CO444: Algebraic Graph Theory

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Contents

Chapter 1

Introduction

1.1 Graphs

Recall that a (simple) *graph* is a pair $X = (V, E)$ consisting of a vertex set and an edge set.

Definition 1.1.1 (Isomorphism)
An isomorphism between graphs X, Y is a function $f : V(X) \to V(Y)$ such that $uv \in E(X) \iff f(u)f(v) \in E(Y)$.

If an isomorphism exists between X and Y, we say X and Y are *isomorphic* and write

$$X \cong Y.$$

Some important classes of graphs are complete graphs, bipartite graphs, empty graphs ($E = \varnothing$), the null graph ($V = \varnothing$), multigraphs, simple graphs, directed graphs, finite graphs, and infinite graphs.

We will focus mainly on finite simple graphs, with some exploration of finite multigraphs.

1.2 Subgraphs

Recall that a *subgraph* Y of the graph X is a graph such that $V(Y) \subseteq V(X), E(Y) \subseteq E(X)$. Some important classes of subgraphs are spannning subgraphs and induced subgraphs. Some other examples of graphs are cliques, independent sets, paths, cycles, and spanning trees.

1.3 Automorphisms

Definition 1.3.1 (Automorphism)
An automorphism of the graph X is an isomorphism $f : X \to X$.

We write $\mathrm{Aut}(X)$ to denote the set of all automorphisms of X.

If $\mathrm{Sym}(V(X))$ is the permutation group of the vertices of X, then it is clear that $\mathrm{Aut}(X) \subseteq \mathrm{Sym}(V(X))$. We will sometimes write $\mathrm{Sym}(n)$ to denote $\mathrm{Sym}(V(X))$ as we can label the vertices using $[n]$.

Proposition 1.3.1
$\mathrm{Aut}(X)$ is a subgroup of $\mathrm{Sym}(V(X))$.

Proof
It is a group and a subset.

For $h \in \mathrm{Sym}(V(X))$ and $v \in V(X)$, we let $v^g := g(v)$ denote the image of v under g. Similarly, for $S \subseteq V(X)$, we let $S^g := \{v^g : v \in S\}$.

Suppose Y is a subgraph of X. We let Y^g be the graph where

$$\begin{aligned} V(Y^g) &:= V(Y)^g \\ E(Y^g) &:= \{u^g v^g : uv \in E(Y)\} \end{aligned}$$

It is not hard to see that Y^g is another subgraph of X that is in fact isomorphic to Y.

Lemma 1.3.2
For $g \in \mathrm{Aut}(X)$ and $v \in V(X)$,

$$\deg_X(v) = \deg_{X^g}(v^g).$$

Hence automorphisms permute the vertices of equivalent degrees.

Proof
Let Y be the subgraph induced by $N(v) + v$. The result follows as $Y^g \cong Y$.

Let $d(u, v)$ denote the length of the shortest path between u, v.

Lemma 1.3.3
For $g \in \mathrm{Aut}(X)$ and $u, v \in V(X)$,

$$d_X(u, v) = d_{X^g}(u^g, v^g).$$

Hence automorphisms preserve distances.

Proof
Using g, g^{-1}, it is not hard to see that there is a uv-path of length ℓ in X if and only if there is a $u^g v^g$-path of length ℓ in X^g.

1.4 Homomorphisms

Definition 1.4.1 (Homomorphism)
A homomorphism between graphs X, Y is a function $f : V(X) \to V(Y)$ such that

$$xy \in E(X) \implies f(x)f(y) \in E(Y).$$

If such a homomorphism exists, we say X is *homomorphic* to Y.

This is a much more relaxed condition than an isomorphism.

Example 1.4.1
Given any bipartite graph X with bipartition $A \cup B$, X is homomorphic to the graph consisting of a single edge. Indeed, simply map A to one endpoint and B to another.

Example 1.4.2
Any subgraph is homomorphic to its supergraph through the identity map.

Recall the definition of a valid *graph colouring*. Let $\chi(X)$ denote the *chromatic number* of X.

Lemma 1.4.3
$\chi(X)$ is equal to the minimum integer r such that there exists a homomorphism from X to K_r.

Proof
First we claim that $\chi(X) \geq r$. Indeed, let $c : V(X) \to [\chi(X)]$ be a $\chi(X)$-colouring of X. If we take $[\chi(X)]$ to be the vertices of $K_{\chi(X)}$, then c is a homomorphism from $X \to K_{\chi(X)}$! Hence $\chi(X) \geq r$.

Conversely, suppose $f : X \to K_r$ is a homomorphism. Then $f^{-1}(\{i\})$ induces an independent set for all $i \in [r]$. Thus X necessarily has an r-colouring. It follows that $\chi(X) \leq r$.

Remark that we have in fact proven that the set of homomorphisms $X \to K_r$ is in fact the set of r-colorings of X with colours $[r]$!

Definition 1.4.2 (Retraction)
A retraction is a homomorphism from X to a subgraph Y of X such that the restriction $f\big|_Y$ is the identity map.

If such a retraction exists, we say that Y is a *retract* of X.

Example 1.4.4
Any clique Y of X that is the size of $\chi(X)$ is a retract of X.

1.5 Examples of Graphs

1.5.1 Circulant Graphs

Consider the cycle C_n. Its automorphism group is exactly the dihedral group on n elements.

We can view circulant graphs as a generalization of cycles.

Definition 1.5.1 (Circulant Graph)
Let the vertex set be $\mathbb{Z}_n$. Choose a subset $C \subseteq \mathbb{Z}_n - 0$ that is closed under inverses. Define

$$E(X) := \{ij : i - j \in C\}.$$

We write $X = X(\mathbb{Z}_n, C)$ to denote the circulant graph parametrized by $\mathbb{Z}_n, C$.

Let $g \in \operatorname{Aut}(X)$ be the cyclic shift and $h \in \operatorname{Aut}(X)$ be the map sending $i \mapsto -i$. Furthermore, put R as the cyclic subgraph generated by g. It follows that R, hR are different cosets of

$\mathrm{Aut}(X)$ and

$$|\mathrm{Aut}(X)| \geq 2n.$$

1.5.2 Johnson Graphs

Definition 1.5.2 (Johnson Graph)
Let $v \geq k \geq i$ be integers. The Johnson graph $J = J(n, k, i)$ is the graph with vertex set

$$V(J) := \{S \subseteq [v] : |S| = k\}$$

and edge set

$$E(J) := \{ST : |S \cap T| = i\}.$$

Proposition 1.5.1
$J(v, k, i)$ is d-regular for

$$d = \binom{k}{i}\binom{v-k}{k-i}.$$

Proof
regularity is clear. It remains only to compute the common degree of all vertices.

Fix a vertex S. Fix a subset of $S' \subseteq S$ with size k. There are $\binom{v-k}{k-i}$ vertices T such that

$$S \cap T = S'.$$

The result follows by letting S' range over all S-subsets of size k.

Lemma 1.5.2
$J(v, k, i) \cong J(v, v-k, v-2k+i)$.

Proof
The map $f(S) := \bar{S}$ yields an isomorphism.

The special case of $J(v, k, 0)$ are the *Kneser Graphs* and $J(5, 2, 0)$ is the *Peterson Graph*.

Lemma 1.5.3
$\text{Aut}(J(v,k,i))$ contains a subgroup isomorphic to $\text{Sym}(v)$.

Proof
Let $g \in \text{Sym}(v)$. Define the map $\sigma_g : V(J(v,k,i)) \to V(J(v,k,i))$ by

$$S \mapsto S^g.$$

It is easy to see that $|S \cap T| = |S^g \cap T^g|$, so $\sigma_g \in \text{Aut}(J(v,k,i))$. But then

$$\{\sigma_g : g \in \text{Sym}(v)\} \subseteq \text{Aut}(J(v,k,i))$$

and $\{\sigma_g : g \in \text{Sym}(v)\} \cong \text{Sym}(v)$ as required.

Remark that we typically have $\text{Aut}(J(v,k,i)) \cong \text{Sym}(v)$ but it is NOT always true.

1.5.3 Line Graphs

Definition 1.5.3 (Line Graph)
The line graph of X is the graph $L(X)$ with vertex set

$$V(L(X)) := E(X)$$

and edge set

$$E(L(X)) := \{ef : e, f \text{ share an endpoint}\}.$$

In general, if $X \cong Y$, then $L(X) \cong L(Y)$. But the converse need not hold! Indeed, $L(C_3) = L(K_{1,3})$ but $C_3 \neq K_{1,3}$.

The converse is true however, if the minimum degrees of X, Y are at least 4.

It is also interesting to observe that not all graphs are line graphs. For example, $K_{1,3}$ is not a line graph since no matter how we distribute the outside vertices according to the endpoints of the central vertex, there are not enough edges for $K_{1,3}$ to be a line graph.

Chapter 2

Group Actions

2.1 Group Actions on Graphs

Recall the definition of a group.

Definition 2.1.1 (Homomorphism)
Let G, H be groups. A map $f : G \to H$ is a homomorphism if

$$f(xy) = f(x)f(y)$$

for all $x, y \in G$.

The *kernel* of a homomorphism is

$$\ker f := f^{-1}(1).$$

Definition 2.1.2 (Group Action)
Let G be a group and V a set. A homomorphism $f : G \to \mathrm{Sym}(V)$ is an action of G on V.

We say G *acts* on V.

If in addition, $f^{-1}(1) = 1$, then we say f is *faithful*.

Suppose G acts on V. Let $g \in G$. We write x^g to denote the image of x under $f(g)$. Similarly, for $S \subseteq V$, we write

$$S^g := \{x^g : x \in S\}.$$

Definition 2.1.3 (Invariant)
Let G act on V. $S \subseteq V$ is G-invariant if

$$S^g = S$$

for all $g \in G$.

Definition 2.1.4 (Orbit)
Let G act on V. The orbit of $x \in V$ is

$$x^G := \{x^g : g \in G\}.$$

It is well-known that B is partitioned into disjoint orbits and each such orbit is G-invariant.

2.1.1 Orbits & Stabilizers

Definition 2.1.5
Let G be a permutation group acting on V. Fix $x \in V$.
The stabilizer of x is

$$G_x := \{g \in G : x^g = x\}.$$

It is not hard to see that G_x is a subgroup of G.

Lemma 2.1.1
Let G be a permutation group acting on V and S an orbit of G.
Suppose $x, y \in S$. Then $H := \{h \in G : x^h = y\}$ is a left coset of G_x.
Conversely, if H is a left coset of G_x, then $x^h = x^{h'}$ for all $h, h' \in H$.

Proof
Since G is transitive on S, there is some $g \in G$ such that

$$x^g = y.$$

But then for any $h \in H$ such that $x^h = y = x^g$,

$$\begin{aligned} x^h &= x^g \\ x^{g^{-1}h} &= x \\ g^{-1}h &\in G_x \\ h &\in gG_x. \end{aligned}$$

Hence H is indeed a left coset of G_x.

Now, consider the left coset gG_x for some $g \in G$. Pick $g\sigma, g\sigma' \in gG_x$ for some $\sigma, \sigma' \in G_x$. We have

$$\begin{aligned} x^h &= x^{g\sigma} \\ &= x^g \\ &= x^{g\sigma'} \\ &= x^{h'} \end{aligned}$$

as desired.

Lemma 2.1.2 (Orbit-Stabilizer)
Let G be a permutation group acting on V and $x \in V$. Then

$$|G_x| \cdot |x^G| = |G|.$$

Proof
Let $\mathcal{H}$ be the collection of left cosets of G_x and $f : x^G \to \mathcal{H}$ be the map

$$f(y) := \{g \in G : x^g = y\}$$

for all $y \in x^G$. By the previous lemma, f is bijective.

It follows that $|x^G| = |\mathcal{H}|$. Since the left cosets of G_x partition G, we must have

$$|G| = |G_x| \cdot |\mathcal{H}| = |G_x| \cdot |x^G|.$$

2.2 Burnside's Lemma

What is the relationship between G_x, G_y for $y \in x^G$?

Definition 2.2.1 (Conjugate)
Let $g, h \in G$. g is conjugate to h if there is some $\sigma \in G$ where

$$g = \sigma^{-1} h \sigma.$$

Conjugacy is an equivalence relation so G is partitioned into conjugacy classes.

For $g \in G$, let $\tau_g : G \to G$ be the map sending

$$h \mapsto ghg^{-1}$$

for all permutations $h \in G$. The conjugacy classes of G are the orbits of G under $\{\tau_g : g \in G\}$.

Let H be a subgroup of G. Then gHg^{-1} is a subgroup of G. Moreover, it is possible to show that $gHg^{-1} \cong H$ and we say gHg^{-1} is conjugate to H.

Lemma 2.2.1
Let G be a permutation group acting on V and $x \in V$. Then for all $g \in G$,

$$gG_x g^{-1} = G_{x^g}.$$

Thus if $y \in x^G$, then G_x, G_y are conjugate.

Proof
$\underline{gG_x g^{-1} \subseteq G_{x^g}}$ Let $y = x^g$ so that $y^{g^{-1}} = x$. For any $ghg^{-1} \in gG_x g^{-1}$, we have

$$\begin{aligned} y^{ghg^{-1}} &= x^{gh} \\ &= x^g \\ &= y. \end{aligned}$$

Hence $gG_x g^{-1} \subseteq G_{x^g}$.

$\underline{G_{x^g} \subseteq gG_x g^{-1}}$ For any $h \in G_y$, we have

$$\begin{aligned} x^{g^{-1}hg} &= y^{g^{-1}h} \\ &= y^{g^{-1}} \\ &= x. \end{aligned}$$

It follows that $g^{-1}hg \in G_x$ and $h \in gG_x g^{-1}$.

Let G be a permutation group acting on V and $g \in G$. We write

$$\operatorname{fix}(g) := \{v \in V : v^g = v\}.$$

Lemma 2.2.2 ("Burnside")
Let G be a permutation group acting on V, then the number of orbits of G is given by

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{fix}(g)|.$$

Proof
Let $\Lambda := \{(g, x) : g \in G, x \in \operatorname{fix}(g)\}$. We will apply a double counting argument.

$$\begin{aligned}
\sum_{g \in G} |\operatorname{fix}(g)| &= |\Lambda| \\
&= \sum_{x \in V} |G_x| \\
&= \sum_{x \in V} \frac{|G|}{|x^G|} && \text{Orbit-Stabilizer Lemma} \\
&= |G| \cdot (\text{number of orbits}).
\end{aligned}$$

2.3 Asymmetric Graphs

Definition 2.3.1 (Asymmetric)
A graph X is asymmetric if its automorphism group $\operatorname{Aut}(X) = \{1\}$ is trivial.

Let $\mathcal{G}_n$ denote the set of graphs with vertex set $[n]$. Then we write

$$\operatorname{Iso}(X) := \{Y \in \mathcal{G}_n : Y \cong X\}$$

to denote the isomorphism class of X.

Lemma 2.3.1
For all $X \in \mathcal{G}_n$,

$$|\operatorname{Iso}(X)| = \frac{n!}{\operatorname{Aut}(X)}.$$

Proof
Let $G = \mathrm{Sym}(n)$. For $g \in G$, let $\tau_g : \mathcal{G}_n \to \mathcal{G}_n$ be the map sending

$$X \mapsto X^g.$$

The group $H := \{\tau_g : g \in G\}$ acts on $\mathcal{G}_n$ and is isomorphic to G. By the Orbit-Stabilizer Lemma

$$\begin{aligned} n! &= |G| \\ &= |H| \\ &= |X^H| \cdot |H_X| \\ &= |\mathrm{Iso}(X)| \cdot |\mathrm{Aut}(X)|. \end{aligned}$$

Lemma 2.3.2
The set $\mathcal{H}$ of all isomorphism classes of $\mathcal{G}_n$ satisfies

$$|\mathcal{H}| = (1 + o(1)) \frac{2^{\binom{n}{2}}}{n!}.$$

Proof
Consider the group
$$\mathcal{P} := \{\tau_g : g \in \mathrm{Sym}(n)\}$$
acting on $\mathcal{G}_n$ where $\tau_g(X) = X^g$. The set of orbits is precisely $\mathcal{H}$. By Burnside's lemma,

$$|\mathcal{H}| = \frac{1}{n!} \sum_{\tau_g \in \mathcal{P}} |\mathrm{fix}(\tau_g)|.$$

Consider some $g \in \mathrm{Sym}(n)$. $\mathrm{fix}(\tau_g)$ is the set of graphs in $\mathcal{G}_n$ fixed by τ_g. Note that g induces a permutation σ_g on $E(K_n)$. Let $\mathcal{C}$ be an orbit of the permutation σ_g and suppose X is fixed by τ_g. Either X contains all edges in C or has none of the edges in $\mathcal{C}$, otherwise, some edge will be mapped to a none-edge. Hence $|\mathrm{fix}(\tau_g)| = 2^{\mathrm{orb}_2(g)}$ where $\mathrm{orb}_2(g)$ is the number of orbits in σ_g.

Among all permutations $g \in \mathrm{Sym}(n)$ whose support is size $2r$, the maximum value of $\mathrm{orb}_2(g)$ is realized by some permutation with r cycles of length 2. Suppose g is such a maximizer. Since $g^2 = \mathrm{id}$, its action on $E(K_v)$ has orbits of size at most 2. There are two ways in which an edge $xy \in E(K_v)$ is NOT fixed by g. Either x, y are in the support of g, but in different orbits, or x is in the support of g and y is not. Hence the number of orbits of length 2 in its action on $E(K_v)$ is

$$r(r-1) - r(n-2r) = r(n-r-1)$$

and

$$\mathrm{orb}_2(g) = \binom{n}{2} - r(n-r-1).$$

We now argue that the main contribution of Burnside's average comes from the identity permutation. Indeed, $\mathrm{orb}_2(\mathrm{id}) = \binom{n}{2}$, thus $\sum_{\tau_g \in \mathcal{P}} |\mathrm{fix}(\tau_g)| \geq 2^{\binom{n}{2}}$.

Fix an even integer $m \leq n-2$. Consider the nonidentity permutations with support of size at most m. There are at most $\binom{n}{m} m! < n^m$ permutations of this form. Let g be such an element. Then $\mathrm{orb}_2(g)$ is maximized if it has a single 2-cycle. If follows that

$$\mathrm{orb}_2(g) \leq \binom{n}{2} - (n-2).$$

Of the rest of the permutations, there are at most $n! < n^n$ of them. Let g be such a permutation. $\mathrm{orb}_2(g)$ is maximized at some permutation with $\frac{m}{2}$ 2-cycles, so

$$\mathrm{orb}_2(g) \leq \binom{n}{2} - \frac{m}{2}\left(n - \frac{m}{2} - 1\right) \leq \binom{n}{2} - \frac{nm}{4}$$

by our work above.

Therefore,

$$\begin{aligned}
\sum_{\tau_g \in \mathcal{P}} |\mathrm{fix}(\tau_g)| &\leq 2^{\binom{n}{2}} + n^m 2^{\binom{n}{2} - (n-2)} + n^n 2^{\binom{n}{2} - \frac{nm}{4}} \\
&= 2^{\binom{n}{2}} \left(1 + n^m 2^{-(n-2)} + n^n 2^{-\frac{nm}{4}}\right).
\end{aligned}$$

By taking $m = \lfloor c \log n \rfloor$ for $c > 4$, we have

$$\begin{aligned}
n^m 2^{-(n-2)} + n^n 2^{-\frac{nm}{4}} &= 2^{m \log n - n + 2} + 2^{n \log n - \frac{nm}{4}} \\
&= 2^{c \log^2 n - n + 2} + 2^{n \log n - \frac{cn \log n}{4}} \\
&= o(1).
\end{aligned}$$

Theorem 2.3.3
Asymptotically, almost all graphs are asymmetric.
That is, if we uniformly pick a graph in $\mathcal{G}_n$ at random, the probability that the chosen graph is asymmetric tends to 1 as $n \to \infty$.

Proof
Let $\mathcal{H}$ be the set of isomorphism classes of graphs on $[n]$. Pick $\mathcal{C} \in \mathcal{H}$. If graphs in $\mathcal{C}$ are asymmetric, then $|\mathcal{C}| = n!$ Otherwise, $|\mathcal{C}| \leq \frac{n!}{2}$.

Let ρ be the proportion of all classes in $\mathcal{H}$ such that $|\mathcal{C}| = n!$ Now,

$$\begin{aligned} 2^{\binom{n}{2}} &= |\mathcal{G}_n| \\ &= \sum_{\mathcal{C}\in\mathcal{H}} |\mathcal{C}| \\ &\leq \rho|\mathcal{H}| \cdot n! + (1-\rho)|\mathcal{H}| \cdot \frac{n!}{2} \end{aligned}$$

By the previous lemma,

$$\begin{aligned} 2^{\binom{n}{2}} &\leq (1+o(1))\frac{2^{\binom{n}{2}}}{n!} \cdot n! \left(\rho + \frac{1-\rho}{2}\right) \\ &= (1+o(1))2^{\binom{n}{2}} \cdot \frac{1+\rho}{2}. \end{aligned}$$

It follows that $\rho \to 1$ as $n \to \infty$ and the proportion of asymmetric graphs in $\mathcal{G}_n$ is

$$\begin{aligned} &\frac{\text{size of isomorphism class} \cdot \text{number of asymmetric isomorphism classes}}{|\mathcal{G}_n|} \\ &= \frac{n! \cdot \rho|\mathcal{H}|}{2^{\binom{n}{2}}} \\ &= (1+o(1))\rho && \text{previous lemma} \\ &\to 1. \end{aligned}$$

2.4 Primitivity

Definition 2.4.1 (Block of Imprimitivity)
Let G be a group acting transitively on V and $S \subseteq V$. S is called a block of imprimitivity for G if $\forall g \in G$, either $S^g = S$ or $S^g \cap S = \varnothing$.

Example 2.4.1 (Trivial Blocks of Imprimitivity)
$S = \{u\}$ for some $u \in V$ and $S = V$ are trivial blocks of imprimitivity.

Definition 2.4.2 (Primitive)
G is primitive if there is no non-trivial blocks of imprimitivity.

If G is not primitive, we say it is *imprimitive*.

Example 2.4.2
$\mathrm{Aut}(K_n)$ has no non-trivial blocks of imprimitivity.

Example 2.4.3
Consider Q_3, the 3-cube. Take any pair of vertices on $S = \{a, b\}$ on oppositive corners of a diagonal. Since automorphisms preserve distances, S is a block of imprimitivity.

Consider $\mathrm{Aut}(C_n)$ for composite n. Suppose $n = xy$ for $2 \leq x \leq y \in \mathbb{Z}_+$. Consider $S = \{0, x, 2x, \dots, (y-1)x\} \subseteq V$. Since $\mathrm{Aut}(C_n)$ is precisely the dihedral group, any $g \in \mathrm{Aut}(C_n)$ is uniquely determined by its action on $0, x$. Thus S is a block of imprimitivity.

For n prime, we claim C_n is primitive. Let $S \subsetneq V, |S| \geq 2$ and fix $v \neq w \in S$ minimizing $d_G(v, w)$ for all distinct pairs in S. Let $r \in \mathrm{Aut}(C_n)$ be the rotation mapping $v \mapsto w$ (along the shortest vw path in G). Clearly $v^r = w$. We claim that $S^g \neq S$. Otherwise, walking from w in the opposite direction to of v, the first vertex of S we see is of distance $d_C(v, w)$. Following this argument, we see that S consists of consecutively equidistance vertices. But then n could not have been prime.

For groups H, K, we write $H \leq k$ if H is a subgroup of K and $H < K$ if H is a proper subgroup of K.

Definition 2.4.3 (Maximal Subgroup)
$H < G$ is a maximal subgroup if there is no subgroup K such that $H < K$.

Lemma 2.4.4
Let G be a transitive permutation group on V and $x \in V$. G is primitive if and only if G_x is a maximal subgroup of G.

Proof
We prove the contrapositives.

$(\neg \implies \neg)$ Let B be a block of imprimitivity where $2 \leq |B| < |V|$. By transitivity, there is some $g \in G$ such that $x \in B^g$. Thus by taking B^g if necessary, we may assume without loss of generality that $x \in B$.

We wish to show that there is a subgroup H where $G_x < H < G$. Suppose G_B is the subgroup of all $g \in G$ such that $B^g = B$. We argue that $G_x < G_B < G$.

Let $g \in G_x$. Then $x \in B \cap B^g$, hence $B = B^g$ and $G_x \leq G_B$. But the fact that $|B| \geq 2$ implies there is some $y \neq x$ such that $y \in B$.

Let $h \in G$ be such that $x^h = g$. This shows that $h \notin G_x$. Then $y \in B \cap B^h$ and $B = B^h$. Thus $h \in G_B$. This shows that $G_x < G_B$.

The fact that $G_B < G$ is obvious since $B \subsetneq V$ and G is transitive.

$(\neg \impliedby \neg)$ Suppose H is a subgroup of G such that $G_x < H < G$. Let B be the orbit of H containing x. We argue that B is a block of imprimitivity.

Now, $G_x < H$ so there is some $g \in H$ such that $x^g \neq x$. But $x^g \in B^g$ then implies $|B| \geq 2$. On the other hand, the orbit-stabilizer formula states that

$$\begin{aligned}
|B| &= \frac{|H|}{|H_x|} \\
&< \frac{|G|}{|H_x|} \cdot \frac{|H_x|}{|G_x|} && H < G, H_x \subseteq G_x \\
&= \frac{|G|}{|G_x|} \\
&= |x^G| \\
&= |V|. && \text{transitivity}
\end{aligned}$$

Fix $g \in G$. We claim that $B \cap B^g \neq \varnothing$ implies $g \in H$. Observe that B is H-invariant by the definition of an orbit. Thus the claim would show that $B \cap B^g \neq \varnothing \implies B = B^g$, concluding the proof.

Suppose $y \in B \cap B^g$. There is some $h \in H$ such that $x^h = y$. Moreover, there is some

$h' \in H$ such that $y = x^{gh'}$. It follows that $x = x^{h^{-1}gh'}$. But then $h^{-1}gh' \in G_x < H$ and $g \in H$ as required.

Chapter 3

Transitive Graphs

3.1 Vertex-Transitive Graphs

Definition 3.1.1 (Vertex Transitive)
A graph X is vertex transitive if $\operatorname{Aut}(X)$ acts transitively on $V(X)$.

Let $H(\cdot, \cdot)$ denote the Hamming distance between two strings in $\mathbb{Z}_2^k$.

Definition 3.1.2 (k-Cube)
The k-cube is the graph with vertex set

$$V(Q_k) := \mathbb{Z}_2^k$$

and edge set

$$E(Q_k) := \{\alpha\beta : H(\alpha, \beta) = 1\}.$$

We write Q_k to denote the k-cube.

Lemma 3.1.1
Q_k is vertex transitive for all $k \geq 1$.

Proof
Fix $v \in Q_k$. For each $x \in V(Q_k)$, let $P_v : V(Q_k) \to V(Q_k)$ be given by

$$x \mapsto x + v.$$

Observe that $H(x,y)=1 \iff H(x+v,y+v)=1$. Write P to denote the subgroup of automorphisms of the form P_v.

It follows that $\mathrm{Aut}(X) \geq P$ acts transitively on $V(Q_k)$ by definition since the map P_{w-v} maps $v \mapsto w$ for any $v,w \in V(Q_k)$.

As a corollary to this lemma, the $|\mathrm{Aut}(Q_k)| \geq 2^k$. We improve upon this lowerbound.

Proposition 3.1.2
$|\mathrm{Aut}(Q_k)| \geq 2^k k!$.

Proof
Pick $g \in \mathrm{Sym}(k)$ and consider $\tau_g : V(Q_k) \to V(Q_k)$ which permutes the entries of $x \in V(Q_k)$ according to g.

We claim that
$$P' := \{\tau_g : g \in \mathrm{Sym}(k)\} \leq \mathrm{Aut}(Q_k).$$
To see this, observe that $H(x,y)=1 \iff H(\tau_g(x),\tau_g(y))=1$ for all $x,y \in \mathbb{Z}_2^k$.

Moreover, we claim that $P' \cap P = \{\mathrm{id}\}$. This is because the ratio of 0-1 entries of a binary string x always remains the same under P', but for all $P_v \in P$, we can find some string x for which the 0-1 ratio of x^{P_v} differs from x.

It follows by elementary group theory that
$$|PP'| = \frac{|P| \cdot |P'|}{|P \cap P'|} = 2^k k!$$
as desired.

Applying a similar proof, we can show that circulant graphs are also vertex transitive by the rotation subgroup to its automorphism group.

Definition 3.1.3 (Cayley Graphs)
Let G be a group and $C \subseteq G$ which is closed under inverses and does NOT contain id.
The Cayley graph $X = X(G,C)$ is defined by
$$V(X) := G$$
$$E(X) := \{gh : g^{-1}h \in C\}.$$

Since $\mathrm{id} \notin C$, Cayley graphs are loopless. If $g^{-1}h \in C$, then $h^{-1}g = (g^{-1}h)^{-1} \in C$ as well.

Thus Cayley graphs are undirected.

Theorem 3.1.3
$X = X(G, C)$ is vertex transitive.

Proof
For each $g \in G$, let $\tau_g : V(X) \to V(X)$ be given by

$$x \mapsto gx.$$

Observe that $\tau_g \in \mathrm{Aut}(X)$ as

$$\begin{aligned} xy \in E(X) &\iff y^{-1}x \in C \\ &\iff (y^{-1}g^{-1})gx \in C \\ &\iff \tau_g(y)^{-1}\tau_g(x) \in C \\ &\iff \tau_g(x)\tau_g(y) \in E(X). \end{aligned}$$

It follows that the set $\{\tau_g : g \in G\}$ acts transitively on $V(X)$.

A special case of this is the k-cube where $Q_k = X(\mathbb{Z}_2^k, \{e_i : i \in [k]\})$.

A circulant graph $X(\mathbb{Z}_n, C)$ is a Cayley graph under the same parameters.

Proposition 3.1.4
For $v \geq k \geq i$, the Johnson graph $J(v, k, i)$ is vertex transitive.

Proof
For each $g \in \mathrm{Sym}(v)$, let $\sigma_g : V(J(v, k, i)) \to V(J(v, k, i))$ be given by

$$S \mapsto S^g.$$

Consider the subgroup

$$\{\sigma_g : g \in \mathrm{Sym}(v)\} \leq \mathrm{Aut}(J(v, k, i)).$$

By observation, the subgroup acts transitively on $V(J(v, k, i))$.

In general, $J(v, k, i)$ are NOT Cayley graphs.

Lemma 3.1.5
The Peterson graph $J(5, 2, 0)$ is NOT a Cayley graph.

Proof
There are only two groups of order 10, the cyclic group $\mathbb{Z}_{10}$ and the dihedral group D_{10}.

Neither of the cubic Cayley graphs on these groups are isomorphic to $J(5, 2, 0)$.

Hence Cayley graphs are a strict subset of vertex transitive graphs.

3.2 Edge-Transitive Graphs

Recall that if G acts on $V(X)$, this induces an action on $E(X)$.

Definition 3.2.1 (Edge Transitive)
A graph X is edge transitive if $\mathrm{Aut}(X)$ acts transitively on $E(X)$.

If G acts on $V(X)$, this also induces an action on $V(X)^2$!

Definition 3.2.2 (Arc Transitive)
A graph X is arc transitive if $\mathrm{Aut}(X)$ acts transitively on $\{(x, y) : xy \in E(X)\}$.

Note that arc transitivity implies both vertex and edge transitivity.

In general, there is no relationship between vertex and edge transitive graphs.

Example 3.2.1
The triangular prism is vertex transitive but not edge transitive.

Example 3.2.2
For $n \neq m$, $K_{n,m}$ is edge transitive but not vertex transitive.

There are also examples of regular edge transitive graphs that are not vertex transitive.

Lemma 3.2.3
Suppose X is edge transitive and has no isolated vertices. If X is NOT vertex transitive, $\mathrm{Aut}(X)$ has exactly 2 orbits and they form a bipartition of X.

Proof
Aut(X) has exactly 2 orbits: First observe that $\mathrm{Aut}(X)$ has at least 2 orbits since it is not vertex transitive.

Fix $xy \in E(X)$. For any $w \in V(X)$, there is some $wz \in E(X)$ since w is not isolated. Hence there is some $\sigma \in \mathrm{Aut}(X)$ such that

$$\{\sigma(x), \sigma(y)\} = \{w, z\}.$$

It follows that $X \subseteq x^G \cup y^G$ by the choice of w and so X has at most 2 orbits.

The orbits yield a bipartition: We argue that there are no edges between vertices of x^G nor y^G. Suppose towards a contradiction that there is some $wz \in E(X)$ where either $w, z \in x^G$ or $w, z \in y^G$.

By the definition of orbits, there is no element of $\mathrm{Aut}(X)$ which maps $wz \mapsto xy$, as the latter has elements from both orbits. This is a contradiction since we assumed that X is edge transitive.

By contradiction, (x^G, y^G) forms a bipartition of X.

In general, edge transitivity and vertex transitivity does not imply arc transitivity.

Definition 3.2.3 (Oriented Graph)
An oriented graph is a directed graph X where $xy \in A(X) \implies yx \notin A(X)$.

The definition of automorphisms of undirected graphs naturally generalises to automorphisms of directed and therefore oriented graphs.

Lemma 3.2.4
If X is both vertex and edge transitive but not arc transitive, then the degrees of X are even.

Proof
For an undirected graph X, let $D(X)$ be the directed graph obtained from X by replacing each $uv \in E(X)$ with the two arcs $(u, v), (v, u) \in A(D(X))$.

Fix $x \in V(X)$ and $y \in N_X(x)$. Put

$$\Omega_1 := (x,y)^G, \Omega_2 := (y,x)^G$$

where $G = \mathrm{Aut}(X)$. Since X is edge transitive, then for any $uv \in E(X)$, either $(u,v) \in \Omega_1$ or $(v,u) \in \Omega_2$. Hence $\Omega_1 \cup \Omega_2 = A(D(X))$.

But X is not arc transitive, thus $\Omega_1 \cap \Omega_2 = \varnothing$. It follows that $(y,x) \notin \Omega_1$. By the arbitrary choice of $xy \in E(X)$,

$$(x,y) \in \Omega_1 \iff (y,x) \in \Omega_2.$$

Thus Ω_1, Ω_2 are oriented graphs.

By construction, $\mathrm{Aut}(X) \leq \mathrm{Aut}(\Omega_1)$ since every automorphism of X yields an automorphism of Ω_1. Moreover, $\mathrm{Aut}(X)$ acts transitively on $V(\Omega_1) = V(X)$ by assumption.

By transitivity, Ω_1 is necessarily a regular oriented graph. Thus for a vertex $u \in V(X)$.

$$\begin{aligned}\deg_X(u) &= d^+_{\Omega_1}(u) + d^-_{\Omega_1} \\ &= 2d^+_{\Omega_1}(u) \\ &\equiv 0 \mod 2.\end{aligned}$$

Corollary 3.2.4.1
If X is vertex and edge transitive and the degrees of X are odd, then X is arc transitive.

3.3 Edge & Vertex Connectivity

Definition 3.3.1 (Edge Connectivity)
The edge connectivity of a graph X, denoted $\kappa_1(X)$, is the minimum cardinality of an edge subset whose deletion increases the number of components in X.

Let $A \subseteq V(X)$. Write

$$\partial A = \delta(A)$$

to be the set of edges leaving A.

Definition 3.3.2 (Edge Atom)
An edge atom of X is a minimum cardinality subset $S \subseteq V(X)$ such that $|\partial(S)| = \kappa_1(X)$.

Lemma 3.3.1
Let X be connected and vertex transitive. Any two distinct edge atoms are vertex disjoint.

Proof
Let $\kappa = \kappa_1(X)$. Suppose $A \neq B$ are edge atoms. By minimality, $|A|, |B| \leq \frac{|V(X)|}{2}$ or else taking $\bar{A}$ or $\bar{B}$ contradicits minimality.

Case I: $A \cup B = V(X)$ Then $|A| = |B| = \frac{|V(X)|}{2}$ and we necessarily have $A \cap B = \varnothing$.

Case II: $A \cup B \subsetneq V(X)$ Since $A \subsetneq A \cup B$, it must be that $|\partial(A \cup B)| \geq \kappa$. On the other hand, $A \cap B \subsetneq A$ implies $|\partial(A \cap B)| > \kappa$.

By the submodularity of the $|\partial(\cdot)|$ function

$$2\kappa < |\partial(A \cup B)| + |\partial(A \cap B)| \leq |\partial a| + |\partial B| = 2\kappa,$$

which is a contradiction.

Lemma 3.3.2
Let X be connected and vertex transitive. Suppose S is a block of imprimitivity for $\mathrm{Aut}(X)$. Then $X[S]$ is regular.

Proof
Let $u \neq v \in S$ and set $Y := X[S]$. Since X is vertex transitive, we can choose some $g \in \mathrm{Aut}(X)$ for which $u^g = v$ so that $S \cap S^g = \varnothing$.

By the choice of S, $S = S^g$. Hence $N_Y(v) = N_Y(u)^g$ and $\deg_Y(u) = \deg_Y(v)$ as required.

Theorem 3.3.3
If X is connected and vertex-transitive, then $\kappa_1(X)$ is equal to the degree of vertices in X.

Proof
Let k be the degrees of vertices in X. We must have $\kappa_1(X) \leq k$. We wish to argue that $\kappa_1(X) \geq k$. Let A be an edge atom. It is sufficient to show that $\kappa_1(X) = |\partial A| \geq k$.

Let $g \in \mathrm{Aut}(X)$ and $B := A^g$. We have shown that either $A = B$ or $A \cap B = \varnothing$ since B is also an edge atom. It follows that A is a block of imprimitivity.

But then $X[A]$ is ℓ-regular for some $0 \le \ell \le k-1$. We know that $\ell < k$ since X is connected. Hence $|A| \ge \ell + 1$ and $\ell \le |A| - 1$. Thus

$$\begin{aligned} |\partial A| &= |A|(k-\ell) && \ell \text{ neighbours within } A \\ &\ge |A|(k+1-|A|) \\ &\ge 1 \end{aligned}$$

and $k \ge |A|$.

Now then,

$$\begin{aligned} |\partial A| - k &\ge |A|k + |A| - |A|^2 - k \\ &= k(|A|-1) - |A|(|A|-1) \\ &= (|A|-1)(k-|A|) \\ &\ge 0 \cdot 0 \\ &= 0. \end{aligned}$$

Notice we have equality if $|A| = k$ or $|A| = 1$.

We conclude that $\kappa_1(X) = k$ as required.

Theorem 3.3.4
A k-regular vertex-transitive graph has vertex connectivity at least

$$\frac{2}{3}(k+1).$$

We can find graphs for which this bound is tight.

3.4 Matchings

We say a vertex is *critical* if it is saturated in every maximum matching.

Lemma 3.4.1
Let $z_1, z_2 \in V(X)$ be vertices such that no maximum matching misses both of them. Suppose M_{z_1}, M_{z_2} are maximum matchings missing z_1, z_2 respectively. Then $M_{z_1} \Delta M_{z_2}$ has an even alternating $z_1 z_2$-path.

Proof
Let M_{z_1}, M_{z_2} be maximum matchings. Thus z_1, z_2 are each on SOME even alternating path in $M_{z_1} \Delta M_{z_2}$.

Let P be the path containing z_1. If it does not contain z_2, then $M_{z_2} \Delta P$ is a maximum matching missing both z_1, z_2. which is a contradiction.

Lemma 3.4.2
Let $u, v \in V(X)$ be vertices in a connected graph X with no critical vertices. Then X contains a matching missing at most 1 vertex.

Proof
To see this we show that for every $u, v \in V(X)$, there is no maximum matching missing both u, v. Then any matching misses at most 1 vertex.

Let P be a uv-path. We argue by induction on the length $|P|$. The case where $|P| = 1$ is trivial.

Suppose $|P| \geq 2$. Let $x \notin \{u, v\}$ be a vertex on P. By the induction hypothesis, no maximum matching misses both u, x nor does a maximum matching miss both v, x.

Since x is not critical, there is a maximum matching M_x missing x. This matching must then saturate u, v.

Suppose now towards a contradiction that some maximum matching M misses both u, v. By the previous lemma, $M_x \Delta M$ has an even alternating path ux-path. Similarly, $M_x \Delta M$ has an even alternating xv-path.

By the parity of the paths and the fact that u, v are not M saturated, this can only happen if $u = v$, which contradicts our assumptions.

By induction, we conclude the result.

Proposition 3.4.3
Let X be connected and vertex transitive. Then X has a matching missing at most 1 vertex.

Proof
Case I: X has a critical vertex Let u be a critical vertex and any maximum matching M. Fix any other vertex v. Since X is vertex transitive, there is some $g \in \mathrm{Aut}(X)$ for which $v^g = u$.

M^g is a maximum matching and thus saturates u. But then M saturates v.

Thus every vertex is critical and X has a perfect matching.

Case II: X has no critical vertices By our previous lemma, X contains a matching missing at most 1 vertex.

Lemma 3.4.4
Let X be connected and vertex transitive. Then every edge lies in some maximum matching.

Proof
We argue by induction on the number of vertices and the number of edges in a connected vertex-transitive graph. The base case of a single isolated vertex is trivial.

Inductively, suppose that some $e \in E(X)$ is not contained in any maximum matching. It follows that X is cannot be edge transitive. Let Y be the subgraph induced by the orbit of e under $\mathrm{Aut}(X)$. Then Y must be a proper subgraph of X and we claim Y is vertex as well as edge transitive.

First, $\mathrm{Aut}(X) \leq \mathrm{Aut}(Y)$. Indeed, for all $uv \in E(Y)$ and $g \in \mathrm{Aut}(X)$,

$$u^g v^g \in e^{\mathrm{Aut}(X)} \implies u^g v^g \in E(Y).$$

But then Y is edge transitive by construction. Moreover, the vertex transitivity is due to the vertex transitivity of X.

Case I: Y is connected Y has a maximum matching missing at most 1 vertex. But then by the edge transitivity of Y, there is a maximum matching missing at most 1 vertex of $V(Y) = V(X)$ that includes e. This is necessarily a maximum matching of X covering e, which is a contradiction.

Case II: Y has components $C_1, C_2, \dots, C_m$ We claim that each C_i is a block of imprimitivity of $\mathrm{Aut}(X)$. Moreover, the components are pairwise isomorphic and are vertex plus edge transitive.

The fact that each component is a block of imprimitivity follows from the fact that automorphisms preserve distances. Hence if a vertex within a component is sent to the some component, all vertices of that component are sent there as well.

The vertex and edge transitivity follows from the each component being blocks of imprimitivity as well as the fact that Y is vertex and edge transitive. Finally, vertex transitivity and the assumption that each component is a block of imprimitivity yields the pairwise isomorphisms.

If each component has an even number of vertices, then each C_i has a perfect matching. But then by edge transitivity, Y and thus X has a perfect matching containing e.

Suppose now that each component is odd. Define

$$V(Z) = \{C_1, \dots, C_m\}$$
$$E(Z) = \{C_iC_j : \exists f \in \partial_X(C_i) \cap \partial_X(C_j)\}$$

Z must be connected since X is connected. We claim that Z is vertex transitive.

To see this claim recall how $\mathrm{Aut}(X)$ permutes the components. Thus $f \in \partial_X(C_i) \cap \partial_X(C_j)$, implies $f^g \in \partial_X(C_i^g) \cap \partial_X(C_i^g)$ for all $g \in \mathrm{Aut}(X)$. Hence $\mathrm{Aut}(X)$ induces a subgroup of $\mathrm{Aut}(Z)$ which acts transitively on the vertices.

Now, Z has a matching missing at most 1 C_i. By vertex transitivity, we may assume this is the component C containing the edge e. By the edge transitivity of C, there is a maximum matching containing e and missing at most 1 vertex.

For each C_iC_j in the matching, let us pick an edge in $\partial_X(C_i) \cap \partial_X(C_j)$. This is a matching M in X. Using the fact that each C_i is vertex transitive with a maximum matching missing 1 vertex, we can thus using M to "connect" the maximum matchings within C_i. This yields a maximum matching of X missing exactly 1 vertex and containing e.

By induction, we conclude that every edge is contained in some maximum matching.

Theorem 3.4.5
Let X be connected and vertex transitive. Then X has a matching missing at most 1 vertex and each edge is contained in some maximum matching.

3.5 More on Cayley Graphs

Definition 3.5.1 (Semiregular)
A permutation group G acting on V is semiregular if $G_x = 1$ for all $x \in V$.

Proposition 3.5.1
If G is semiregular, then

$$|G| = |x^G|$$

for all $x \in V$.

Proof
Orbit-Stabilizer lemma.

Definition 3.5.2 (Regular)
If G is semiregular and transitive, then G is regular.

Proposition 3.5.2
If G acting on V is regular, then $|G| = |V|$.

Proof
$|G| = |x^G| = |V|$.

Lemma 3.5.3
Let G be a group and $C \subseteq G \setminus \{1\}$ be inverse-closed. Then $\mathrm{Aut}(X(G, C))$ contains a regular subgroup isomorphic to G.

Proof
Let $X = X(G, C)$. For $g \in G$, define $\tau_g : V(X) \to V(X)$ given by

$$x \mapsto gx.$$

Write $T := \{\tau_g : g \in G\}$. By our initial work with Cayley graphs, we know that $T \leq \mathrm{Aut}(X)$. Moreover, T acts transitively on X. In addition $T \cong G$ by the map $\tau_g \mapsto g$. Finally, T is semiregular, as any non-identity element does not fix any vertices.

Thus by definition, T is a regular subgroup of $\mathrm{Aut}(X)$.

Lemma 3.5.4
If $G \leq \mathrm{Aut}(X)$ acts regularly on $V(X)$, then $X = X(G, C)$ for some inverse-closed $C \subseteq G \setminus \{1\}$.

Proof
Since G is regular, then $|G| = |V(X)|$. We aim to construct C so that $X \cong X(G, C)$.

Fix $u \in V(X)$. Since G is regular, there is a unique $g_v \in G$ such that $u^{g_v} = v$. Define

$$C := \{g_v : v \sim u\}$$

as the subset of G sending u to a neighbour of u.

It is clear that $1 \notin C$, as X does not have loops. It suffices to show that

$$xy \in E(X) \iff g_x^{-1} g_y \in C \iff g_y^{-1} y_x \in C.$$

$\underline{xy \in E(X) \iff g_x^{-1} g_y \in C}$ Recall that $g_x \in G \leq \mathrm{Aut}(X)$. Hence $g_x^{-1} \in \mathrm{Aut}(X)$. Thus

$$x^{g_x^{-1}} y^{g_x^{-1}} \in E(X) \iff xy \in E(X).$$

Since $u^{g_x} = x$, we have $x^{g_x^{-1}} = u$. Similarly, $u^{g_y} = y$ so $y^{g_x^{-1}} = u^{g_x^{-1} g_y}$. So

$$xy \in E(X) \iff uu^{g_x^{-1} g_y} \in E(X) \iff g_x^{-1} g_y \in C$$

as desired.

$\underline{xy \in E(X) \iff g_y^{-1} g_x \in C}$ This is identical except we consider g_y instead of g_x.

Hence C is inverse closed and $X \cong X(G, C)$ with an isomorphism given by $x \mapsto g_x$.

We are now concerned with Cayley graphs on the same group.

Definition 3.5.3 (Group Automorphism)
Let G be a group. $\theta : G \to G$ is an automorphism if θ is bijective and $\theta(gh)\theta(g)\theta(h)$ for all $g, h \in G$.

Lemma 3.5.5
Let θ be an automorphism of G. Then $X(G, C) \cong X(G, \theta(C))$.

Proof
We claim that θ is an isomorphism $X(G, C) \to X(G, \theta(C))$.

We have

$$\begin{aligned} g^{-1}h \in C &\iff \theta(g^{-1}h) \in \theta(C) \\ &\iff \theta(g^{-1})\theta(h) \in \theta(C) \\ &\iff \theta(g)^{-1}\theta(h) \in \theta(C) \end{aligned}$$

as desired.

We will note that if $X(G, C) \cong X(G, \vartheta(C))$, then it is not necessary that ϑ is an automorphism.

Definition 3.5.4 (Generating Set)
Let G be a group. $C \subseteq G$ is a generating set if every $g \in G$ is a product of elements in C.

Lemma 3.5.6
$X = X(G, C)$ is connected if and only if C is a generating set for G.

Proof
<u>$(\implies)$</u> If X is connected, there is a eg-path for all $g \in G$. Fix $g \in G$. Enumerate some eg-path

$$e = h_0, h_1, h_2, \dots, h_k, g.$$

Observe that $h_{i-1}^{-1} h_i \in C$ for $i \in [k]$, and $h_k^{-1} g \in C$.

But then

$$g = \left(\prod_{i=1}^{k} h_{i-1}^{-1} h_i \right) h_k^{-1} g$$

is a product of elements of C.

<u>$(\impliedby)$</u> Suppose C generates G. We argue there is a eh-path for all $h \in G$.

Write h as a product of g_i's from C:

$$h = c_1 c_2 \dots c_k.$$

Notice that

$$\begin{aligned} e &\sim c_1 & \qquad ec_1 &= c_1 \in C \\ c_1 &\sim c_1 c_2 & \qquad c_1^{-1} c_1 c_2 &= c_e \in C \\ \prod_{i=1}^{\ell} c_i &\sim \prod_{i=1}^{\ell+1} c_i & \qquad \left(\prod_{i=1}^{\ell} c_i \right)^{-1} \prod_{i=1}^{\ell+1} c_i &= c_{\ell+1} \in C \end{aligned}$$

This e is indeed connected to $\prod_{i=1}^{k} c_i = h$ and we are done.

3.6 Retracts

Definition 3.6.1 (Retract)
A subgraph Y of X is a retract if there is some homomorphism $f : X \to Y$ such that

$$f\Big|_Y = 1_Y.$$

Theorem 3.6.1
Every connected vertex-transitive graph is isomorphic to a retract of a Cayley graph.

Proof
Let X be connected and vertex transitive. Fix $x \in V(X)$ and

$$C := \{g \in \mathrm{Aut}(X) : x \sim x^g\}.$$

Let $G \leq \mathrm{Aut}(X)$ as the subgroup generated by C. We claim that G acts transitively on $V(X)$.

It suffices to show that there is an element sending $x \mapsto y$ for all $y \in V(X)$. We argue by induction on $d(x, y)$ that $y \in x^G$. The case where $d(x, y) = 1$ is trivial.

Suppose $d(x, y) \geq 2$. There is some neighbour of y, say $z \neq x, y$, on the shortest xy-path in X. By induction, there is some $g \in G$ sending $x \mapsto z$. Since X is vertex transitive, we can find some $h \in \mathrm{Aut}(X)$ sending $z \mapsto y$.

We have $z \sim z^h$. Since automorphisms preserve edges, $z^{g^{-1}} \sim z^{g^{-1}h}$ and $g^{-1}h \in C \subseteq G$ by definition. But then $h = g \cdot g^{-1}h \in G$ as both are elements of G.

Let $Y := X(G, C)$. We argue that X is isomorphic to a retract of Y.

Recall from our work with group actions on graphs that $H := \{h \in G : x^h = y\}$ is a left coset of the stabilizer G_x. Thus for every $y \in V(X)$,

$$C_y := \{g \in G : x^g = y\}$$

is a left coset of G_x. Hence $C = \bigcup_{y \sim x} C_y$ by definition is a union of left cosets of G_x. Moreover, $C \cap G_x = \varnothing$ as $x \not\sim x$.

Observe that for any $a, b \in \mathrm{Aut}(X)$,

$$x^a \sim x^b \iff x \sim x^{a^{-1}b} \iff a^{-1}b \in C.$$

Let $A_1, \dots, A_k$ be the left cosets of G_x. Pick a representative $a_i \in A_i$ for each $i \in [k]$. We claim that $Y[a_1, \dots, a_k] \cong X$ and that $Y[a_1, \dots, a_k]$ is a retract of Y. This would terminate the proof.

First, we show that $C = G_x C G_x$. Indeed, it is clear that $C \subseteq G_x C G_x$. Pick $h, h' \in G_x$ and $g \in C$. We have

$$\begin{aligned} x &\sim x^g \\ x = x^h &\sim (x^h)^g = x^{gh} \\ x = x^{h'} &\sim x^{h'gh}. \end{aligned}$$

Then $h'gh \in C$ and $G_x C G_x \subseteq C$ as required.

Next, we claim that in $Y = X(G, C)$, $E(A_i) = \varnothing$ for all i. Moreover, for all $1 \leq i < j \leq k$, either $\partial A_i \cap \partial A_j = 0$ or A_i, A_j induces a complete bipartite graph.

For any $g' \in G$, there is some j for which $g' \in A_j$. Thus we can write $g' = a_j g$ for some $g \in G_x$. Suppose $g, h \in G_x$. Then

$$\begin{aligned} a_i g \sim a_j h &\iff (a_i g)^{-1} a_j h \in C \\ &\iff g^{-1} a_i^{-1} a_j h \in C \\ &\iff a_i^{-1} a_j \in gCh^{-1} \in G_x C G_x = C. \end{aligned}$$

This shows that the adjacency between any vertices in different cosets depend on the cosets and not the specific vertices. Moreover, since $1 \notin C$, we know that $a_i^{-1} a_i \notin C$ and hence $a_i g \not\sim a_i h$ for any $g, h \in G_x$.

We now claim that $Y[a_1, \dots, a_k] \sim X$. By the previous claim, $a_i a_j \in E(Y[a_1, \dots, a_k])$ if and only if $a_i^{-1} a_j \in C$. Consider the map $\rho : V(X) \to \{a_i, \dots, a_i\}$ given by

$$y \mapsto a_j \in C_y.$$

Select $u, v \in V(X)$. Since G acts transitively, there are $g, h \in G$ such that $x^g = u, v^h = v$.

$$\begin{aligned} u \sim v &\iff x^g \sim x^h \\ &\iff x \sim x^{g_{-1} h} \\ &\iff g^{-1} h \in C. \end{aligned}$$

On the other hand, $\rho(u)\rho(v) \in E(Y[a_1, \dots, a_k])$ if and only if $\rho(u)^{-1}\rho(v) \in C$. Since $u = x^g, v = x^h$, we know that g, h are representatives of the cosets to which $\rho(u), \rho(v)$ belong, respectively. Thus we may assume without loss of generality that $h = \rho(u), h = \rho(v)$. So $\rho(u) \sim \rho(v) \iff g^{-1} h \in C$ and ρ preserves adjacencies.

Finally, we show that $Y[a_1, \ldots, a_k]$ is a retract of Y. This is given by the homomorphism $\tau : V(Y) \to \{a_1, \ldots, a_k\}$ given by

$$g \mapsto g_j, g \in A_j.$$

Clearly then this is a homomorphism by our work above and acts as the identity on $\{a_1, \ldots, a_k\}$.

Chapter 4

Generalized Polygons

4.1 Incidence Graphs

Definition 4.1.1 (Incidence Structure)
Let $\mathcal{P}$ be a set of points, $\mathcal{L}$ a set of lines, and $I \subseteq \mathcal{P} \times \mathcal{L}$ an incidence relation. The triple $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ defines an incidence structure.

If $(p, L) \in I$, then we say point p and line L are *incident*.

The triple $\mathcal{I}^* = (\mathcal{L}, \mathcal{P}, I^*)$ where

$$I^* = \{(L, p) : (p, L) \in I\}$$

is the *dual* of $\mathcal{I}$.

Definition 4.1.2 (Incidence Graph)
Given an incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$, the incidence graph $X(\mathcal{I})$ is defined as the bipartite graph $(\mathcal{P}, \mathcal{L}, E)$ where

$$pL \in E \iff (p, L) \in I.$$

Remark that $X(\mathcal{I}) \cong X(\mathcal{I}^*)$.

Definition 4.1.3 (Automorphism)
An automorphism of $(\mathcal{P}, \mathcal{L}, I)$ is a permutation of $\mathcal{P}, \mathcal{L}$ such that $\mathcal{P}^\sigma = \mathcal{P}$ and $\mathcal{L}^\sigma = \mathcal{L}$ with

$$(p^\sigma, L^\sigma) \in I \iff (p, L) \in I.$$

4.1.1 Partial Linear Space

Definition 4.1.4 (Partial Linear Space)
$\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ is a partial linear space if for all $x, y \in \mathcal{P}$, there exists at most one $L \in \mathcal{L}$ such that $(x, L) \in I, (y, L) \in I$.

Lemma 4.1.1
If $\mathcal{I}$ is a partial linear space, then any two lines are incident with at most 1 point.

If lines L_1, L_2 are incident with p, then we say these two lines are *concurrent* and meet at p.

Lemma 4.1.2
If $\mathcal{I}$ is a partial linear space, then $X(\mathcal{I})$ has girth at least 6.

Proof
$X(\mathcal{I})$ has no 4-cycle as otherwise, there are two points incident with two lines.

4.2 Projective Planes

Definition 4.2.1 (Projective Plane)
A projective plane is a partial linear space satisfying

(C1) Every two lines meet in a unique point

(C2) Every two points lie in a unique line

(C3) There exists 3 noncollinear points (they form a triangle)

Theorem 4.2.1
Let $\mathcal{I}$ be a partial linear space containing a triangle. Then $\mathcal{I}$ is a projective plane if and only if $X(\mathcal{I})$ has diameter 3 and girth 6.

Proof
$(\implies)$ Let $\mathcal{I} = (\mathcal{P}, \mathcal{L}, I)$ be a projective plane with a triangle. Then every $x, y \in \mathcal{P}$ are at distance 2 and every $L_1, L_2 \in \mathcal{L}$ are at distance 2.

Pick $x \in P, L \in \mathcal{L}$. Either $(x, L) \in I$, or there is some $L' \neq L$ such that $(x, L') \in I$. In the latter case, there is $y \in P$ such that L, L' meet at y. So $d(x, L) \leq 3$ and the diameter is at most 3. But since $\mathcal{I}$ has a triangle, $X(\mathcal{I})$ is NOT a complete bipartite graph. Thus $X(\mathcal{I})$ has diameter at least 3.

Now, $\mathcal{I}$ is a partial linear space. Hence we know the girth is at least 6.

Let $\{x, y, z\}$ be a triangle. Suppose x, y meet at L_1, x, z meet at L_2, and y, z meet at L_3. Since x, y, z are not collinear, L_1, L_2, L_3 are distinct. Hence the girth is at most 6.

$(\impliedby)$ Conversely, suppose $\mathcal{I}$ is a partial linear space and $X(\mathcal{I})$ has diameter 3 as well as girth 6.

To see (C2), note that for any $x \neq y \in \mathcal{P}$, the distance is even and at most 3. Hence they are distance 2 apart. It follows that there is $L \in \mathcal{L}$ such that $(x, L), (y, L) \in I$. Moreover, the girth being 6 (strictly more than 4) implies that such L is unique.

(C1) is satisfied with an identical argument.

(C3) is given by the assumption of girth being 6.

4.3 A Family of Projective Planes

Consider $\mathbb{F}_q$, the finite field of order q. Let $V := \mathbb{F}_q^3$. Define

$$\mathrm{PG}(2, q) := (\mathcal{P}, \mathcal{L}, I)$$

where $\mathcal{P}$ are the 1-dimensional subspaces of V, $\mathcal{L}$ are the 2-dimensional subspaces of V, and $(p, \mathcal{L}) \in I$ if p is a subspace of L.

Alternatively, L can be represented by the orthogonal complement of L_+, which is $\mathrm{span}\{a\}$. Here $a^T p_1, a^T p_2 = 0$.

Now, V has $q^3 - 1$ non-zero vectors. There are $q - 1$ non-zero vectors in each 1-dimensional subspace. Thus

$$|\mathcal{P}| = \frac{q^3 - 1}{q - 1} = 1 + q + q^2.$$

By the bijection between 1-dimensional subspaces and 2-dimensional subspaces,

$$|\mathcal{L}| = 1 + q + q^2.$$

Each line has $q^2 - 1$ non-zero vectors. Hence each line is incident with $\frac{q^2-1}{q-1} = 1 + q$ points. Moreover, for $(p, L) \in I$ to hold, we have $p \subseteq L$. There are $q^2 - 1$ linearly independent vectors to p, each spanning $q - 1$ non-zero points. Hence each point has $1 + q$ lines passing through it.

Observe that $X(\mathrm{PG}(2, q))$ has $2(q^2 + q + 1)$ vertices and is $(q + 1)$-regular.

Remark that $\mathrm{PG}(2, 2)$ is the Fano plane.

Lemma 4.3.1
$\mathrm{PG}(2, q)$ is a projective plane.

Proof
Let $L_1 := \operatorname{span}\{u, v\}$ and $L_2 := \operatorname{span}\{u', v'\}$ such that $L_1 \neq L_2$. We have $\dim(L_1 \cap L_2) \geq 1$ since $\dim V = 3$. But $\dim(L_1 \cap L_2) < 2$ since $L_1 \neq L_2$. It follows that $\dim(L_1 \cap L_2) = 1$. Thus there is a unique point where L_1, L_2 meet. This verifies (C1).

Let $p_1 = \operatorname{span}\{u\}$ and $p_2 = \operatorname{span}\{v\}$, where $v \notin p_1$. Suppose L is a line incident with both points. Then $\operatorname{span}\{u, v\} \subseteq L$ and the fact that $\dim L = 2$ implies that $L = \operatorname{span}\{u, v\}$. This shows (C2).

Let u, v, w be 3 linearly independent vectors. Then $\operatorname{span}\{u, v, w\} = V$ and so u, v, w are not contained in a 2-dimensonal subspace. Hence the spans of each vector form a triangle. This demonstrates (C3).

4.3.1 Automorphisms

Consider $\mathrm{GL}(3, q)$, the set of invertible 3×3 matrices over $\mathbb{F}_q$. Note that $\mathrm{GL}(3, q)$ is a group which acts on vectors of $\mathbb{F}_q^3$ through multiplication.

Lemma 4.3.2
Let $A \in \mathrm{GL}(3, q)$ and $X = X(\mathrm{PG}(2, q))$. Then A induces a permutation on $V(X)$.

Lemma 4.3.3
$\mathrm{GL}(3,q) \leq \mathrm{Aut}(\mathrm{PG}(2,q))$.

Proof
Let $A \in \mathrm{GL}(3,q)$. Assume that $p \sim L$, ie $p \subseteq L$. Suppose that $p = \mathrm{span}\{u\}$ and $L = \mathrm{span}\{u,v\}$. Thus

$$\begin{aligned} p^A &= \mathrm{span}\{Au\} \\ L^A &= \mathrm{span}\{Au, Av\} \\ p^A &\sim L^A. \end{aligned}$$

Finally, $\mathcal{P}^A = \mathcal{P}$ and $\mathcal{L}^A = \mathcal{L}$.

Lemma 4.3.4
$X(\mathrm{PG}(2,q))$ is arc-transitive.

Note this implies the incidence graph is both vertex and edge-transitive.

Proof
Suppose $p_1 \sim L$ and $p_2 \sim L_2$. Write

$$\begin{aligned} p_1 &= \mathrm{span}\{u_1\} \\ L_1 &= \mathrm{span}\{u_1, v_1\} \\ p_2 &= \mathrm{span}\{u_2\} \\ L_2 &= \mathrm{span}\{u_2, v_2\}. \end{aligned}$$

By elementary linear algebra, there is some invertible A for which $Au_1 = u_2, Av_1 = v_2$. Then $(p_1, L_2)^A = (p_2, L_2)$. Hence $\mathrm{GL}(3,q)$ acts transitively on $\mathcal{P}, \mathcal{L}$.

It suffices to show that there is some $\pi \in \mathrm{Aut}(X)$ such that $\pi(\mathcal{P}) = \mathcal{L}$. This is given by

$$\pi \,\mathrm{span}\{u\} := \mathrm{span}\{u\}^\perp.$$

Suppose $(p, L) \in I$ so $p = \mathrm{span}\{u\}, L = \mathrm{span}\{u,v\}$. We have

$$p^\pi = \mathrm{span}\{u\}^\perp, L^\pi = \mathrm{span}\{u,v\}^\perp.$$

But then $L^\pi \subseteq p^\pi$ hence $L^\pi \sim p^\pi$ and $\pi \in \mathrm{Aut}(X)$.

Chapter 5

Homomorphisms

5.1 Definitions & Basic Results

We write $X \to Y$ to denote the existence of a homomorphism from X to Y.

Lemma 5.1.1
The relation $\to$ is reflexive and transitive.
But neither symmetric nor antisymmetric. Thus $X \to Y$ does not in general imply $Y \to X$ and it is possible for $X \ncong Y$ to satisfy $X \to Y, Y \to X$.

Since $\to$ is not symmetric, it is NOT a equivalence relation. Since $\to$ is not antisymmetric, it is not a partial order, unless we restrict to a specific family of graphs.

Definition 5.1.1 (Homomorphically Equivalent)
We say X, Y are homomorphically equivalent if $X \to Y, Y \to X$.

This clearly an equivalence relation.

A homomorphism is *surjective* if $V(Y) = f(V(X))$.

Lemma 5.1.2
If surjective homomorphisms exist from X to Y and Y to X, then $X \cong Y$.

Definition 5.1.2 (Odd Girth)
The odd girth of X is the length of the shortest odd cycle in X. If X is bipartite, then its odd girth is ∞.

Lemma 5.1.3
Suppose $X \to Y$. Then
(a) $\chi(X) \le \chi(Y)$
(b) the odd girth of X is at least the odd girth of Y

Corollary 5.1.3.1
(a) There is no homomorphism from C_{2n+1} to K_2
(b) There is no homomorphism from the Peterson graph to C_4
(c) There is no homomorphism from the Peterson graph to any of its proper subgraph

5.2 Cores

Definition 5.2.1 (Core)
A graph X is a core if any automorphism from X to itself is an automorphism.

Definition 5.2.2 (Core Of)
A subgraph Y of X is a core of X if Y is a core and $X \to Y$.

Thus the cores of X are minimal subgraphs that are homomorphic images of X.

Example 5.2.1
K_n is a core.

Example 5.2.2
A graph is *critical* if any of its proper subgraphs has strictly smaller chromatic number.

Then $\chi(X) > \chi(Y)$ for every proper subgraph Y and $X \not\to Y$. Thus every critical graph is a core.

Lemma 5.2.3
If Y is a core of X, then Y is a retract of X.

Proof
Let $f : X \to Y$ be a homomorphism. Since Y is a core, $g := f\big|_Y$ is an automorphism of Y.

Then $g^{-1} \circ f : X \to Y$ is a retraction.

Lemma 5.2.4
Let Y_1, Y_2 be cores. Then Y_1, Y_2 are homomorphically equivalent if and only if they are isomorphisms.

Proof
Let $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$ be homomorphisms. Then both $f \circ g, g \circ f$ are homomorphisms. But Y_1, Y_2 are cores. So $f \circ g$ and $g \circ f$ are both bijective. Hence f, g must both be surjective.

Lemma 5.2.5
Every graph X has a core which is an induced subgraph and unique up to isomorphism.

Due to this result, we may refer to THE core $X^\bullet$ of X.

Proof
Existence: Let $\mathcal{F}$ be the set of subgraphs of X that are homomorphic images of X. $\mathcal{F} \neq \varnothing$ since X is a homomorphic image of X.

Let Y be a minimal graph in $\mathcal{F}$ with respect to the subgraph relation. By the minimality, any homomorphism from Y to itself must be an automorphism of Y. Hence Y is a core of X.

Since Y is a retract of X, Y must be induced.

Uniqueness: Let Y_1, Y_2 be cores of X. Let $f_i : X \to Y_i$ be a homomorphism from X to Y_i for $i = 1, 2$.

Consider $f_1\big|_{Y_2} : V(Y_2) \to V(Y_1)$. This is a homomorphism from Y_2 to Y_1. Similarly,

$f_2\big|_{Y_1}$ is a homomorphism from Y_1 to Y_2. Hence $Y_1 \to Y_2$ and $Y_2 \to Y_1$ and they are homomorphically equivalent.

Since Y_1, Y_2 are cores, they are homomorphically equivalent if and only if they are isomorphic.

Lemma 5.2.6
Two graphs X, Y are homomorphically equivalent if and only if their cores are isomorphic.

Proof
Suppose $X \to Y, Y \to X$. Then

$$X^\bullet \to X \to Y \to Y^\bullet$$
$$Y^\bullet \to Y \to X \to X^\bullet$$

By the a lemma, $X\bullet \cong Y\bullet$.

Suppose now that $X^\bullet \cong Y^\bullet$. Then $X^\bullet \to Y^\bullet$ and $Y^\bullet \to X^\bullet$.

It follows by a previous lemma that

$$X^\bullet \to X \to Y \to Y^\bullet$$
$$Y^\bullet \to Y \to X \to X^\bullet$$

Corollary 5.2.6.1
$\to$ is a partial order on the set of isomorphism classes of cores.

Proof
We know $\to$ is reflexive and transitive. It is antisymmetric on the set of isomorphism classes of cores since if X, Y are cores that are homomorphically equivalent, then $X \cong Y$.

Lemma 5.2.7
Let X be connected. If every path of length 2 lies in a shortest odd cycle of X, then X is a core.

Proof
Suppose towards a contradiction that $X^\bullet \neq X$ and $f : X \to X^\bullet$ is a retraction. Then by connectedness, there are $u, v \in V(X)$ such that $u \sim v$, $v \in V(X^\bullet)$, and $u \notin V(X^\bullet)$.

Let $w := f(u) \in V(X^\bullet)$. Then $w \sim v$ and $w \not\sim u$. By assumption uvw is contained in a shortest odd cycle C. The homomorphic image of C under f is an odd closed walk where $f(u) = f(w) = w$.

Hence $f(X)$ has a strictly shorter odd cycle, which is a contradiction by earlier lemma. By contradiction, $X^\bullet = X$.

Corollary 5.2.7.1
The Peterson graph is a core.

5.3 Products

Definition 5.3.1 (Product Graph)
For graphs Y, Z, the (direct) product $Y \times Z$ is defined by $V(Y \times Z) = V(Y) \times V(Z)$ where

$$(y, z) \sim (y', z') \iff y \sim y' \wedge z \sim z'.$$

Lemma 5.3.1

(a) If Y, Z are both connected. Then $Y \times Z$ are disconnected if and only if both Y, Z are bipartite.

(b) Let $Y_1 + Y_2$ denote the disjoint union of Y_1, Y_2. Then $(Y_1 + Y_2) \times Z = Y_1 \times Z + Y_2 \times Z$.

(c) $Y \times Z \cong Z \times Y$ (commutative). $(Y_1 \times Y_2) \times Y_3 \cong Y_1 \times (Y_2 \times Y_3)$ (associative). However, $Y \times Z_1 \cong Y \times Z_2$ does not imply $Z_1 \cong Z_2$ [ie $K_1 \times (K_3 + K_3) \cong K_2 \times C_6 \cong C_6 + C_6$].

(d) $p_X : V(X \times Y) \to V(X)$ and $p_Y : V(X \times Y) \to V(Y)$ given by $(x, y) \mapsto x, (x, y) \mapsto y$, respectively, are homomorphisms from $X \times Y$ to X, Y.

Theorem 5.3.2
Let X, Y, Z be graphs. If $f : Z \to X$ and $g : Z \to Y$ are homomorphisms, there is a unique homomorphism $\phi : Z \to X \times Y$ such that

$$f = p_X \circ \phi$$
$$g = p_Y \circ \phi.$$

Proof
Define $\phi(z) := (f(z), g(z))$ for all $z \in Z$. If $u \sim v \in Z$, then $f(u) \sim f(v)$ in X and $g(u) \times g(v)$ in Y. Hence
$$(f(u), g(u)) \sim (f(v), g(v))$$
in $X \times Y$ by construction. By definition, ϕ is a homomorphism such that $f = p_X \circ \phi, g = p_Y \circ \phi$.

Suppose now that $\phi(z) = (\phi_X(z), \phi_Y(z))$ is a function such that $p_X \circ \phi(z) = \phi_x(z) = f(z)$ and $p_Y \circ \phi(z) = \phi_Y(z) = g(z)$ for all $z \in Z$. But then $\phi = \phi_X \times \phi_Y$ is determined by f, g.

We write
$$\mathrm{Hom}(X, Y)$$
to denote the set of homomorphisms from X to Y.

Corollary 5.3.2.1
$|\mathrm{Hom}(Z, X \times Y)| = |\mathrm{Hom}(Z, X)| \cdot |\mathrm{Hom}(Z, Y)|$.

Proof
By the previous lemma, there is an injective map
$$\mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X \times Y).$$
Thus $|\mathrm{Hom}(Z, X \times Y)| \leq |\mathrm{Hom}(Z, X)| \cdot |\mathrm{Hom}(Z, Y)|$.

For each $\phi \in \mathrm{Hom}(Z, X \times Y)$ $p_X \circ \phi \in \mathrm{Hom}(Z, X)$ and $p_Y \circ \phi \in \mathrm{Hom}(Z, Y)$. Thus we have an injective map
$$\mathrm{Hom}(Z, X \times Y) \to \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y).$$
Thus $|\mathrm{Hom}(Z, X \times Y)| \geq |\mathrm{Hom}(Z, X)| \cdot |\mathrm{Hom}(Z, Y)|$.

Definition 5.3.2 (Lattice)
A partially ordered set Λ is a lattice, if for each $x, y \in \Lambda$, there is a least upper bound $x \vee y$, as well as a greatest lower bound $x \wedge y$.

Lemma 5.3.3
The set of isomorphism classes of cores with partial order $\to$ is a lattice.

Proof
Least Upper Bound: Let X, Y be cores. We claim that $X \vee Y = (X+Y)^\bullet$.

It is clear that $X \to X+Y \to (X+Y)^\bullet$ as well as $Y \to X+Y \to (X+Y)^\bullet$. Thus $(X+Y)^\bullet$ is an upper bound of $X \times Y$.

Suppose now that Z is a core such that $X \to Z, Y \to Z$. Then $X+Y \to Z$.

Naturally, $(X+Y)^\bullet \to Z$ through the identity map and hence $X \vee Y = (X+Y)^\bullet$.

Greatest Lower Bound: We show that $X \wedge Y = (X \times Y)^\bullet$.

Indeed, $(X \times Y)^\bullet \to X \times Y \to X$ and $(X \times Y)^\bullet \to X \times Y \to Y$.

Suppose Z is a core such that $Z \to X, Z \to Y$. Then $Z \to (X \times Y) \to (X \times Y)^\bullet$. Hence $X \wedge Y = (X \times Y)^\bullet$ as desired.

Chapter 6

Matrix Theory

6.1 Spectral Graph Theory

6.1.1 The Adjacency Matrix

Definition 6.1.1 (Adjacency Matrix)
Let X be an undirected graph. Its adjacency matrix $A = A(X) \in \{0,1\}^{V \times V}$ is the square matrix such that

$$A_{u,v} = 1\{u \sim v\}.$$

Note that $A(X)$ is symmetric.

Definition 6.1.2 (Graph Characteristic Polynomial)
The characteristic polynomial of X is

$$\phi(A, x) := \det(xI - A(X)).$$

Note that characteristic polynomials do not uniquely determine X.

Definition 6.1.3 (Graph Spectrum)
The spectrum of X is the list of eigenvalues of $A(X)$ together with their algebraic multiplicities.

If $X \cong Y$, then X, Y have the same spectrum.

Lemma 6.1.1
Let $A = A(X)$. Then $(A^r)_{u,v}$ is the number of walks from u to v in X of length r.

Corollary 6.1.1.1
Suppose X has m edges and t triangles. Let $A = A(X)$. Then

$$\begin{aligned} \operatorname{tr} A &= 0 \\ \operatorname{tr} A^2 &= 2m \\ \operatorname{tr} A^3 &= 6t. \end{aligned}$$

Let $n := |V(X)|$. Remark that the number of closed v-walks is

$$\operatorname{tr} A^r = \sum_{i=1}^{n} \lambda_i^r.$$

6.1.2 Incidence Matrix

Definition 6.1.4 (Incidence Matrix)
The incidence matrix $B = B(X) \in \{0,1\}^{V \times E}$ of X defined by

$$(B)_{ue} = 1\{u \in e\}.$$

Theorem 6.1.2
Let X be a graph on n vertices and c bipartite components. Let $B = B(X)$. Then $\operatorname{rank} B = n - c$.

Proof
Remark that

$$\operatorname{rank} B = \operatorname{rank} B^T = n - \operatorname{null} B^T.$$

Hence it suffices to show that $\operatorname{null} B^T = c$.

Let $z \in \mathbb{R}^n$ such that $B^T z = 0$. By the definition of B,

$$z_u + z_v = 0$$

for all u, v such that $u \sim v$. In particular,

$$z_u = (-1)^r z_v$$

if u, v are joined by a path of length r.

If follows that $z_u = 0$ if u is in a non-bipartite component and z takes inverse values on vertices from opposite classes in a bipartite component.

Thus $\text{null}(B^T) = c$ as desired.

Lemma 6.1.3
Let $B = B(X)$ and L the line graph of X. Then

$$B^T B = 2I + A(L).$$

Definition 6.1.5 (Degree Matrix)
The degree matrix of X, $D = D(X) \in \mathbb{Z}_+^{V \times V}$, is the diagonal matrix where

$$(D)_{uu} = \deg(u).$$

for each $u \in V(X)$.

Lemma 6.1.4
Let $B = B(X), D = D(X)$, and $A = A(X)$. Then

$$BB^T = D(X) + A(X).$$

6.2 Symmetric Matrices

Proposition 6.2.1
Let A be a real symmetric $n \times n$ matrix.

(a) If u, v are eigenvectors of different eigenvalues, then $u \perp v$.

(b) All eigenvalues are real.

(c) If U is a subspace of $\mathbb{R}^n$, then U is A-invariant implies $U^\perp$ is A-invariant.

(d) U is a non-zero A-invariant subspace of $\mathbb{R}^n$ implies that U contains a real eigenvector of A.

(e) $\mathbb{R}^n$ has an orthonormal basis consisting of eigenvectors of A.

(f) $A = PDP^T$, where P is orthogonal, and columns of P are orthonoral eigenvectors of A.

(g) $A = \sum_{i=1}^n \lambda_i v_i v_i^T$ where λ_i is the eigenvalue corresponding to the orthonormal eigenvector v_i.

6.3 Eigenvectors of the Adjacency Matrix

We wish to find the eigenvalues of $A = A(X)$. Suppose there is some $f : V(X) \to \mathbb{R}$ such that $Af = \lambda f$. By the definition of A,

$$\begin{aligned}(Af)(u) &= \sum_v A_{uv} f(v) \\ &= \sum_{v \sim u} f(v).\end{aligned}$$

Hence if we can find (a vector/function) f such that

$$\sum_{v \sim u} = \lambda f(u)$$

for all $u \in V(X)$, Then λ is by definition an eigenvalue of $A(X)$.

Example 6.3.1 (Regular Graphs)
For a r-regular graph X, by setting $f = 1_n$,

$$\sum_{v \sim u} f(v) = r$$

Hence 1_n is an eigenvector of X.

Note that the converse is also true. 1_n is an eigenvector if and only if X is regular.

Example 6.3.2 (Cycles)
Let τ be an n-th root of unity. Then let $f(u) := \tau^u$.

$$\sum_{v\sim u} f(v) = (\tau^{-1} + \tau)\tau^u$$

so $\tau^{-1} + \tau$ is a real eigenvalue of X.

Lemma 6.3.3
Let X be k-regular with n vertices and eigenvalues

$$k, \theta_2, \dots, \theta_n.$$

Then $X, \bar{X}$ have the same eigenvectors and the eigenvalues of $\bar{X}$ are

$$n - k - 1, -\theta_2 - 1, \dots, -\theta_n - 1.$$

Proof
Let J be the all 1' matrix. By observation,

$$A(\bar{X}) = J - I - A(X).$$

Recall that 1_n is the eigenvector of $A(X)$ corresponding to eigenvalue k. Hence 1_n is the eigenvalue of $A(\bar{X})$ corresponding to the eigenvalue $n - 1 - k$.

Let $\{1_n, v_2, \dots, v_n\}$ be the orthogonal eigenvectors of A. For each $2 \leq j \leq n$,

$$\begin{aligned} A(X)v_j &= \theta_j v_j \\ 1^T v_j &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} A(\bar{X})v_j &= (J - I - A(X))v_j \\ &= -v_j - \theta_j v_j \\ &= (-1 - \theta_j)v_j. \end{aligned}$$

6.4 Positive Semidefinite Matrices

Definition 6.4.1 (Positive Semidefinite Matrix)
A real symmetric matrix A is positive semidefinite (PSD) if for all $u \in \mathbb{R}^n$,

$$u^T A u \geq 0.$$

If in addition, $u^T A u = 0 \iff u = 0$, then A is *positive definite.*

Proposition 6.4.1
Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

(a) A is PSD

(b) All eigenvalues of A are non-negative

(c) $A = B^T B$ for some B

Lemma 6.4.2
If L is a line graph, then $\lambda_{\min}(L) \geq -2$.

Proof
Suppose L is the line graph of X. Let $B = B(X)$. We know that $B^T B = A(L) + 2I$. But $B^T B$ is PSD. Hence $A(L) + 2I$ has minimum eigenvalue 0 and $\lambda_{\min}(L) \geq -2$.

Lemma 6.4.3
Let Y be an induced subgraph of X. Then

$$\lambda_{\min}(X) \leq \lambda_{\min}(Y) \leq \lambda_{\max}(Y) \leq \lambda_{\max}(X).$$

Proof
Let $A = A(X)$ and $\tilde{A} = A(Y)$. Put $\lambda = \lambda_{\max}(X)$. Thus $\lambda I - A$ is PSD.

For any $\tilde{f} : V(Y) \to \mathbb{R}$, extend it to a function $f : V(X) \to \mathbb{R}$ such that $f(u) = 0$ for each $u \in V(X) \setminus V(Y)$. Then

$$\begin{aligned} 0 &\leq f^T(\lambda I - A) f \\ &\leq \tilde{f}^T(\lambda I - \tilde{A})\tilde{f}. \end{aligned}$$

Thus $\lambda I - \tilde{A}$ is PSD and $\lambda_{\max}(\tilde{A}) \leq \lambda$.

Similarly, working on the PSD matrix $A(X) - \lambda_{\min}(X) \cdot I$, we cna show $\lambda_{\min}(\tilde{A}) \geq \lambda_{\min}(A)$.

Remark that this lemma follows from a more general interlacing theorem.

Definition 6.4.2 (Laplacian Matrix)
The Laplacian matrix of the graph X, is

$$L(X) := D(X) - A(X).$$

Proposition 6.4.4
$L := L(X)$ is positive semidefinite.

Proof
Let $n := |V(X)|$. For any $v \in \mathbb{R}^n$,

$$\begin{aligned}
x^T L x &= \sum_{u,v} x_u L_{uv} x_v \\
&= \sum_u x_u^2 \deg(u) - \sum_u x_u \sum_{v \sim u} x_v \\
&= \sum_{uv \in E} (x_u^2 + x_v^2) - \sum_{uv \in E} 2 x_u x_v \\
&= \sum_{uv \in E} (x_u - x_v)^2 \\
&\geq 0.
\end{aligned}$$

Remark that $x^T L x$ measures the "smoothness" of x on X.

Since $L(X)$ is PSD, its smallest eigenvalue is at least 0. Observe that

$$(L(X) 1^n)_u = \deg(u) - \deg_u = 0.$$

Hence 1_n is an eigenvector of $L(X)$ with eigenvalue 0 and the minimum eigenvalue of $L(X)$ is 0.

Proposition 6.4.5
Let $L = L(X)$ and

$$0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$$

be the eigenvalues of L. Then $\mu_2 > 0$ if and only if X is connected.

Proof

$(\implies)$ Suppose X is the disjoint union of 2 graphs X_1, X_2 with n_1, n_2 vertices. Then

$$L = \begin{bmatrix} L(X_1) & 0 \\ 0 & L(X_2) \end{bmatrix}.$$

Both $\begin{bmatrix} 0_{n_1} \\ 1_{n_2} \end{bmatrix}$ and $\begin{bmatrix} 1_{n_1} \\ 0_{n_2} \end{bmatrix}$ are eigenvectors with eigenvalue 0 and $\mu_2 = 0$.

$(\impliedby)$ Let f be an eigenvector with eigenvalue 0. But then

$$\begin{aligned} 0 &= f^T(Lf) \\ &= \sum_{uv \in E(X)} (f(u) - f(v))^2 \\ &\iff \forall uv \in E(X), f(u) = f(v). \end{aligned}$$

Since X is connected, f is constant on $V(X)$. The eigenspace corresponding to eigenvalue 0 has dimension 1. It follows that $\mu_2 > 0$.

Suppose X is k-regular. Let $A = A(X)$. Then

$$L = L(X) = kI - A(X).$$

If $k = \lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of A and $0 = \mu_1 \leq \cdots \leq \mu_n$ are the eigenvalues of L, then $\mu_i = k - \lambda_i$ for each $1 \leq i \leq n$.

6.5 Courant-Fisher Theorem

Definition 6.5.1 (Rayleigh Quotient)
The Rayleigh quotient of $X \in \mathbb{R}^n$ with respect to $A \in \mathbb{R}^{n \times n}$ is

$$\frac{x^T A x}{x^T x}.$$

Observe that if x is an eigenvector of A with respect to the eigenvalue λ,

$$\frac{x^T A x}{x^T x} = \lambda.$$

The Courant-Fisher theorem gives a relation between the extreme values of the Rayleigh quotients and the eigenvalues of A.

Theorem 6.5.1 (Courant-Fisher)
Let $A \in \mathbb{R}^{n\times n}$ be symmetric with eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

Then for all $k \in [n]$,

$$\lambda_k = \max_{S\subseteq \mathbb{R}^n : \dim S = k} \min_{x\in S : x\neq 0} \frac{x^T A x}{x^T x} = \min_{S\subseteq \mathbb{R}^n : \dim S = n-k+1} \min_{x\in T : x\neq x} \frac{x^T A x}{x^T x}.$$

Proof
Let $f_1, \dots, f_n$ be an orthonormal eigenvector basis of A where f_i corresponds to λ_i. Consider $S := \operatorname{span}\{f_1, \dots, f_k\}$.

We first show that

$$\max_{S\subseteq \mathbb{R}^n : \dim S = k} \min_{x\in S : x\neq 0} \frac{x^T A x}{x^T x} \geq \lambda_k.$$

For any $x = \sum_{i=1}^k c_i f_i \in S$,

$$\begin{aligned} x^T A x &= \left(\sum_{i=1}^k c_i f_i\right)^T A \left(\sum_{i=1}^k c_i f_i\right) \\ &= \left(\sum_{i=1}^k c_i f_i\right)^T \left(\sum_{i=1}^k c_i \lambda_i f_i\right) \\ &= \sum_{i=1}^k \lambda_i c_i^2. \end{aligned}$$

Moreover,

$$x^T x = \sum_{i=1}^k c_i^2.$$

Thus the Rayleigh quotient of x is

$$\begin{aligned} \frac{x^T A x}{x^T x} &= \frac{\sum_{i=1}^k \lambda_i c_i^2}{\sum_{i=1}^k c_i^2} \\ &\geq \lambda_k \frac{\sum_{i=1}^k c_i^2}{\sum_{i=1}^k c_i^2} \\ &= \lambda_k. \end{aligned}$$

We then show that

$$\max_{S \subseteq \mathbb{R}^n : \dim S = k} \min_{x \in S : x \neq 0} \frac{x^T A x}{x^T x} \leq \lambda_k.$$

Let S be a subspace of dimension k and $T := \operatorname{span}\{f_k, \dots, f_n\}$ be a subspace of dimension $n - k + 1$.

But then by dimension, $\dim(S \cap T) \geq 1$.

$$\begin{aligned}
\min_{x \in S : x \neq 0} \frac{x^T A x}{x^T x} &\leq \min_{x \in S \cap T : x \neq 0} \frac{x^T A x}{x^T x} \\
&\leq \max_{x \in S \cap T : x \neq 0} \frac{x^T A x}{x^T x} \\
&\leq \max_{x \in T : x \neq 0} \frac{x^T A x}{x^T x} \\
&\leq \lambda_k.
\end{aligned}$$

The last inequality follows from an identical calculation to our work above.

The other equality is identical and we omit the proof.

Theorem 6.5.2
Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $x \neq 0$ a vector that maximizes or minimizes the Rayleigh quotient. Then x is an eigenvector of λ_1 or λ_n respectively.

6.6 Maximum Graph Eigenvalue

Theorem 6.6.1
Let $\bar{d}, \Delta,$ be the average and maximum degrees of x. Then

$$\bar{d} \leq \lambda_1 \leq \Delta.$$

Proof
$\underline{\bar{d} \leq \lambda_1}$: By the Courant-Fisher theorem,

$$\begin{aligned}\lambda_1 &= \max_{x \in \mathbb{R}^n : x \neq 0} \frac{x^T A x}{x^T x} \\ &\geq \frac{1^T A 1}{1^T 1} \\ &= \frac{\sum_u \sum_{v \sim u} 1}{n} \\ &= \frac{2|E(X)|}{n} \\ &= \bar{d}.\end{aligned}$$

$\underline{\lambda_1 \leq \Delta}$: Let f be the unit eigenvector corresponding to λ_1. Let $u := \operatorname{argmax}_{v \in V} f(v)$. Without loss of generality, by taking $-f$ if necessary, we may assume that $f(u) > 0$.

Then

$$\begin{aligned}\lambda_1 &= \frac{(Af)(u)}{f(u)} \\ &= \frac{\sum_{v \sim u} f(v)}{f(u)} \\ &\leq \frac{\deg(u) \cdot f(u)}{f(u)} \\ &\leq \Delta\end{aligned}$$

as desired.

Lemma 6.6.2
If X is connected with $\lambda_1 = \Delta$, then X is Δ-regular.

Proof
Let f be the unit eigenvector corresponding to λ_1. Let $u := \operatorname{argmax}_{v \in V} f(v)$. Without loss of generality, by taking $-f$ if necessary, we may assume that $f(u) > 0$.

Then

$$\begin{aligned}\lambda_1 &= \frac{(Af)(u)}{f(u)} \\ &= \frac{\sum_{v\sim u} f(v)}{f(u)} \\ &\leq \frac{\deg(u)\cdot f(u)}{f(u)} \\ &\leq \Delta \\ &= \lambda_1.\end{aligned}$$

Now this holds with equality if and only if $f(v) = f(u)$ for all $v \sim u$ and $\deg(u) = \Delta$.

By connectivity, we can apply the same argument in breadth-first search fashion to realize that f is constant on $V(X)$. It must then be that each $v \in V$ satisfies $\deg(v) = \Delta$ as we argued for u above.

Theorem 6.6.3 (Wilf '67)
The chromatic number of a graph X satisfies

$$\chi(X) \leq \lfloor \lambda_1 \rfloor + 1.$$

Proof
We argue by induction on $n = |V(X)|$. The base case of $n = 1$ is trivial as $\chi(X) = 1, \lambda_1 = 0$. So the theorem is true.

Suppose $n \geq 1$ and the theorem holds for all graphs with n vertices. Let X be a graph on $n+1$ vertices and $\lambda_1 := \lambda_1(X)$. By a previous lemma, the average degree of X is at most λ_1. Hence there is at least one vertex u for which $\deg(u) \leq \lfloor \lambda_1 \rfloor$. Define $S := V(X - u)$.

Let $\lambda = \lambda_1(X[S])$. By our work earlier, $\lambda \leq \lambda_1(X) = \lambda_1$ since $X[S]$ is an induced subgraph of X. By the induction hypothesis,

$$\chi(X[S]) \leq \lfloor \lambda \rfloor + 1 \leq \lfloor \lambda_1 \rfloor + 1.$$

Consider any $(\lfloor \lambda_1 \rfloor + 1)$-colouring for $X[S]$. We can extend this to a colouring of X since u has at most $\lfloor \lambda_1 \rfloor$ neighbours. By induction, the theorem holds.

6.7 Eigenvalue Interlacing

Definition 6.7.1 (Principal Submatrix)
A submatrix B of $A \in \mathbb{R}^{n\times n}$ is principal if B is obtained by deleting the same set fo rows and columns of A.

Theorem 6.7.1 (Cauchy)
Let $A \in \mathbb{R}^{n\times n}$ be symmetric and B an $(n-1)\times(n-1)$ principal submatrix of A. Let $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_{n-1}$ be the spectra of A, B respectively.
Then

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1} \geq \alpha_n.$$

Proof
Without loss of generality, assume that B is obtained from A by deleting the first row and column. By the Courant-Fisher theorem,

$$\begin{aligned}
\alpha_k &= \max_{S\subseteq\mathbb{R}^n:\dim S=k}\ \min_{x\in S:x\neq 0} \frac{x^T A x}{x^T x} \\
\beta_k &= \max_{S\subseteq\mathbb{R}^{n-1}:\dim S=k}\ \min_{x\in S:x\neq 0} \frac{x^T B x}{x^T x} \\
&= \max_{S\subseteq\mathbb{R}^{n-1}:\dim S=k}\ \min_{x\in S:x\neq 0} \frac{\begin{bmatrix}0\\x\end{bmatrix}^T A \begin{bmatrix}0\\x\end{bmatrix}}{\begin{bmatrix}0\\x\end{bmatrix}^T \begin{bmatrix}0\\x\end{bmatrix}} \\
&= \max_{S\subseteq\mathbb{R}^{n-1}:\dim S=k, S\perp e_1}\ \min_{x\in S:x\neq 0} \frac{x^T A x}{x^T x} \\
&\leq \alpha_k.
\end{aligned}$$

With an identical argument applied to $-A, -B$, we get that $-\beta_k \leq -\alpha_{k+1}$ and hence

$$\beta_k \geq \alpha_{k+1}.$$

6.7.1 Perron-Frobenius Theorem

Theorem 6.7.2 (Perron-Frobenius; Undirected Graphs)
Let X be connected and $A := A(X)$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the spectrum of A. Then

(a) λ_1 has a strictly positive eigenvector

(b) $\lambda_1 > \lambda_2$

(c) $\lambda_1 \geq -\lambda_n$ with equality if and only if X is bipartite.

Proof (a)
We first show that λ_1 has a non-negative eigenvector. Let f be a unit eigenvector of λ_1 and consider

$$f^+ := |f|.$$

We have

$$\begin{aligned}\lambda_1 &= f^T A f \\ &= \sum_{uv \in E(X)} 2f(u)f(v) \\ &\leq \sum_{uv \in E(X)} 2f^+(u)f^+(v) \\ &= (f^+)^T A f^+.\end{aligned}$$

But we must have equality since f is a maximizer of the Rayleigh quotient. Hence f^+ is also a maximizer. By a previous theorem, this implies that f^+ is also an eigenvector of λ_1.

Now suppose there is some u for which $f^+(u) = 0$. Since $f \not\equiv 0$ and X is connected, there is some $vw \in E(X)$ for which $f^+(v) = 0 < f^+(w)$. But then

$$\sum_{w \sim v} f^+(w) > 0 = \lambda_1 f^+(v)$$

which is a contradiction.

Hence f^+ is a strictly positive eigenvector of λ_1.

Proof (b)
Let g be a unit eigenvector of λ_2 which is orthogonal to f^+. Hence

$$\begin{aligned}\lambda_1 &= g^T A g \\ &= \sum_{vw \in E(X)} g(v)g(w) \\ &\leq \sum_{vw \in E(X)} |g|(v)|g|(w) \\ &= |g|^T A |g| \\ &\leq \max_{g \in \mathbb{R}^n : \|g\|=1} g^T A g \\ &= \lambda_1.\end{aligned}$$

If we have $\lambda_1 = \lambda_1$ then $|g|$ is strictly positive by the same argument as in the proof of (a). Moreover, for each $uv \in E(X)$,

$$g(u)g(v) = |g(u)||g(v)|.$$

Now, f^+ is strictly positive, $g^T f^+ = 0$, and g has non-zero entries. Hence there must be two entries of g at which the signs differ. But since X is connected, there is an edge $uv \in E(X)$ at which $g(u) < 0 < g(v)$. But then

$$g(u)g(v) \neq |g(u)||g(v)|$$

which contradicts our conclusion above.

It follows that $\lambda_2 < \lambda_1$.

Proof (c)
Let g be a unit eigenvector of λ_n. Then

$$\begin{aligned}|\lambda_n| &= |g^T A g| \\ &= |\sum_{uv \in E(X)} 2g(u)g(v)| \\ &\leq \sum_{uv \in E(X)} 2|g(u)| \cdot |g(v)| \\ &= |g|^T A |g| \\ &\leq \max_{g \in \mathbb{R}^n : \|g\|=1} g^T A g \\ &= \lambda_1.\end{aligned}$$

$\underline{X \text{ bipartite implies } \lambda_n = -\lambda_1}$. Let S, T be a bipartition of X. Suppose X has eigenvalue λ with eigenvector f. Define

$$\bar{f} := \begin{cases} f(u), u \in S \\ -f(u), u \in T \end{cases}$$

Then for $u \in S$,

$$\begin{aligned} (A\bar{f})(u) &= \sum_{v \sim u} -f(v) \\ &= -\lambda f(u) \\ &= -\bar{f}(u). \end{aligned}$$

For $u \in T$,

$$\begin{aligned} (A\bar{f})(u) &= \sum_{v \sim u} f(v) \\ &= -\lambda \bar{f}(u). \end{aligned}$$

Thus $-\lambda$ is an eigenvalue of X as well. It follows that the spectrum of X is symmetric around 0. In particular, $\lambda_1 = \lambda_n$.

$\underline{\lambda_1 = \lambda_n \text{ implies } X \text{ is bipartite}}$ If $|\lambda_n| = \lambda_1$, then $|g|$ is a strictly positive eigenvector of λ_1. Moreover, for equality to hold in the initial inequality above, every edge $uv \in E(X)$ satisfies

$$g(u)g(v) = -|g(u)||g(v)| < 0.$$

Let S be the vertices v for which $g(v) > 0$ and T the vertices v for which $g(v) < 0$. Both are non-empty by our work above.

It follows that all edges in X joins S and T so it forms a bipartition of X as desired.

Chapter 7

Strongly Regular Graphs

7.1 Parameters

Definition 7.1.1 (Strongly Regular)
X is strongly regular with parameters (n, k, a, c) if

(i) $|V(X)| = n$

(ii) X is k-regular

(iii) Every pair of adjacent vertices has a common neighbours

(iv) Every pair of non-adjacent vertices has c common neighbours

Example 7.1.1
C_5 is a $(5, 2, 0, 1)$-strongly regular graph.

Proposition 7.1.2
Suppose X is (n, k, a, c) strongly regular. Then $\bar{X}$ is $(n, \bar{k}, \bar{a}, \bar{c})$ strongly regular for

$$\begin{aligned}\bar{k} &= n - 1 - k\\ \bar{a} &= n - 2 - 2k + c\\ \bar{c} &= n - 2k + a\end{aligned}$$

Definition 7.1.2 (Primitive)
A strongly regular graph X is primitive if both $X, \bar{X}$ are connected.

If X is not primitive, then it is *imprimitive*.

Lemma 7.1.3
Let X be (n, k, a, c) strongly regular. Then the following are equivalent:

(a) X is disconnected

(b) $c = 0$

(c) $a = k - 1$

(d) $X \cong mK_{k-1}$. That is, it is isomorphic to m copies of K_{k-1} for some m

Proof
$\underline{\text{(a)} \implies \text{(b)}}$ Suppose u, v are in different components. Then $N(u) \cap N(v) = \varnothing$ implies $c = 0$.

$\underline{\text{(b)} \implies \text{(c)}}$ Fix any $u \in V(X)$. We claim that all neighbours of u must be pairwise adjacent Let $v, w \in N(u)$. Then $u \in N(v) \cap N(w)$. Since $c = 0$, it must be that $v \sim w$.

Thus $a = k - 1$.

$\underline{\text{(c)} \implies \text{(d)}, \text{(d)} \implies \text{(a)}}$ Obvious.

Example 7.1.4
The line graph of K_n is

$$\left(\binom{n}{2}, 2n-4, n-2, 4\right)$$

strongly regular.

The line graph of $K_{n,n}$ is

$$(n^2, 2n-2, n-2, 2)$$

strongly regular.

7.1.1 Parameter Relation

Proposition 7.1.5
A (n, k, a, c) strongly regular graph X satisfies

$$c(n-k-1) = k(k-a-1).$$

Proof
Fix $u \in V(X)$. Let U denote the neighbours of u and V the non-neighbours of u.

Clearly, $|U| = k, |V| = n - k - 1$.

Now, for each $v \in V$, $v \not\sim u$ so they must share exactly c neighbours. But all neighbours of u are in U and

$$E(U, V) = c(n - k - 1).$$

Similarly, for all $x \in U$, $x \sim u$ hence they share a common neighbours. Since all neighbours of u are in U, x shares a neighbours with u in U. Then x has $k - 1 - a$ other neighbours in V. Hence

$$E(U, V) = k(k - 1 - a).$$

7.2 Eigenvalues

Lemma 7.2.1
Suppose X is (n, k, a, c) strongly regular and X is not complete. Put $A := A(X)$. Then A has 3 distinct eigenvalues k, θ, τ, where

$$\theta = \frac{a - c + \sqrt{\Delta}}{2}$$
$$\tau = \frac{a - c - \sqrt{\Delta}}{2}$$
$$\Delta := (a - c)^2 + 4(k - c).$$

These are the roots of the quadratic polynomial

$$\lambda^2 - (a - c)\lambda - (k - c).$$

Moreover, the algebraic multiplicities of θ, τ are

$$m_\theta = \frac{1}{2}\left((n - 1) - \frac{2k + (n - 1)(a - c)}{\sqrt{\Delta}}\right)$$
$$m_\theta = \frac{1}{2}\left((n - 1) + \frac{2k + (n - 1)(a - c)}{\sqrt{\Delta}}\right).$$

Proof
Consider the entries of A^2. There are k closed 2-walks at each vertex. If $u \sim v$, there are

precisely a 2-walks from u to v. If $u \not\sim v$, there are exactly c 2-walks from u to v.

It follows that

$$\begin{aligned} A^2 &= kI + aA + c(J - I - A) \\ A^2 - (a-c)A - (k-c)I = cJ. \end{aligned}$$

We know that k is an eigenvalue of A with the all-1s vector 1_n being an eigenvector. Thus if $\lambda \neq k$ is an eigenvalue with eigenvector v, then $v \perp 1_n$.

Moreover,

$$\begin{gathered} A^2v - (a-c)Av - (k-c)Iv = cJv \\ \lambda^2 v - (a-c)\lambda v - (k-c)v = 0 \\ \lambda^2 - (a-c)\lambda - (k-c) = 0. \end{gathered}$$

Clearly $\lambda \in \theta, \tau$.

It remains to verify m_θ, m_τ as well as $m_\theta, m_\tau > 0$ with $\theta, \tau \neq k$ and $\theta \neq \tau$.

$\underline{\theta \neq \tau}$ We have $\Delta := (a-c)^2 + 4(k-c) = 0$ if and only if $a = k = c$. But since $a \leq k-1$, this is impossible.

By the definitions of θ, τ, they must differ.

$\underline{\theta, \tau \neq k}$ This follows directly from the Perron-Frobenius theorem.

$\underline{m_\theta, m_\tau}$ There are n eigenvalues, hence

$$m_\theta + m_\tau = n - 1.$$

Moreover, $\operatorname{tr} A = 0$ implies that

$$k + m_\theta \theta + m_\tau \tau = 0.$$

Solving these linear equations yield

$$\begin{aligned} m_\theta &= -\frac{(n-1)\tau + k}{\theta - \tau} \\ m_\tau &= \frac{(n-1)\theta + k}{\theta - \tau} \end{aligned}$$

$\underline{m_\theta, m_\tau > 0}$ We leave this as an exercise.

Lemma 7.2.2
A connected regular graph with exactly 3 distinct eigenvalues is strongly regular.

Proof
Suppose X is k-regular, connected, and has eigenvalues k, θ, τ. Put $A := A(X)$ and

$$M := (k-\theta)^{-1}(k-\tau)^{-1}(A-\theta I)(A-\tau I).$$

Then M has eigenvalue 1 with multiplicity 1 by the Perron-Frobenius theorem with corresponding eigenvector 1_n. Moreover, we can also show that M has eigenvalue 0 with multiplicity $n-1$.

Now, rank $M = 1$. Moreover, M is symmetric and $M1_n = 1_n$. Thus we must have

$$M = \frac{1}{n}J.$$

By computation,

$$\begin{aligned}(A-\theta I)(A-\tau I) &= (k-\theta)(k-\tau)\frac{1}{n}J \\ A^2 - (\theta+\tau)A + \theta\tau I &= (k-\theta)(k-\tau)\frac{1}{n}J \\ A^2 &= (\theta+\tau)A - \theta\tau I + (k-\theta)(k-\tau)\frac{1}{n}J.\end{aligned}$$

It follows taht the number of 2-walks from u to v in X depends only on whether $u = v, u \sim v$, or $u \neq v, u \not\sim v$. Hence X is strongly regular.

Theorem 7.2.3
Suppose X is connected and regular and is NOT complete. Then X has 3 distinct eigenvalues if and only if X is strongly regular.

Recall that $\Delta := (a-c)^2 + 4(k-c)$ and

$$\begin{aligned} m_\theta &= \frac{1}{2}\left((n-1) - \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right) \\ m_\theta &= \frac{1}{2}\left((n-1) + \frac{2k+(n-1)(a-c)}{\sqrt{\Delta}}\right).\end{aligned}$$

But m_θ, m_τ are integers. Hence this is a condition on the parameters n, k, a, c.

7.3 Payley Graphs

Definition 7.3.1 (Payley Graphs)
Suppose q is a prime power and $q \equiv 1 \mod 4$. Then $P(q)$ is the graph with

$$\begin{aligned} V(P(q)) &= \mathrm{GF}(q) \\ E(P(q)) &= \{uv : u - v \text{ is a nonzero square}\}. \end{aligned}$$

Proposition 7.3.1
$P(q)$ is well-defined.

Proof
It suffices to show that $u - v$ is a non-zero square if and only if $v - u$ is a non-zero square.

It is not hard to see that exactly half of elements in $\mathrm{GF}(q) \setminus \{0\}$ are squares. In particular, $P(q)$ is $\frac{q-1}{2}$-regular.

Lemma 7.3.2
$P(q)$ is arc transitive.

Proof
Let $\phi_{a,b} : V(P(q)) \to V(P(q))$ be given by

$$x \mapsto ax + b.$$

Then $\phi_{a,b} \in \mathrm{Aut}(P(q))$ if a is a non-zero square (exercise).

For any two arcs $(x, y), (x', y')$, let $a = (x' - y')(x - y)^{-1}$ and $b = x' - ax$. It follows that

$$(x', y') = (x, y)^{\phi_{a,b}}$$

and $P(q)$ is arc transitive.

Theorem 7.3.3
$P(q)$ is self-complementary.

Proof
Define $\sigma : x \mapsto ax$ where $a \neq 0$ is NOT a square. This is an isomorphism to its complement.

Theorem 7.3.4
$P(q)$ is $\left(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4}\right)$ strongly regular.

Proof
Exercise.

7.4 More Strongly Regular Graphs

We have seen that $C_n, P(q)$ are strongly regular.

Definition 7.4.1 (Lattice Graph)
Let L_n denote the graph where

$$\begin{aligned} V(L_n) &= [n] \times [n] \\ E(L_n) &= \{(a,b)(a',b') : a = a', b = b'\} \end{aligned}$$

Observe that L_n is $(n^2, 2(n-1), n-2, 2)$ strongly regular.

Definition 7.4.2 (Latin Square)
An $n \times n$ Latin square is an $n \times n$ matrix such that each row and column is a permutation of $[n]$.

Definition 7.4.3 (Latin Square Graph)
Let A be an $n \times n$ latin square. The Latin square graph $X(A)$ associated with A_n is defined by

$$\begin{aligned} V(X(A)) &= [n] \times [n] \\ E(X(A)) &= \{(a,b)(a',b') : a = a' \vee b = b' \vee A_{ab} = A_{a'b'}\} \end{aligned}$$

The latin graph $X(A)$ for an $n \times n$ Latin square is $(n^2, 3(n-1), n, 6)$ strongly regular (exercise).

7.5 The Second Eigenvalue

Let λ_2 denote the second largest eigenvalue of $A(X)$ and μ_2 denote the second smallest eigenvalue of $L(X)$. They are closely related. Indeed, if X is k-regular, then $\mu_2 = k - \lambda_2$.

While λ_1 gives information about the degrees of X, λ_2, μ_2 measure the expansion property of X (how well connected X is). We know that $\mu_2 = 0$ if and only if X is disconnected. If X has a small λ_2, or large μ_2, then X is a good expander.

Definition 7.5.1 (Isoperimetric Ratio)
Let $S \subseteq V(X)$. The isoperimetric ratio of S is

$$\theta(S) := \frac{e(S, \bar{S})}{|S|}.$$

The isoperimetric ratio of X is

$$\theta(X) := \min_{S:|S| \leq \frac{V(X)}{2}} \theta(S).$$

Definition 7.5.2 (α-Expander)
X is an α-expander if $\theta(X) \geq \alpha$.

Theorem 7.5.1
$\theta(X) \geq \frac{\mu_2}{2}$.

Proof
We argue that for every $S \subseteq V(X), |S| \leq \frac{n}{2}$, the isoperimetric ratio satisfies

$$\theta(S) \geq \mu_2 \left(1 - \frac{|S|}{n}\right).$$

The result then follows.

Recall that $L = L(X)$ has $\mu_1 = 0$ with eigenvector 1_n. Hence μ_2 is the smallest eigenvalue of $L - \frac{1}{n}J$. Moreover, the Rayleigh quotient is minimized at an eigenvector v of μ_2 where $v \perp 1_n$.

That is,

$$\begin{aligned}\mu_2 &= \min_{x \perp 1_n} \frac{x^T \left(L - \frac{1}{n} J\right) x}{x^T x} \\ &= \min_{x \perp 1_n} \frac{x^T L x}{x^T x}.\end{aligned}$$

Let $\alpha := \frac{|S|}{n}$ and $x := \chi_S - \alpha 1_n$. Then $x \perp 1_n$ and

$$\|x\|_2^2 = \alpha n(1-\alpha).$$

Moreover,

$$\begin{aligned}\mu_2 &\leq \frac{x^T L x}{x^T x} \\ &= \frac{1}{\alpha n(1-\alpha)} \sum_{uv \in E} (x_u - x_v)^2 \\ &= \frac{1}{\alpha n(1-\alpha)} e(S, \bar{S}) \\ &= \frac{\theta(S)}{1-\alpha}.\end{aligned}$$