphs is towards the stable/independent set problem. The *stability number* of G is

$$\alpha(G) := \max\{|S| : S \text{ is a stable set in } G\}.$$

The stable set polytope is defined by

$$\mathrm{STAB}(G) := \mathrm{conv}\big\{\chi_S \in \{0,1\}^V : S \text{ is a stable set of } G\big\}.$$

We can also consider the fractional stable set polytope

$$\mathrm{FRAC}(G) := \big\{x \in [0,1]^V : \forall ij \in E, x_i + x_j \leq 1\big\}.$$

Notice that

$$\mathrm{STAB}(G) = \mathrm{conv}\left(\mathrm{FRAC}(G) \cap \{0,1\}^V\right).$$

For every clique C in G the clique inequality $x(C) \leq 1$ is a valid inequality for $\mathrm{STAB}(G)$. Let $A_{\mathrm{clq}}(G)$ denote the clique-node incidence matrix of G. Then the clique polytope of G is given by

$$\mathrm{CLQ}(G) := \big\{x \in \mathbb{R}_+^V : A_{\mathrm{clq}}(G)x \leq \bar{e}\big\}.$$

We define the *theta body* of a graph G as

$$\begin{aligned} \mathrm{TH}(G) := \{x \in \mathbb{R}_+^V : &\forall c \in \mathbb{R}^n, \|c\|_2^2 = 1, \\ &\forall u : V \to \mathbb{R}^n, \text{orthonormal representation of } G, \\ &\sum_{j=1}^n [c^T u(j)]^2 x_j \leq 1\}. \end{aligned}$$

Theorem 5.2.2
For every graph G, $\mathrm{TH}(G)$ is nonempty, compact, convex, and satisfies

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{CLQ}(G) \subseteq \mathrm{FRAC}(G).$$

Proof
By definition, every pair $ij \in E$ is a clique. Thus, $\mathrm{CLQ}(G) \subseteq \mathrm{FRAC}(G)$ for every graph G.

$\underline{\mathrm{TH}(G) \subseteq \mathrm{CLQ}(G)}$: Let $C \subseteq V$ be a clique in G. Pick any $c \in \mathbb{R}^n$ with $\|c\|_2 = 1$. Define $u(i) := c$ for each $i \in C$. For all other nodes, $i \in V \setminus C$, choose an orthonormal system of vectors orthogonal to c. Then $u : V \to \mathbb{R}^n$ is an orthonormal representation of G and hence the inequality

$$\begin{aligned} 1 &\geq \sum_{j=1}^{n} [c^T u(j)]^2 x_j \\ &= \sum_{j \in C} (c^T c)^2 x_j \\ &= x(C) \end{aligned}$$

is valid for $\mathrm{TH}(G)$. Since $\mathrm{TH}(G) \subseteq \mathbb{R}^V_+$, all constraints defining $\mathrm{CLQ}(G)$ are valid for $\mathrm{TH}(G)$. Thus $\mathrm{TH}(G) \subseteq \mathrm{CLQ}(G)$.

Since $\mathrm{CLQ}(G) \subseteq [0,1]^V$, we conclude that $\mathrm{TH}(G)$ is bounded. Since $\mathrm{TH}(G)$ is defined as the intersection of closed convex sets, it is closed and convex.

$\underline{\mathrm{STAB}(G) \subseteq \mathrm{TH}(G)}$: Let $S \subseteq V$ be a stable set in G and $\chi_X \in \{0,1\}^V$ denote its incidence vector. Let $u : V \to \mathbb{R}^n$ be any orthonormal representation of G and let $c \in R^n$ satisfy $\|c\|_2^2 = 1$. Then

$$\begin{aligned} \sum_{j=1}^{n} [c^T u(j)]^2 (\chi_S)_j &= \sum_{j \in S} [c^T u(j)]^2 \\ &\leq \|Q^T c\|_2^2 \\ &= \|c\|_2^2 \\ &= 1 \end{aligned}$$

where $Q \in \mathbb{R}^{n \times n}$ is an orthonormal matrix whose columns are formed by $u(i), i \in S$ and extending to an orthonormal basis.

Since $\mathrm{TH}(G)$ is convex and contains χ_S for stable sets S, $STAB(G) \subseteq \mathrm{TH}(G)$ by the definition of the convex hull.

Note that since $0 \in \mathrm{STAB}(G)$, $\mathrm{TH}(G)$ is nonempty.

Given $w \in \mathbb{R}_+^V$, the *Lovász Theta Function* is given by

$$\theta(G, w) := \max_{x \in \mathrm{TH}(G)} w^T x.$$

Define $W \in \mathbb{S}^V$ by

$$W_{ij} := \sqrt{w_i \cdot w_j}$$

for all $i, j \in V$.

Theorem 5.2.3
Let $G = (V, E)$ and $w \in \mathbb{R}_+^V$. The following are equal:

(i) $\theta(G, w)$

(ii) $\min_{u: V \to \mathbb{R}^n, \text{orthonormal representation} \wedge c \in \mathbb{R}^n, \|c\|_2 = 1} \max_{i \in V} \frac{w_i}{[c^T u(i)]^2}$

(iii) $\min\{\eta : \operatorname{diag}(S) = 0 \wedge \forall ij \in \bar{E}, S_{ij} = 0 \wedge \eta I - S \succeq W\}$

(iv) $\max\{\mathrm{Tr}(WX) : \forall ij \in E, X_{ij} = 0 \wedge \mathrm{Tr}(X) = 1 \wedge X \in \mathbb{S}_+^V\}$

The theorem above shows that we can effciently compute $\theta(G, w)$ to any precision via approximately solving an SDP.

Recall that $\omega(\cdot), \chi(\cdot)$ denotes the clique and chromatic numbers respectively.

Definition 5.2.2 (Perfect Graph)
A graph $G = (V, E)$ is perfect if for every node induced subgraph H of G,

$$\omega(H) = \chi(H).$$

Definition 5.2.3 (Odd-Hole)
An odd-hole is a chordless cycle of length at least 5.

An *odd-antihole* is the complement of an odd-hole.

Define

$$\begin{aligned}
&(\hat{\mathrm{TH}}(G)) \\
Y_{00} &= 1 \\
Y_{ij} &= 0 && \forall ij \in E \\
\operatorname{diag}(Y) &= Y_{e_0} \\
Y &\in \mathbb{S}_+^{\{0\} \cup V}
\end{aligned}$$

Theorem 5.2.4
Let G be a graph. Then the following are equivalent:

(i) G is perfect

(ii) $\bar{G}$ is perfect

(iii) G does not contain an odd-hole or odd-antihole

(iv) $\mathrm{STAB}(G) = \mathrm{CLQ}(G)$

(v) $\mathrm{STAB}(G) = \mathrm{TH}(G)$

(vi) $\mathrm{TH}(G) = \mathrm{CLQ}(G)$

(vii) $\mathrm{TH}(G)$ is a polytope

(viii) $\{x : A_{\mathrm{clq}}(G)x \leq \bar{e}, x \geq 0\}$ is Totally Dual Integral (TDI)

(ix) $\hat{\mathrm{TH}}(G)$ is SDP-TDI

Recall that the polar set

$$[\mathrm{TH}(G)]^o = \{s \in \mathbb{R}^V : \forall x \in \mathrm{TH}(G), x^T s \leq 1\}.$$

Theorem 5.2.5
For every graph $G = (V, E)$, the theta body of $\bar{G}$ is equal to the antiblocker of the theta body of G:

$$[\mathrm{TH}(G)]^o \cap \mathbb{R}^V_+ = \mathrm{TH}(\bar{G}).$$

Theorem 5.2.6
For every graph $G = (V, E)$, we have

$$\mathrm{TH}(G) = \left\{ x \in \mathbb{R}^V : \exists Y \in \hat{\mathrm{TH}}(G), \begin{bmatrix} 1 \\ x \end{bmatrix} = Y e_0 \right\}.$$

5.3 Product of Graphs & Kronecker Products

Let graphs $G = (V, E)$ and $G = (W, F)$. We define the *strong product* of G, H as

$$G \otimes H := (V(G \otimes H), E(G \otimes H))$$

where $V(G \otimes H) := V \times W$ and

$$\begin{aligned}(i,u)(j,v) \in E(G \otimes H) \iff & ij \in E, uv \in F \\ & \vee ij \in E, u = v \\ & \vee i = j, uv \in F\end{aligned}$$

5.4 Stable Sets & Shannon Capacity

Suppose we are trying to communicate through a noisy channel. We are using an alphabet and some letters may be confused with each other. Let $G = (V, E)$ model this situation: V is the set of letters in this alphabet and $ij \in E$ if and only if letters i, j may be confused with each other.

The maximum number of letters which will not be confused with each other is precisely $\alpha(G)$, the size of the largest independent/stable set.

We say words w_1, w_2 may not be confused with each other if there is some position at which w_1, w_2 have different letters AND these different letters do not share an edge.

Suppose we wish to know the maximum number of k-letter words so that no pair of words may be confused with each other. This is precisely then given by $\alpha(G^{\otimes k})$!

Definition 5.4.1 (Shannon Capacity)
The Shannon capacity of a graph G is

$$\Theta(G) := \lim_{k\to\infty} \sup[\alpha(G^k)]^{\frac{1}{k}}.$$

We can show that $\alpha(G^k) \geq [\alpha(G)]^k$ for every $k \in \mathbb{Z}_{++}$ for every graph G. Thus $\Theta(G) \geq \alpha(G)$.

Suing the fact that Kronecker products of orthonormal representation of G, H give rise to orthonormal representations of $G \otimes H$, we can prove the following theorem.

Theorem 5.4.1
For every pair of graphs G, H,

$$\theta(G \otimes H) = \theta(G) \cdot \theta(H).$$

Corollary 5.4.1.1
For every graph G and every $k \in \mathbb{Z}_{++}$,

$$\theta(G^k) = [\theta(G)]^k.$$

As a shorthand, we write

$$\theta(G) := \theta(G, \bar{e}).$$

Theorem 5.4.2
For every graph $G = (V, E)$, $\theta(G)$ is equivalent to

$$\begin{aligned} \max \bar{e}^T X \bar{e} & \\ X_{ij} = 0 & \qquad \forall ij \in E \\ \operatorname{Tr}(X) = 1 & \\ X \in \mathbb{S}_+^V & \end{aligned}$$

as well as

$$\begin{aligned} \min t & \\ \operatorname{diag}(Z) = (t-1)\bar{e} & \\ Z_{ij} = -1 & \qquad \forall ij \in \bar{E} \\ Z \in \mathbb{S}_+^V & \end{aligned}$$

Moreover,

$$\alpha(G) \le \Theta(G) \le \theta(G) \le \chi(\bar{G}).$$

Finally, we have equality all the way through if G is a perfect graph.

The SDPs in the statement are duals to each other and they are special cases of the SDP from the characterization of the theta function. Hence they both have Slater points.

Let $S \subseteq V$ be a stable set in G. Define

$$\bar{X}_{ij} := \begin{cases} \frac{1}{|S|}, & i, j \in S \\ 0, & \text{else} \end{cases}$$

Then $\bar{X}$ yields a feasible solution of the first SDP and $\bar{e}^T \bar{X} \bar{e}^T = |S|$. Thus $\alpha(G) \le \theta(G)$.

Suppose we have a colouring c of $\bar{G}$ with k colours. Then for the dual SDP, define $\bar{t} := k$ and $\bar{Z} \in \mathbb{S}^V$ such that

$$\bar{Z}_{ij} := \begin{cases} -1, & c(i) \ne c(j) \\ (k-1), & c(i) = c(j) \end{cases}$$

Then $(\bar{Z}, \bar{t})$ is a feasible solution of the dual SDP with objective value k. Thus $\theta(G) \leq \chi(\bar{G})$.

Note that to check Z_{ij} is PSD, we can choose a nice permutation of the rows and columns so that it is diagonally dominant.

The rest of the theorem follows from the characterization of perfect graphs. Note that $\alpha(G) = \omega(\bar{G})$ since a set is stable if and only if it is a clique in the complement.

Chapter 6

Lift-and-Project Methods

The previous section established a representation of the theta body of G (compact convex set described by infinitely many linear inequalities) as a projection of a *spectrahedron* in $\mathbb{S}^{n+1}$. This spectrahedron only requires $O(n^2)$ linear equations and a single PSD constraint.

6.1 Lift-and-Project Methods

Given a polytope $P \subseteq [0,1]^d$ and suppose we are interested in

$$P_I := \operatorname{conv}(P \cap \{0,1\}^d).$$

For instance: $P = \mathrm{FRAC}(G)$ and $P_I = \mathrm{STAB}(G)$.

Suppose $P = \{x \in \mathbb{R}^d : Ax \leq b, 0 \leq x \leq \bar{e}\}$. Introduce a new variable x_0 and define

$$K := \left\{ \begin{bmatrix} x_0 \\ x \end{bmatrix} \in \mathbb{R}^{1+d} : Ax \leq x_0 b, 0 \leq x \leq x_o \bar{e} \right\}.$$

Consider the set $M_+(K)$ defined as the feasible region of

$$\begin{aligned} Ye_0 &= \operatorname{diag}(Y) & \\ Ye_i &\in K & \forall i \in [d] \\ Y(e_0 - e_i) &\in K & \forall i \in [d] \\ Y &\in \mathbb{S}^{1+d}_+ & \end{aligned}$$

Suppose $\bar{x} \in P \cap \{0,1\}^d$. Define

$$\bar{Y} := \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \begin{bmatrix} 1 & \bar{x}^T \end{bmatrix}.$$

Then

$$
\begin{aligned}
\bar{Y}e_0 &= \operatorname{diag}(\bar{Y}) && x \in \{0,1\}^d \\
\bar{Y}e_i &= \bar{x}_i \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \in K && \forall i \in [d] \\
\bar{Y}(e_0 - e_i) &= (1 - \bar{x}_i) \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix} \in K && \forall i \in [d] \\
\bar{Y} &\in \mathbb{S}_+^{1+d}
\end{aligned}
$$

It follows that

$$
\begin{aligned}
P \cap \{0,1\}^d &\subseteq \mathrm{LS}_+(P) \\
&:= \left\{ x \in \mathbb{R}^d : \exists Y \in M_+(K), \begin{bmatrix} 1 \\ x \end{bmatrix} = Ye_0 \right\}.
\end{aligned}
$$

Since $M_+(K)$ is a spectrahedron, it is convex. Since $\mathrm{LS}_+(P)$ is a projection of a convex set, it itself is convex. Hence

$$
\operatorname{conv}(P \cap \{0,1\}^d) \subseteq \mathrm{LS}_+(P).
$$

Define

$$
\begin{aligned}
H_j^0 &:= \{x \in \mathbb{R}^d : x_j = 0\} \\
H_j^1 &:= \{x \in \mathbb{R}^d : x_j = 1\}.
\end{aligned}
$$

Lemma 6.1.1
Let $P \subseteq [0,1]^d$ be a convex set. Then

$$
\begin{aligned}
\operatorname{conv}(\mathcal{P} \cap \{0,1\}^d) &\subseteq \mathrm{LS}_+(P) \\
&\subseteq \bigcap_{j=1}^d \operatorname{conv}\left[(P \cap H_j^0) \cup (P \cap H_j^1) \right].
\end{aligned}
$$

Proof
It suffices to prove the second inclusion.

Let $\bar{x} \in \mathrm{LS}_+(P)$. Then there is some $\bar{Y} \in M_+(K)$ such that $\bar{Y}e_0 = \begin{bmatrix} 1 \\ \bar{x} \end{bmatrix}$. By the definition of $M_+(K)$,

$$\begin{aligned}\begin{bmatrix}1\\ \bar{x}\end{bmatrix} &= \bar{Y}e_0\\ &= \underbrace{\bar{Y}e_i}_{\in K\cap\{y\in\mathbb{R}^{d+1}:y_i=y_0\}} + \underbrace{\bar{Y}(e_0-e_i)}_{\in K\cap\{y\in\mathbb{R}^{d+1}:y_i=0\}}\end{aligned}$$

But notice that the 0-th entry is 1, which forces $y_0 = 1$. Moreover, the summation corresponds to a union in the lower space.

Thus

$$\bar{x} \in \operatorname{conv}\left[(P\cap H_j^0)\cup(P\cap H_j^1)\right]$$

for each $j \in [n]$. Therefore,

$$\bar{x} \in \bigcap_{j=1}^{d} \operatorname{conv}\left[(P\cap H_j^0)\cup(P\cap H_j^1)\right].$$

For $k \geq 2$, recursively define

$$\mathrm{LS}_+^k(P) := \mathrm{LS}_+(\mathrm{LS}_+^{k-1}(P)).$$

Theorem 6.1.2
Let $P \subseteq [0,1]^d$ be a convex set. Then

$$P \supseteq \mathrm{LS}_+(P) \supseteq \mathrm{LS}_+^2(P) \supseteq \cdots \supseteq \mathrm{LS}_+^d(P) = \operatorname{conv}(P\cap\{0,1\}^d).$$

Moreover, if for some $k \in \{0,1,\ldots,d-1\}$, $\mathrm{LS}_+^k(P) \neq \operatorname{conv}(P\cap\{0,1\}^d)$, then

$$\mathrm{LS}_+^k(P) \subset \mathrm{LS}_+^{k+1}(P).$$

The last theorem indicates that every 0-1 IP can be solved by solving a convex optimization problem based on SDPs. Unfortunately, the number of variables and constraints can increase significantly. We can also derive methods achieving the same goal via LP problems without SDPs. What is interesting about this approach is that these strictloy improving convex relaxations are generated automatically.

6.2 Lift-and-Project Operator Applied to FRAC(G)

Recall that the stable set polytope STAB(G) is the convex hull of characteristic vectors of stable sets of G. and FRAC(G) is the fractional stable set polytope.

Let H be the vertex set of an odd-cycle in G. Then the inequality

$$\sum_{i \in H} x_i \leq \frac{|H| - 1}{2}$$

is valid for $\mathrm{STAB}(G)$. Define

$$\mathrm{OC}(G) := \{x \in \mathrm{FRAC}(G) : x \text{ satisfies all odd-cycle constraints}\}.$$

An odd-hole is a chord-less cycle of length at least 5. Recall that an odd-antihole is the complement of an odd-hole. Let H be the vertex set of an odd-antihole in G. The inequality

$$\sum_{i \in H} x_i \leq 2$$

is valid for $\mathrm{STAB}(G)$. Put

$$\text{ANTI-HOLE}(G) := \{x \in \mathrm{FRAC}(G) : x \text{ satisfies all odd-antihole constriants}\}.$$

An *odd-wheel* the union of an odd cycle (*rim*) with a *hub* vertex which is then connected to all vertices on the rim. Suppose we have an odd-wheel in G with hub vertex indexed by $2k + 2$. Then the odd-wheel inequality

$$k x_{2k+2} + \sum_{i=1}^{2k+1} x_i \leq k$$

is valid for $\mathrm{STAB}(G)$. Define

$$\mathrm{WHEEL}(G) := \{x \in \mathrm{FRAC}(G) : x \text{ satisfies all odd-wheel constraints}\}.$$

Theorem 6.2.1
For every graph G,

$$\begin{aligned}\mathrm{STAB}(G) &\subseteq \mathrm{LS}_+(\mathrm{FRAC}(G)) \\ &\subseteq \mathrm{OC}(G) \cap \text{ANTI-HOLE}(G) \cap \mathrm{WHEEL}(G) \cap \mathrm{CLQ}(G) \cap \mathrm{TH}(G).\end{aligned}$$

The last inclusion is sometimes strict.

Problem 5 (Open)
Given a full, elegant, combinatorial characterization for $\mathrm{LS}_+(G)$, for all G.

6.3 Successive Convex Relaxation Methods

We now generalize our approach to lift-and-project methods to compute the convex hull of any set, hence in principle solve any optimization problem by solving a possibly very very large scale SDP.

6.3.1 Fundamental Framework

Let f be continuous on a compact set F. We wish to find

$$\inf f(x)x \qquad\qquad \in F$$

Introduce a new variable x_{n+1} and consider

$$\begin{aligned} &\inf x_{n+1} \\ f(x) &\leq x_{n+1} \\ x &\in F \\ x_{n+1} &\geq \ell \\ x_{n+1} &\leq u \end{aligned}$$

We may as well assume that we are optimizing a linear function over a compact set $F \oplus [u, \ell]$.

Lemma 6.3.1
Any compact set in $F \subseteq \mathbb{R}^d$ can be expressed as the feasible region of a system of quadratic inequalities.

Proof
$\mathbb{R}^d \setminus F$ is open and hence can be expressed as a union of open Euclidean balls. Then

$$F = \mathbb{R}^n \setminus (\mathbb{R}^n \setminus F)$$

which the intersection of quadratic inequalities $\|x - \bar{x}\|_2^2 \geq r^2$.

Lemma 6.3.2
For every triple $(Q, q, \gamma) \in \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$,

$$\begin{aligned}
&\{x \in \mathbb{R}^d : x^T Q x + 2q^T x + \gamma \le 0\} \\
&\subseteq \left\{ x \in \mathbb{R}^d : \operatorname{Tr} \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \le 0, \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{d+1}_+ \right\}.
\end{aligned}$$

In fact if $\operatorname{rank} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} = 1$, then equality holds above.

Suppose we are given a set $\mathcal{P} \subseteq \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$ such that

$$F = \{x \in \mathbb{R}^d : \forall (Q, q, \gamma) \in \mathcal{P}, x^T Q x + 2q^T x + \gamma \le 0\}.$$

We may as well replace $\mathcal{P}$ by $\operatorname{cone}(\mathcal{P})$ or even just the generators of $\operatorname{cone}(\mathcal{P})$.

We know that the inequality $x^T Q x + 2q^T x + \gamma$ is convex if and only if Q is PSD. Define

$$\mathcal{P}_+ := \operatorname{cone}(\mathcal{P}) \cap (\mathbb{S}^d_+ \oplus \mathbb{R}^d \oplus \mathbb{R}).$$

Thus we collect all convex inequalities. This is a convex relaxation of F.

Theorem 6.3.3
Let $\mathcal{P} \subseteq (\mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R})$ be a closed convex cone containing $(I, 0, R)$ for some $R > 0$. Then the convex sets

$$\left\{x \in \mathbb{R}^d : \forall (Q, q, \gamma) \in \mathcal{P}_+, x^T Q x + 2q^T x + \gamma \le 0\right\}$$

and

$$\left\{ x \in \mathbb{R}^d : \forall (Q, q, \gamma) \in \mathcal{P}, \operatorname{Tr} \begin{bmatrix} \gamma & q^T \\ q & Q \end{bmatrix} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \le 0; \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{d+1}_+ \right\}$$

are identical. Moreover, in the second description we may replace $\mathcal{P}$ by its generators.

6.3.2 Successive Convex Relaxation Method

Given $\mathcal{P} \subseteq \mathbb{S}^d \oplus \mathbb{R}^d \oplus \mathbb{R}$ containing $(I, 0, R)$ for $R > 0$, define

$$\begin{aligned}
C_0 &:= \{x \in \mathbb{R}^d : \forall (Q, q, \gamma) \in \mathcal{P}_+, x^T Q x + 2q^T x + \gamma \le 0\} \\
D_1 &:= \{d \in \mathbb{R}^d : \|d\|_2 = 1\} \\
D_2 &:= \{e_i, -e_i : i \in [d]\} \\
k &:= 0
\end{aligned}$$

At iteration k, we set

$$\begin{aligned}
\alpha(a) &:= \max_{x \in C_k} a^T x && \forall a \in D_1 \\
\beta(b) &:= \max_{x \in C_k} b^T x && \forall b \in D_2 \\
\mathcal{P}_k &:= \text{coefficients of } (\alpha - a^T x)(b^T x - \beta) \leq 0 \\
C_{k+1} &:= \left\{ x \in \mathbb{R}^d : \forall (Q, q, \gamma) \in (\mathcal{P} \cup \mathcal{P}_k)_+, x^T Q x + 2q^T x + \gamma \leq 0 \right\}
\end{aligned}$$

Theorem 6.3.4
With the above definitions, the sequence of convex relaxations C_k of F generated by SCRM satisfies

(a) $\forall k \in \mathbb{Z}_+$, $\operatorname{conv}(F) \subseteq C_{k+1} \subseteq C_k$, moreover, $C_{k+1} = C_k$ if and only if $C_k = \operatorname{conv}(F)$

(b) $\bigcap_{k=1}^{\bar{k}} C_k = \varnothing$ for some finite k if $F = \varnothing$

(c) $\bigcap_{k=1}^{\infty} C_k = \operatorname{conv}(F)$

Theorem 6.3.5
Let $F \subseteq \{0,1\}^d$ and let C_0 be defined by quadratic inequalities such that

$$\operatorname{conv}(F) \subseteq C_0 \subseteq [0,1]^d.$$

Suppose the quadratic inequalities $x_i^2 - x_i \leq 0, -x_i^2 + x_i \leq 0$ for all $i \in [d]$ are included in the quadratic inequality system. Let $\{C_k\}$ denote the sequence of compact convex sets generated by the SCRM. Then

$$C_k = \mathrm{LS}_+^k(C_0)$$

for all $k \in \mathbb{Z}_+$.

An observation leading to the proof of the previous theorem is the following: Recall the definition of LS_+ via M_+:

$$M_+(C_k) := \{\dots Y e_i, Y(e_0 - e_i) \in \operatorname{cone}(1 \oplus C_k) =: K\}.$$

Then for every $i \in [d]$ and $(s_0, s) \in K^*$,

$$\begin{aligned}
\begin{bmatrix} s_0 & s^T \end{bmatrix} Y e_i &\geq 0 \\
\left[s_0, s^T \right] Y(e_0 - e_i) &\geq 0.
\end{aligned}$$

Also

$$\begin{aligned}
(s_0, s) \in K^* &\iff \begin{bmatrix} s_0 & s^T \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0 && \forall x \in C_k \\
&\iff s_0 \geq -s^T x && \forall x \in C_k
\end{aligned}$$

Chapter 7

Convex Algebraic Geometry

7.1 Motivation

Given a polynomial $f : \mathbb{R}^4 \to \mathbb{R}$, how can we decide that $f(x) \geq 2$ for every $x \in \mathbb{R}^4$? We can do so by re-expressing the formula for f as a sum of squared terms plus a constant 2.

Given a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$ for $d \in \mathbb{Z}_{++}$, let $h(x) \in \mathbb{R}^N$ be the vector of monomials for $N := \binom{n+d}{d}$. We are interested in the set

$$F(f) := \{X \in \mathbb{S}^N : h(x)^T X h(x) = f(x)\}.$$

Theorem 7.1.1
Let $z \in \mathbb{R}$ and f be a multivariate polynomial over the reals. Then $[f(x) - \bar{z}]$ is a sum of squares if and only if

$$\left\{X \in F(f) : X \succeq \bar{z} e_1 e_1^T\right\} \neq \varnothing.$$

7.2 Polynomial Optimization

We can extend the idea of using sum of squares relaxations of nonnegativity of polynomials to polynomial optimization problems:

$$\begin{aligned} &\inf p_0(x) \\ p_1(x) &\geq 0 \\ p_2(x) &\geq 0 \\ &\dots \\ p_m(x) &\geq 0 \end{aligned}$$

We already know how to handle such problems via reformulation the problem by quadratic polynomials. We can also bypass this method. Let us first consider the feasibility version for simplicity: Is

$$F := \{x \in \mathbb{R}^n : \forall i \in [m], p_i(x) \geq 0\} = \varnothing?$$

Theorem 7.2.1
Let $p_1, \dots, p_m$ be given multivariate polynomials over n real variables. Then

$$F := \{x \in \mathbb{R}^n, \forall i \in [m], p_i(x) \geq 0\} = \varnothing$$

if and only if there are $s_0, \dots, s_j \in \mathrm{SoS}(n, d)$ (d is some finite degree) such that

$$g := \sum_{J \subseteq [m]} s_J \left(\prod_{i \in J} p_i \right) = -1.$$

In some sense, this is a generalization of Farkas' lemma and Hilbert's Nullstellenstaz.

Theorem 7.2.2 (Hilbert's Nullstellenstaz)
Given multivariate polynomials $p_1, \dots, p_m : \mathbb{C}^n \to \mathbb{C}$, exactly one of the following systems has a solution in $\mathbb{C}^n$:

(i) $p_i(x) = 0$ for all $i \in [m]$

(ii) there are polynomials h_i such that $\sum_{i \in [m]} h_i(x) p_i(x) = -1$

Note that the previous theorem can be implemented computationally (although not necessarily efficiently). Guess an upper bound on the degree of the polynomials s_j's and treat the coefficients of the monomials of s_j's as variables so we have an SDP to solve. Considering larger degree certificates of s_j's lead to very large scale SDP problems.

Chapter 8

Extension Complexity

Given a polyhedron (or family of polyhedron), what is the smallest number of linear inequalities necessary to represent this polyhedron as a projection of another polyhedron?

Similarly, given a closed convex set expressed as a spectrahedron, what is the smallest size and number of matrix variables and PSD contraints which allow us to represent the given convex set as a projection of a spectrahedron?

8.1 Definitions

Let $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$. Consider

$$P := \left\{ x \in \mathbb{R}^d : Ax \leq b \right\}$$

and suppose that $\dim P = d$, P is bounded, and P has m facets, n extreme points.

The *Slack matrix of* P is $S \in \mathbb{R}^{m \times n}$ given by

$$S_{ij} := b_i - \langle a^{(i)}, v^{(j)} \rangle$$

for all facets i and extreme points j. Here $a^{(i)}$ is the i-th row of A and $v^{(j)}$ is an extreme point of P.

The *nonnegative rank of* S *and* P is the smallest integer k such that

$$S = FV^T$$

where $F \in \mathbb{R}_+^{m \times k}$ and $V \in \mathbb{R}_+^{n \times k}$. Then $\operatorname{rank}_+(P) := \operatorname{rank}_+(S) := k$.

Theorem 8.1.1 (Yannakakis '91)
Let $P \subseteq \mathbb{R}^d$ be a polytope with $k := \text{rank}_+(P)$. Then every lifted representation (*extended formulation*) of P has at least k costraints. Moreover, there exists a lifted representation of P with at most $(k+d)$ constraints and $(k+d)$ variables.

Note that by lifted representation / extended formulation refers to polyhedral constraints. Hence the number of constraints refers to the number of linear equations / inequalities.

A sketch of the proof involves the fact that every valid inequality for P is a linear consequence of facet defining inequalities for P. Suppose all facets of P are expressed as $Ax \leq b$. Let S be the slack matrix of P and

$$\begin{aligned} \text{rank}_+(S) &= k \\ S &= FV^T \\ F &\in \mathbb{R}_+^{m\times k} \\ V &\in \mathbb{R}_+^{n\times k} \end{aligned}$$

Consider

$$\hat{P} := \{(x,u) \in \mathbb{R}^d \oplus \mathbb{R}^k : Ax + Fu = b, u \geq 0\}.$$

We claim that

$$P = \left\{x \in \mathbb{R}^d : \exists u \in \mathbb{R}_+^k, (x,u) \in \hat{P}\right\}.$$

Thus $\hat{P}$ is a lifted representation with $(k+d)$ variables and $(m+k)$ linear constraints. We can eliminate $(m-d)$ constraints from the description of $\hat{P}$ which completes one direction of the proof.

Conversely, suppose there is a polytope $\tilde{P}$ with q facets so that the projection of $\tilde{P}$ is P. Consider the slack matrix $\tilde{S}$ of $\tilde{P}$, but only focus on the submatrix of $\tilde{S}$ whose columns correspond to extreme points of $\tilde{P}$ projecting to extreme points of P. Every facet inducing inequality for P comes from a valid inequality for $\tilde{P}$ which by our first observation is a nonnegative linear combination of facet inducing inequlities for $\tilde{P}$. Collecting these facets of $\tilde{P}$ in a matrix $\tilde{F}$ and defining the submatrix of $\tilde{S}$ as $\tilde{V}$, we have

$$S = \tilde{F}\tilde{V}^T$$

which shows that $\text{rank}_+(S) \leq q$.

Definition 8.1.1 (Extension Complexity)
The smallest number of facets needed in a lifted polyhedral representation is called the extension complexity of P, and is denoted by $\text{xc}(P)$.

8.2 A Generalization

Given a convex set G and a convex cone K, does there exist an affine subspace V and linear subspace W such that

$$G = \Pi_W(K \cap V)?$$

Π_W denotes the projection onto subspace W. If so we say G *admits a lifted representation by* K and *proper* if $V \cap \operatorname{int}(K) \neq \varnothing$.

Suppose G is a compact convex set with nonempty interior. We may assume $0 \in \operatorname{int}(G)$. Recall the *polar*

$$G^o := \{s : \forall x \in G, \langle x, s\rangle \leq 1\}.$$

The *slack function of* G is given by

$$S_G : \operatorname{ext}(G) \oplus \operatorname{ext}(G^o) \to \mathbb{R}$$
$$S_G(x, s) := 1 - \langle x, s\rangle.$$

A K*-factorization of* S_C is a pair of maps $V : \operatorname{ext}(G) \to K$ and $F : \operatorname{ext}(G^o) \to K^*$ such that for all $(x, s) \in \operatorname{ext}(G) \oplus \operatorname{ext}(G^o)$,

$$S_G(x, y) = \langle V(x), F(s)\rangle.$$

Theorem 8.2.1 (Gouveia, Parrilo, Thomas '13)
If S_G has a K-factorization, then G has a lifted K-representation. If G has a proper lifted K-representation, then S_G has a K-factorization.

Set $K := \mathbb{S}^n_+$ or $K := \oplus_{i=1}^r \mathbb{S}^{n_i}_+$. Then $K^* = K$. For combinatorial optimization applications, G is a polytope in $[0, 1]^d$, such as STAB.