CO342: Graph Theory

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Preface

Grading Scheme

- 5 assignments 20%
- 2 in class tests 20%
- final exam 60%

Office Hours

- Peter 10-11 AM Fri, MC5128
- Sabrina 2:30-4:30 PM Wed, MC5464
- Amena 2-4 PM, MC5131
- Zach 1-2 PM Wed, 2-3 PM Fri, MC6487

Assignments

(1) Sep 27
(2) Oct 11
(3) Oct 1
(4) Nov 15
(5) Nov 29

Tests

(1) Oct 21
(2) Nov 29

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1 Review

1.1 Trees & Forests

Proposition 1.1.1
Every forest $G = (V, E)$ with at least 1 edge has a leaf vertex.

Consider the following procedure.

1. start at a non-isolated vertex
2. given the current vertex v, choose a neighbour of v that was not just visited. Go there.
3. If no such neighbour exists, then STOP. The current vertex is a leaf. Elsewise, repeat.

One way of proving that the proposition holds is by proving the correctness of the algorithm above. A better solution is to consider the fact that we are constructing a path and simply consider the longest path.

Proof
Let $v_1, v_2, \ldots, v_k$ be the vertices of a longest path of G.

Note that the longest path must exist since by definition, a path is composed of unique nodes and there are only finitely many paths in G.

Since there is at least one edge, $k \geq 2$ and $v_1 \neq v_k$.

Consider v_k, we claim it only has one neighbour v_{k-1}.

Suppose otherwise it has a neighbour w, if it on the path, we would have a circuit, elsewise the path would not be the longest. The claim clearly holds.

1.2 Graph Colorings

Definition 1.2.1 (Bounding χ)
$\chi(G)$ is the chromatic number of G.
This is the minimum $k \geq 0$ such that the vertices of G can be assigned colours from $[k]$ such that adjacent vertices always receive different colours.

Proposition 1.2.1 (Greedy Bound)
If G is a graph if every vertex has degree at most k, then $\chi(G)$ is at most $k + 1$.

We first note that any complete graph K_t requires at least $t = k + 1$ colours and any odd cycle C_t requires $3 = k + 1$ colours.

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Proof
Suppose the proposition fails.

Let G be a graph for which it fails, with as few vertices as possible. Clearly $|G| \neq 0$

This means the proposition holds for all $P - v$ where $v \in V(G)$.

Every vertex of $G - v$ has degree at most k, so $G - v$ is $(k+1)$ colourable.

There must be some colour in a $(k+1)$-colouring of $G - v$ not used by any color of v. Assigning this colour to v gives a $(k+1)$-colouring of G.

This contradicts the fact that the proposition fails for G. In addition, this means there is no minimum example and the proposition holds.

1.3 Matchings

1.3.1 Definitions

Definition 1.3.1 (Matching)
A matching M in a simple graph G is a subset of the edges $E(G)$ such that no two share a vertex ie

$$\{x, y\} \in M \implies \forall z \neq x, y, \{x, z\}, \{y, z\} \notin M$$

We denote the size of the maximum matching with $v(G)$.

Definition 1.3.2 (Satisfy)
We say a matching M satisfies a vertice $v \in V(G)$ if it is incident to an edge in M

Definition 1.3.3 (Perfect Matching)
A matching M in a graph G is perfect if it satisfies every vertice $v \in V(G)$.

Note that if a graph has a perfect matching, then $|V(G)|$ is even

1.3.2 Hall's Theorem

Definition 1.3.4 (neighbourhood)
A neighbourhood of a subset $S \subseteq V(G)$ of the vertices of a graph is

$$\Gamma_G(S) := \{w \in V(G) : v \in S, \{v, w\} \in E(G)\}$$

Definition 1.3.5 (Halls's Condition)
Let G be a bipartite graph with vertice classes X, Y.
Hall's Condition is the statement that

$$\forall S \subseteq X, |\Gamma(S)| \geq |S|$$

Theorem 1.3.1 (Hall)
Let G be a bipartite graph with vertice classes X, Y.
There is a matching satisfying X if and only if Hall's Condition holds.

Proof ($\Longrightarrow$)
This is trivial.

Take any subset of X and note that a matching satisfying X must satify this subset as well.

By the definition of a matching and a bipartite graph, Hall's Condition holds. ∎

Proof ($\Longleftarrow$)
We argue by induction on the size of X.

The case where X is a singleton is trivial.

Suppose now that the Hall's Theorem holds true for all $n < |X|$.

<u>Case I</u>

Let $x \in X$ be arbitrary and choose $y \in Y$ such that $xy \in E(G)$.

If Hall's Condition holds in the graph induced by the removal of x, y, namely $(X - x) + (Y - y)$, then we are done as the induction hypothesis gives us a matching satifying $X - x$ and with the addition of xy, X is satisfied.

<u>Case II</u>

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Otherwise there must be some subset $S_0 \subseteq X - x$ such that

$$|\Gamma_{G-x-y}(S_0)| < |S_0|$$

which contradicts Hall's Condition for $X - x$.

But since by assumption Hall's Condition holds for X and we have removed at most one vertice in the neighbourhood of S_0, we have

$$|\Gamma_G(S_0) \setminus \Gamma_{G-x-y}(S_0)| \leq 1$$

In particular, this means that $|\Gamma_G(S_0)| = |S_0|$.

We will now show there are two disjoint matchings satisfying S_0 and $X \setminus S_0$.

To get the matching for S_0, note that by assumption, Hall's Condition holds for X and therefore also for S_0. Coupled with the induction hypothesis, there must be a matching which satisfies both S_0 and $\Gamma_G(S_0)$.

Now, it remains to find a disjoint matching for $X \setminus S_0$ in $G \setminus S_0 \setminus \Gamma_G(S_0)$. We will achieve this by showing that Hall's Condition holds for $X \setminus S_0$ in the proposed subgraph. Note that for all $S' \subseteq X \setminus S_0$ we have by assumption that

$$|\Gamma_G(S' \cup S_0)| \geq |S' \cup S_0|$$

But since S', S_0 are disjoint with $|\Gamma_G(S_0)| = |S_0|$, we must have

$$|\Gamma_G(S') \setminus \Gamma_G(S_0)| \geq |S'|$$

so

$$|\Gamma_{G \setminus S_0 \setminus \Gamma_G(S_0)}(S')| \geq |S'|$$

This means that Hall's Condition holds for $X \setminus S_0$ in $Y \setminus \Gamma_G(S_0)$ so we have found a disjoint matching satisfying $X \setminus S_0$. ■

Corollary 1.3.1.1
Any regular bipartite graph with degree ≥ 1 has a perfect matching.

Proof
Let X, Y be the vertex classes of G and $S \subseteq X$ be arbitrary.

Let $E(S, \Gamma(S))$ be the set of edges from a vertice in S to one of its neighbours.

Then we have

$$|E(S, \Gamma(S))| = k|S|$$

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On the other hand, since the the number of vertex degree is bounded, we also have

$$|E(S, \Gamma(S))| \leq k|\Gamma(S)|$$

But then Hall's Theorem holds and there is a matching satisying X. This means that $|X| \leq |Y|$.

By similar logic, there is a matching satisfying Y, meaning $|Y| \leq |X|$.

All in all, $|X| = |Y|$ and we have a perfect matching. ∎

1.3.3 Berge's Theorem

Definition 1.3.6 (Alternating Path)
Let M be a matching of a graph G.
A M-alternating path is a path in G with every other edge in M.

Definition 1.3.7 (Exposed)
A vertex $v \in V(G)$ is M-exposed if it is not M-saturated.

Definition 1.3.8 (Augmenting)
An M-augmenting path is an M-alternating path that start and end with M-exposed vertices.

Theorem 1.3.2 (Berge)
A matching M in a graph G is maximal if and only if it has no M-augmenting paths.

1.4 Vertex Covers

1.4.1 Basic Definitions & Results

We denote the maximum size of a matching in a graph G with $v(G)$.

Definition 1.4.1 (Vertex Cover)
A vertex cover is a subset $C \subseteq V(G)$ of the vertices such that every edge is incident to a vertex in C ie

$$\forall xy \in E(G), x \in C \vee y \in C$$

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We denote with $\tau(G)$ the size of the minimum cover in graph G.

Proposition 1.4.1

$$v(G) \leq \tau(G) \leq 2v(G)$$

Proof

To see the lower bound, note that a vertex cover necessarily includes one of the vertices of each edge of a maximum matching.

To see the upper bound, we note that by taking all vertices indicent with an edge in a maximal matching necessarily yields a vertex cover. If not, there is an edge which is not in the proposed matching with both endpoints not inside the proposed vertex cover. This would contradict the maximality of the proposed matching.

1.4.2 König's Theorem

Theorem 1.4.2 (König)

If G is a bipartite graph, then $v(G) = \tau(G)$.

Proof

By the proposition above, it suffices to show that the maximal matching has equivalent size to the minimum vertex cover.

Let A, B be the vertex classes and let $S \subseteq A, T \subseteq B$ such that $S + T$ is a minimum vertex cover of G.

Note that there cannot be any edges between $A - S, B - T$, or else this would contradict the definition of a vertex cover.

We will now proceed to find disjoint matchings satisfying S, T so that

$$|S| + |T| \leq v(g) \leq \tau(G) = |S| + |T|$$

We will apply Hall's Theorem on S. Indeed, let $Q \subseteq S$ be arbitrary.

Suppose Q has less than $|Q|$ neighbours in $B - T$. Note that we can then remove Q from S and add $\Gamma_G(Q) \setminus T$ to T to maintain a valid vertex cover of strictly smaller size.

In other words, $|S - Q + T + (\Gamma_G(Q) \setminus T)| < |S + T|$ with the former being a valid vertex cover.

This contradicts the minimality of our vertex cover and is thus a contradicion.

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By contradiction, Hall's Condition holds and there must be a matching satisfying S with only neighbours in $B - T$.

We can apply similar logic and arrive at a matching satisfying T with only neighbours in $A - S$.

These two matchings are necessarily disjoint and have cardinality equivalent to that of the the proposed minimum cover.

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2 Introduction

2.1 Definitions

Definition 2.1.1 (Graph)
A graph $G = (V, E, i)$ is a 3-tuple where

- V is a finite set of vertices
- E is a finite set of edges with $V \cap E = \emptyset$
- $i : V \times E \to \{0, 1, 2\}$

$$i(v, e) = \# \text{ of times } e \text{ is incident to } v$$

  such that
$$\forall e \in E, \sum_{v \in V} i(v, e) = 2$$

We say $G = (V, E)$ is a simple graph under the Math249 definitions.

Definition 2.1.2 (Adjacent)
$u, v \in V$ are adjacent in G if either
 (1) $i(u, e) = i(v, e) = 1$, if $u \neq v$
 (2) $i(u, e) = 2$
for some $e \in E$.

Definition 2.1.3 (Incident)
$v \in V, e \in E$ are incident in G if $i(u, e) \neq 0$ for some $e \in E$.

Definition 2.1.4 (Degree)
The degree of a vertex v is

$$\deg(v) = d(v) = \sum_{e \in E} i(v, e)$$

Definition 2.1.5 (Ends)
The ends of an edge e are the vertices u, v such that $i(u, i > 0, i(v, i) > 0$.

To see why we define graphs this way consider the following example

Example 2.1.1 (Dual)
If $G(V, E, i)$ is the primal planar graph with a fixed planar embedding then

$$H = (F, E, j)$$

is the dual planar graph with F being the faces of the embedding of G and j the incidence function determined by adjacent faces of

our definition of a graph makes it easier to work with the dual graph.

Definition 2.1.6

- walk
- path
- cycle / circuit
- forest
- isolated vertex
- leaf vertex

Definition 2.1.7 (Connected)
$u, v \in V(G)$ are connected in a graph G, if there is a walk from $u \to v$ in G.

Definition 2.1.8 (Connected)
G is connected if $G(V) \neq \varnothing$ and for every pair of vertices is connected in G.

Proposition 2.1.2
Connectedness is an equivalence relation on V.

Proof

1. reflexive
2. symmetric
3. transitive

Definition 2.1.9 (Subgraph)
A subgraph of $G = (V, E, i)$ is a 3-tuple

$$H = (V', E', i')$$

where $V' \subseteq V, E' \subseteq E$ and i' is the restriction of i to the domain $V' \times E'$

Definition 2.1.10 (Induced Subgraph)
If $X \subseteq V$, the subgraph $G[X]$ induced by X is the subgraph

$$(X, E', i')$$

where E' consists of all edges with both ends in X and

$$i' = i\Big|_{X \times E'}$$

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3 Connectedness

3.1 Definition

Definition 3.1.1 (Components)
A component of G is an induced subgraph of the form $G[X]$ where X is an equivalence class under connectedness.

Proposition 3.1.1
Components are maximal connected subgraphs.

Definition 3.1.2 (Union)
Let $G_1 = (V_1, E_1, i_1)$ and $G_2 = (V_2, E_2, i_2)$.
Suppose that the subgraph obtained from G_1 by restricting to $V_1 \cap V_2$ and $E_1 \cap E_2$ is the same as the subgraph obtained from G_2 by restricting to these sets. In other words, G_1, G_2 agree on their common vertices / edges.
Then $G_1 \cup G_2$ is defined to be the graph with vertex sets $V_1 \cup V_2$, edge set $E_1 \cup E_2$, in which a vertex v is incident to an edge e if and only if e and v are incident in either G_1 or G_2.

Definition 3.1.3 (Direct Sum)
We write $G_1 \oplus G_2$ to denote the direct sum of G_1, G_2 which is

$$G_1 \cup G_1$$

when $V_1 \cup E_1, V_2 \cup E_2$ are disjoint.

Definition 3.1.4 (Subtraction)
For a set $X \subseteq V \cup E$ and where $G = (V, E, i)$ is a graph, we write

$$G - X$$

for the subgraph of G with vertex set $V \setminus X$ ad edge set $E \setminus X'$, where X' is the set of edges that are either in X or incident with a vertex in X.

Proposition 3.1.2
Every graph is uniquely the direct sum of connected graphs if we define the empty graph as disconnected.

Definition 3.1.5 (Path)
A path is a graph whose edges and vertices form a path.

Definition 3.1.6 (Ends)
The ends of a path are its degree 1 vertices, or the sole vertex if there are no edges.

Definition 3.1.7 (Circuit)
A graph which forms a circuit.

Definition 3.1.8 (AB-path)
Give disjoint sets of vertices A, B in a graph G, an AB-path is a path with one end in A, another in B and all other vertices in $V(G) \setminus (A \cup B)$

Definition 3.1.9 (ab-path)
An $\{a\}\{b\}$-path.

Definition 3.1.10 (Seperates)
A set $X \subseteq V \cup E$ separates A, B in G if there is no AB-path in $G - X$.

Definition 3.1.11 (cut edge / bridge)
$e \in E$ is a cut edge / bridge if there are vertices u, v of G that are not separated by $\emptyset$, but separated by $\{e\}$.

Definition 3.1.12 (k-Connectedness)
Let $k \geq 1$. A graph G is k-connected if $|V(G)| > k$ and there is no set $X \subseteq V(G)$ with $|X| < k$ such that $G - X$ is disconnected.
In other words, it is the minimal size of a subset of the vertices which separates G.

Definition 3.1.13 (cut-vertex)
A cut vertex of G is a vertex v such that there is some pair a, b not separated by $\emptyset$ but separated by $\{v\}$.

Note that

- 1-connected $\iff$ connected and G is not the singleton graph
- 2-connected $\iff$ connected and there is no cut vertex, except $G = o - o$.

3.2 Basic Results

We can augment a 2-connected graph by connecting a pair of vertices u, v of G by taking its union with a path whose ends are u, v.

Definition 3.2.1 (Adding a Path)
G' arises from G by adding a path if there is a non-trivial path P such that

$$G' = G \cup P$$

and

$$(E(P) \cup V(P)) \cap (E(G) \cup V(G))$$

is the set of two ends of P

Proposition 3.2.1 (Ear Decomposition)
A loopless graph G is 2-connected if and only if G can be obtained starting from a circuit by successively adding paths (ears).
In other words there must be graphs $G_1, \dots, G_k$ such that

(1) G_1 is a circuit, $G_k = G$

(2) for each $1 \leq i < k$, G_i is connected and G_{i+1} arises from G_i by adding a path.

Proof ($\Longleftarrow$)
Assignment.

Proof ($\Longrightarrow$)
Let G be a loopless 2-connected graph, and let l be maximal so that there are subgraphs $G_1, \dots, G_l$ of G so that each G_i is 2-connected and arises from G_{i-1} by adding a path, while G_1 is a circuit.

Note that l is at least 1 since every 2-connected graph has a circuit or else G is a tree and has leaf vertices which contradicts the definition of 2-connected.

Suppose $G_l \neq G$.

Since adding a single edge between vertices of G_l adds a path, the maximality of l implies that every edge of G between two vertices of G_l is an edge of G_l. In otherwords, G_l is an induced subgraph of G.

If $V(G_l) = V(G)$, then $G_l = G$ so we may assume some vertex $u \in V(G) \setminus V(G_l)$ which is adjacent to some $v \in V(G_l)$ by connectedness.

Because G is 2-connected, there must be a path from u to $V(G_l)$ in the graph $G - v$.

Pv is a path of G with both ends in $V(G_l)$ and no other vertices in $V(G_l)$.

But then

$$G_l \cup (Pv)$$

is a subgraph of G obtained from G_l by adding a path. It is also 2-connected by $\Longleftarrow$.

This contradicts the maximality of l.

3.3 The Block Graph

Definition 3.3.1 (Separator)
A separator of a connected graph G is a set X such that $G - X$ is nonempty and disconnected.

Definition 3.3.2 (Cut Vertex)
$v \in V(G)$ such that $\{v\}$ is a separator.

Definition 3.3.3 (Block)
A block of G is a maximal connected subgraph of G which has no cut vertex.
So it is either 2-connected or has at most 2 vertices.

Definition 3.3.4 (Simple Graph)
A graph with no multiple edges or loops.

Definition 3.3.5 (Block Graph)
The block graph of a graph G is a simple bipartite graph with bipartition (B, K) where B is the set of blocks of G and K is the set of cut vertices of G.
A block b is adjacent to a cut vertex v if its vertex set in G contains v.

The block graph encodes how the blocks are pieced together to form G.

Proposition 3.3.1
Each graph G is the union of its blocks.

Proof
It suffices to show that each vertex and edge of G is contained in a block of G.

This is true as for every edge e from x to y, the subgraph on x, y, e has no cut vertex and so is contained in some maximal subgraph with no cut vertex (ie block).

Proposition 3.3.2
Any two blocks intersect in at most one vertex (and no edges).

Proof
Let B_1, B_2 be blocks and

$$x, y \in V(B_1) \cap V(B_2)$$

be distinct.

By maximality of B_1, B_2, the subgraph $B_1 \cup B_2$ has a cut vertex w. Let a, b be vertices separated by w.

Let $z \in \{x, y\} \setminus \{w\}$.

Since B_1 has no cut vertex, each of a, b is connected to z in either $B_1 - w$ or $B_2 - w$ and hence in $B_1 \cup B_2 - w$.

So a, b are connected in $B_1 \cup B_2 - w$, a contradiction.

3.4 Contractions

Definition 3.4.1 (Edge Contraction)
For a graph $e \in E(G)$ we write G/e for "G contract e".
For the same graph with edge set

$$E \setminus \{e\}$$

and vertex set

$$V \setminus \{u, v\} \cup \{x_{uv}\}$$

where x_{uv} is a new vertex, in which each edge with an end equal to u or v in G now has an end equal to the new vertex x_{uv}.

Note that any edge parallel to e becomes a loop at the new vertex.

Also note that if G is connected, then so is G/e.

However, the same does not hold for 2-connectedness.

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Lemma 3.4.1
If G is k-connected and $X \subseteq V(G)$ with $|X| = k$, then each vertex in X has a neighbour in each component of $G - X$.

Proof
Suppose for a contradiction that $x \in X$ fails to have the proposed property.

Consider $G - X \setminus \{x\}$ and if x in that graph has no neighbours in some component of $G - X$, then $X \setminus \{x\}$ is a separator with cardinality strictly less than k.

Proposition 3.4.2
If e is an edge of G with ends u, v and $X \subseteq V \cup E$ containing neither e, u, v then

$$(G - X)/e = (G/e) - X$$

Proof
Exercise.

Proposition 3.4.3
If G is a 2-connected graph, $e \in E$ and $|V(G)| > 3$, then either G/e or $G - e$ is 2-connected.

Proof
Suppose neither are 2-connected. Let y, z be vertices so that

$$G/e - y, G - e - z$$

are both disconnected graphs.

Let u, v be the ends of e. Note that $u \neq v$ or else 2-connectedness is preserved when deleting e.

If $y \neq x_{uv}$ then $(G/e) - y = (G - y)/e$ but $(G - y)$ is connected so $(G - y)/e$ is connected, a contradiction to the choice of y.

Otherwise, suppose $y = x_{uv}$, we have

$$(G/e) - y = G - \{u, v\}$$

is disconnected.

We will show that $G - e$ must be connected. To do this, we argue that z is not a cut vertex of $G - e$.

First, remark that if $z \in \{u, v\}$

$$G - e - z = G - z$$

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which cannot be disconnected so $z \neq u, v$.

Note that $(G/e) - y$ is disconnected. Let C be a component of $(G/e) - y = G - \{u, v\}$ that does NOT contain z. By the lemma, C contains a neighbour u' of u, v' of v, and since C does not contains z, u, v, then u', v' are connected in $G - \{e, z\}$.

Therefore, u, v are connected in $G - \{e, z\}$.

Let a be a vertex of $G - z$. Since $G - z$ is connected, there is a path from a to u or v in $G - \{z, e\}$.

So

- a is connected to u or v in $G - \{z, e\}$
- u and v are connected in $G - \{z, e\}$

so a, u are connected in $G - \{z, e\}$ and by the arbitrary choice of a we have $G - \{z, e\}$ is connected. This contradicts the choice of z.

Proposition 3.4.4
Any graph with a degree 2 vertex is not 3-connected.

Proof
Removing its neighbours disconnects the graph.

Proposition 3.4.5
k-connected graphs have a minimum vertex degree of k.

Example 3.4.6
Consider $K_{3,3}$, removing any vertex would lower the degrees of all vertices adjacent to it.

More generally, if all vertices of a graph G has degree 3, then $G - e$ is not 3-connected for any edge e, as $G - e$ has degree-2 vertices.

Example 3.4.7
Consider the 8-wheel, and in particular any spoke edge of the 8-wheel. We cannot delete nor contract it and remain 3-connected.

Proposition 3.4.8
An n-wheel for $n \geq 3$ is 3-connected. But its “spoke” edges cannot be deleted nor contracted while maintaining 3-connectedness.

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Theorem 3.4.9 (Tutte)
Let G be a 3-connected graph with $|V(G)| > 4$, then G has an edge so that G/e is 3-connected.

Lemma 3.4.10
If Tutte's Theorem does not hold then we claim that for every edge e with distinct ends x, y, there is a vertex

$$z \notin \{x, y\}$$

so that $G - \{x, y, z\}$ is disconnected, and each of the vertices x, y, z has a neighbour in each component of

$$G - \{x, y, z\}$$

Proof
To see the claim, let a_{xy} be the vertex of G/e created by the contraction. Since G/e is not 3-connected, there is a set Z with $|Z| = 2$ such that

$$G/e - Z$$

is not connected.

If $a_{xy} \notin Z$, then

$$G/e - Z = (G - Z)/e$$

but the RHS is connected by the size of Z and contraction does not affect connectivity.

So $a_{xy} \in Z$, let $z \in Z$ be the other element of Z.

The disconnected graph $(G/e) - Z$ is equal to

$$G - \{x, y, z\}$$

so the other part follows from what was proved last lecture.

Proof (Tutte's Theorem)
Suppose that the statement is false.

Choose e with ends x, y so that the number of vertices of a smallest component C of

$$G - \{x, y, z\}$$

is as small as possible, where z is given by the claim.

Let v be a neighbour of z in C.

Now, by the claim, there is a vertex w so that $G - \{v, z, w\}$ is disconnected and each of

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v, z, w has a neighbour in each component of $G - \{v, z, w\}$.

Since x, y are adjacent in G and $G - \{v, z, w\}$ has at least 2 components, there is a component of D of $G - \{v, z, w\}$ containing neither x nor y.

So D is also a component of $G - \{v, z, w, x, y\}$.

We argue that $V(D) \subseteq V(C)$. This would be a contradiction as $v \in V(C) \setminus V(D)$ and C was chosen to be minimal.

To see this, let $b \in V(D)$. Since v has a neighbour in $V(D)$ and D is a component of $G - \{v, w, z, x, y\}$, there is a path from b to v in $G - \{z, w, x, y\}$.

So there is a path from b to v in $G - \{x, y, z\}$, this means that b is in the same component as v in the graph

$$G - \{x, y, z\}$$

so $b \in V(C)$.

Thus $V(D) \subseteq V(C)$ and we are done.

3.5 Menger's Theorem

Definition 3.5.1 (Internally Disjoint Paths)
Suppose that $a \neq b$.
ab-paths are internally disjoint if the sets

$$V(P_1) \setminus \{a, b\}, \ldots, V(P_k) \setminus \{a, b\}$$

and if a, b are adjacent,

$$E(P_1), \ldots, E(P_k)$$

are disjoint sets.

Theorem 3.5.1 (Menger I)
If G is a graph with $|V(G)| > k$, then the following are equivalent:

(1) G is k-connected.

(2) for every $a \neq b \in V(G)$ there are k internally disjoint ab-paths in G.

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Theorem 3.5.2 (Menger II)
Let a, b be distinct vertices in a graph G. For $k \geq 1$, exactly one of the following is true:

(1) There exists k internally disjoint ab-paths in G.

(2) There is a set $X \subseteq V(G) \setminus \{a, b\}$ so that $|X| < k$ and there are no ab-paths in $G - X$.

Theorem 3.5.3 (Menger III)
Given sets $A, B \subseteq V(G)$ and an integer $k \geq 1$, exactly one of the following holds:

(1) There are k vertex disjoint AB-paths in G.

(2) There is some $X \subseteq V(G)$ with $|X| < k$ such that $G - X$ has no AB-paths.

Theorem 3.5.4 (Menger III*)
Given sets $A, B \subseteq V(G)$ and k being the cardinality of a minimal set X such that $G - X$ has no AB-paths, then there are k vertex disjoint AB-paths in G.

Proof
Suppose the statement is false and let G, A, B, k specify a counterexample where $|E(G)|$ is as small as possible.

We argue by induction on $|E|$.

If every edge of G is a loop, then every AB-separator contains $A \cap B$ and $A \cap B$ is itself a separator. This means

$$k = |A \cap B|$$

But each vertex in $A \cap B$ is an AB-path, so there are k vertex disjoint AB-paths. Therefore G is not a counterexample.

So G has an edge e with ends u, v where $u \neq v$.

Let x_e be the identified vertex of G/e.

Let

$$A' := \begin{cases} A, & u, v \notin A \\ (A \setminus \{u, v\}) \cup \{x_e\}, & \{u, v\} \cap A \neq \varnothing \end{cases}$$

and

$$B' := \begin{cases} B, & u, v \notin B \\ (B \setminus \{u, v\}) \cup \{x_e\}, & \{u, v\} \cap A \neq \varnothing \end{cases}$$

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Claim I Let H be a subgraph of G containing e. Let $H' = H/e$.

We claim there is an AB-path in H if and only if there is an $A'B'$-path in H'.

Let C be a component of H and C' be the corresponding component of $H' = H/e$.

Note that $C' = C$ if $e \notin E(C)$ and $C' = C/e$ otherwise.

Then C contains a vertex in A if and only if C' contains a vertex in A', by the definition of A'.

Same for B, B'.

So H contains an AB-path if and only if some component of C of H contains a vertex in A and a vertex in B if and only if some component of C' of H' contains a vertex in A' and a vertex in B' if and only if H' contains an $A'B'$-path.

Claim II Now we claim that there do not exist k disjoint $A'B'$-paths in G/e.

Suppose that disjoint $A'B'$-paths $P_1, \dots, P_k$ existed in G/e.

Each path P_i not containing x_e is also an AB-path in G. If none of the P_i contain x_e, then G has k disjoint A, B-paths, contradicting the choice of G.

Now suppose one of the P_i, say P_1, contains x_e. Let H be the subgraph of G with vertex set and edge set

$$(V(P_1) \setminus \{x_e\}) \cup \{u, v\}, E(P) \cup \{e\}$$

Now $H/e = P$. By claim I, since P has (is) an $A'B'$-path, H contains an AB-path $\hat{P}_1$.

Now, since $P_1, \dots, P_k$ are vertex-disjoint, the paths

$$\hat{P}_1, \dots, P_k$$

are vertex-disjoint AB-paths in G, which is again a contradiction.

By the minimality of $E(G)$, G/e is not a counterexample. This would be a contradiction unless the smallest $A'B'$-separator X' in G/e has size less than k.

Claim III $x_e \in X'$

If $x_e \notin X'$ then

$$(G/e) - X' = (G - X')/e$$

so $(G - X')/e$ contains no $A'B'$-path.

Applying claim 1 with $H = G - X'$ shows that H has no AB-paths. So X' is a separator of size less than k in G, which is a contradiction.

Let us "uncontract" G/e.

$$X := (X' \setminus \{x_e\}) \cup \{u, v\}$$

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Note that $(G/e) - X' = G - X$ so X separates A from B in G.

This gives $|X| \geq k$. Since $|X|' < k$ and $|X| = |X|' + 1$ we must have

$$|X| = k$$

Let $X = \{x_1, \cdots x_k\}$. If there were an AX-separator Y in G with $|Y| < k$ in G, then $G - Y$ would contain no AX-paths. Since every AB-path contains an AX-path, this would imply that Y is an AB-separator, so $|Y| < k$ gives a contradiction.

So the smallest size of an AX-separator is at least k, similarly for a XB-separator.

Since $G - e$ is not a counterexample by the minimality of $E(G)$, it follows that there are k disjoint AX-paths

$$P_1, \dots, P_k$$

in $G - e$. Say that x_i is the end in X of P_i.

Similarly, we can find k disjoint XB-paths

$$Q_1, \dots, Q_k$$

where x_i is the end in X of Q_i.

If there were a vertex z in

$$(P_i - X) \cap (Q_j - X)$$

for some i, j, then let a be the end of P_i in A and b the end of Q_j in B. So $a \sim z \sim b$ contradicts the definition of a separator.

This shows that any P_i, Q_j can only intersect at their ends in X.

So

$$P_i x_i Q_i : 1 \leq i \leq k$$

gives a collection of k disjoint AB-paths in G, a contradiction.

Corollary 3.5.4.1 (Menger's Theorem II)

Proof

Let k be the size of the smallest a, b-separator X which does not include a, b.

Consider $A = N(a), B = N(b)$, the neighbours of a, b respectively.

Then let k' be the smallest size of an A, B-separator X' in G.

Since $G - X'$ contains no A, B-paths, and every ab-path contains an AB-path, X' is

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necessarily an ab-separator and so

$$k' = |X'| \geq k$$

Apply Menger's Theorem III* and we have k vertex-disjoint paths

$$P_1, \dots, P_k$$

Without loss of generality, assume $a, b \notin V(P_i)$ for every i. But then we can augment the ends of the paths with a, b and arrive at internally disjoint a, b-paths as desired.

Note that all (vertex) versions of Menger's Theorem apply for k-connected graphs, since k-connected graphs do not contain any separators of size less than k.

Lemma 3.5.5 (Fan)
If a is a vertex of a graph G, and $B \subseteq V(G)$ with $a \notin B$. Then either

(1) There are k paths $P_1, \dots, P_k$ each starting at a and disjoint with the exception of a, which all intersect B only at the end vertex.

(2) there is a set $X \subseteq V(G) \setminus \{a\}$ so that $|X| < k$ and $G - X$ has no aB-paths.

Proposition 3.5.6
If G is a k-connected graph for $k \geq 2$ and A is a set of at most k vertices of G, then G has a circuit containing each vertex in A.

Proof
Let C be a circuit containing the maximal number of vertices in A.

Notice that 2-connected graphs have circuits so such a C exist.

Let $a \in A \setminus V(C)$.

Since G is k-connected, there is no set $X \subseteq V(G) \setminus \{a\}$ of size less than $\min(|C|, k)$ such that there are no aC-paths in $G - X$ by Menger's Theorem.

Therefore, there are at least $\min(|C|, k)$ paths from a to C, say $\mathcal{Q}$ is the collection of paths.

Let $P_1, P_2, \dots, P_\ell$ be the paths formed by C between the elements of $V(C) \cap A$.

We have $\ell = |A \cap V(C)| < k$.

By the Fan Lemma, there are $\min(|V(C)|, k)$ paths from a to C that only intersect at a.

Since $\ell \leq |V(C)|$, in either case ($k < |V(C)|$ or $k \geq |V(C)|$) there is some i so that P_i contains the end in C of two of these paths $Q \neq Q' \in \mathcal{Q}$.

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Now, there is a circuit C' contained in $C \cup Q \cup Q'$ containing all vertices in $V(C) \cap A$ and also the vertex a.

This contradicts the maximality of C.

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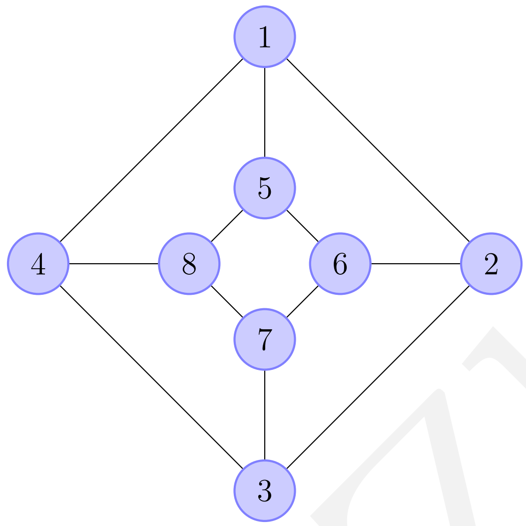
4 Planar Graphs



4.1 Definitions

Definition 4.1.1 (Plane Graph)
a pair $G = (V, E)$ where

(i) V is a finite subset of $\mathbb{R}^2$.

(ii) each $e \in E$ is an "arc" whose "endpoints" are in V

(iii) the interior of the edges in E are disjoint from each other, and from V.

Note that the word disjoint gives us non-crossing edges.

Definition 4.1.2 (Curve)
A subset of $\mathbb{R}^2$ which is homeomorphic to the unit interval

$$[0, 1]$$

Example 4.1.1 (Curve)
$X := f([0, 1])$ where $f : [0, 1] \to \mathbb{R}^2$ which is a continuous injection.

The endpoints of the curve is $f(0), f(1)$.

Definition 4.1.3 (Closed Curve)
A set of the form $f([0,1])$ where $f : [0,1] \to \mathbb{R}^2$ is continuous and injective on the domain $[0,1)$ and $f(0) = f(1)$.

Definition 4.1.4 (Polygonal)
A curve (or closed curve) is polygonal if it is a union of a finite number of straight line segments.

Definition 4.1.5 (arc)
Polygonal curve.

Definition 4.1.6 (polygon)
Polygonal and closed.

Theorem 4.1.2 (Thomassen)
The class of graphs that have a plane drawing where the edges are curves is equal to the class where the edges are required to be polygonal.

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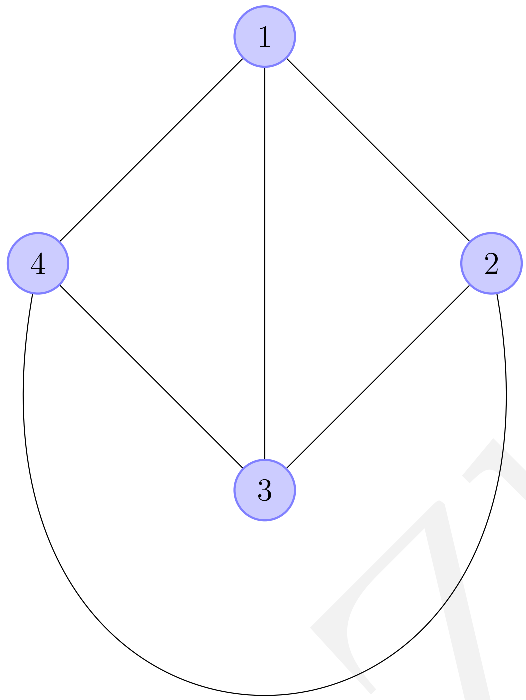

The above is a plane graph. There is a corresponding abstract graph.

Definition 4.1.7 (Abstract Graph)
A plane graph $G = (V, E)$ naturally corresponds to a graph

$$G' = (V, E, i)$$

G' is the abstract graph defined by G.

G is a plane drawing or plane embedding of G'.

Definition 4.1.8 (Planar)
A graph G is planar if it has a plane drawing.

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Definition 4.1.9 (Open Disc)
A subset of $\mathbb{R}^2$

$$D = \{x \in \mathbb{R}^2 : \|x - a\| < r\}$$

with radius r, center a.

Definition 4.1.10 (Open)
$X \subseteq \mathbb{R}^2$ is open if every $x \in X$ is contained in an oen disc $D \subseteq X$.

Proposition 4.1.3
open sets do not contains boundary points.

Definition 4.1.11 (Closed)
$X \subseteq R^2$ is closed if $\mathbb{R}^2 \setminus X$ is open.

Proposition 4.1.4
Closed sets contains all boundary points.

Definition 4.1.12 (Bounded)
$X \subseteq \mathbb{R}^2$ is bounded if it sits inside some disk D.

Definition 4.1.13 (Compact)
$X \subseteq \mathbb{R}^2$ is compact if it is closed and bounded.

Proposition 4.1.5
Finite union of open/closed/compact/bounded sets in is still open/closed/compact/bounded.

Proposition 4.1.6 (Topogical Compactness)
If $\mathcal{U}$ is a collection of open sets such that

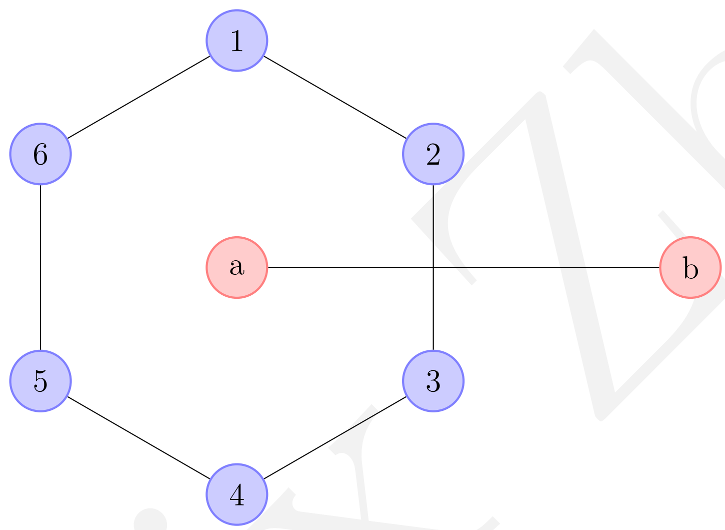
$$X \subseteq \bigcup_{U \in \mathcal{U}} U$$

for some set $X \subseteq \mathbb{R}^2$ that is compact.
Then there there is a finite subcover.

Proposition 4.1.7 (Sequential Compactness)
If X is compact, then any sequence in X has a convergent subsequence which converges within X.

Definition 4.1.14 (Linked)
$x_1, x_2 \in X$ are linked in a set $X \subseteq \mathbb{R}^2$ if there is an arc contained in X with endpoints x_1, x_2.

Below, a, b are NOT linked.



Note that connectedness in plane graphs gives us an equivalence relationship.

Its equivalence classes are the components of X.

4.2 Basic Results

Definition 4.2.1 (Faces)
Let G be a plane graph.
The faces of G are the components of $\mathbb{R}^2 \setminus G$

$$\mathbb{R}^2 \setminus \left(\left(\bigcup V\right) \cup \left(\bigcup E\right)\right)$$

We can "abuse" notation by writing

$$\mathbb{R}^2 \setminus G$$

Proposition 4.2.1
Every plane graph has exactly one unbounded face.

Proposition 4.2.2
Faces are always open.

Definition 4.2.2 (Frontier)
The frontier of a face f of a plane graph G is the set

$$\{x \in \underbrace{G}_{\subseteq \mathbb{R}^2} : \text{every disc centered at } x \text{ contains a point in } f\}$$

Theorem 4.2.3 (Jordan Edge Theorem for Polygons)
If G is a plane graph whose abstract graph is a circuit, then G has exactly two faces.

Proposition 4.2.4
If e is a cut edge of a plane graph G, then the interior of e is on the frontier of exactly one face of G.

Proposition 4.2.5
Let ė denote the set of points in e that are not endpoints.

(1) If X is the frontier of a face f of a plane graph G, then every edge of e either satisfies $\dot{e} \subseteq X$ or $\dot{e} \cap X = \varnothing$.

(2) For each edge e, the set ė is contained in the frontier of at most 2 faces.

Proof (2)
Consider the set $G \setminus \dot{e} \subseteq \mathbb{R}^2$. Since $G \setminus \dot{e}$ is also a plane graph, it is a closed (compact) set.

Now, each $x \in \dot{e}$ is NOT in $G \setminus \dot{e}$, so since $\mathbb{R}^2 \setminus (G \setminus e')$ is open, there is a disc D_x centered at x that does not intersect $G \setminus e'$.

Since e is a finite union of line segments, by choosing D_x small enough, we can guarantee that $D_x \cap e$ is the union of two radii of D_x.

Since $D_x \setminus G$ contains 2 equivalence classes with respect to linkedness, it follows that x is the frontier of at most 2 faces.

Proof (1)
Let $x_1, x_2 \in e'$ be distinct.

Let $e[x_1, x_2]$ be the sub-arc of e with x_1, x_2 as endpoints.

Let L_1, L_2 be the perpendiculars at x_1, x_2 to the initial/final segments.

Let R be the region of the plane bounded by L_1, L_2 containing $e[x_1, x_2]$.

Now, by the compactness of $e[x_1, x_2]$ and the existence of the discs D_x.

There is a finite collection $D_1, \ldots, D_k$ of discs whose union contains $e[x_1, x_2]$. Each D_i only intersects G in e, and in the union of two radii of D_i.

By choosing D_i appropriately, we may assume that their centers appear "in order" along the arc from x_1 to x_2 and that each D_i intersects D_{i+1}.

Let f be a face with x_1 on its frontier and choose points

$$a_0, \ldots, a_{k-1}$$

such that

- $a_0 \in f$
- $a_i \in D_i \cap D_{i+1}$ for each $i \geq 1$
- a_i is linked to a_{i+1} in $(D_i \cup D_{i+1} \setminus e)$ for each $i \geq 0$

Inductively, a_0 is linked to a_{k-1} in $\mathbb{R}^2 \setminus G$ so f contains a point in a face of G containing x_2 on its frontier.

So f has x_2 on its frontier.

If e' is contained in the frontier of f, then e is also contained in the frontier of f, as each disc around an endpoint of e contains a point x in the interior of e.

Therefore, by shrinking the ball around x sufficiently small, we fit that ball around the disc, and we are done.

Corollary 4.2.5.1
The frontier of a face f is a subgraph of G (union of vertices and edges of G).

Definition 4.2.3 (boundary)
The subgraph corresponding to the frontier of a face f the boundary of f.

Proposition 4.2.6
If e is a cut edge, it is contained in exactly one face boundary.
Otherwise e is in a circuit, and is contained in two face boundaries.

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Corollary 4.2.6.1
If G is a plane forest, then G has exactly one face.
The boundary of this face is G.

Proposition 4.2.7
Let G be a plane graph and P be a path of G so that G is obtained from a graph H by adding the path P.

- there is a single face f of H that contains the interior of P
- each face of H other than f is a face of G
- The face of H containing P is the union of two faces f_1, f_2 of G and the interior of P

Moreover, if f is bounded by a circuit then so is f_1, f_2.

Corollary 4.2.7.1
G has exactly one more face than H.

Proposition 4.2.8
In a 2-connected, loopless graph, every face boundary is a circuit.

Proof
Recall that there are 2-connected plane graphs

$$G_1, \ldots, G_k$$

so that G is a circuit, $G_k = G$ and each G_{i+1} is obtained from G_i by adding a path.

This shows that if each face of G_i is bounded by a circuit, the same is true of G_{i+1}. An inductive argument shows the proof.

4.3 Euler's Formula

Let $F(G)$ denote the set of faces of a plane graph G.

Theorem 4.3.1 (Euler's Formula)
If G is a connectd plane graph the

$$|V(G)| - |E(G)| + |F(G)| = 2$$

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Note that the formula works in S^2 as well since we can take draw planar graphs on a neighbourhood of the sphere and take the stereographical projection of of $S^2 \setminus \{(0,1,0)\}$ onto $\mathbb{R}^2$. Note that this is a homeomorhpism between the plane and the unit sphere less one point.

Proof
Recall that a tree on n vertices has $n-1$ edges.

Let G be a counterexample with as few edges as possible. If G has no circuit, then G is a tree so

$$|G(E)| = |V(G)| - 1$$

Moreover, trees have exact one face, so $|F(G)| = 1$ and the formula holds so G is not a tree by choice of G.

So G has a circuit. Let $e \in E(G)$ be an edge contained in a ciruit of G.

Note that $G - e$ is connected. By the lemma,

$$|F(G)| = |F(G-e)| + 1$$

as $G - e$ is not a counter example.

So

$$\begin{aligned} 2 &= |V(G-e)| - |E(G-e)| + |F(G-e)| \\ &= |V(G)| - (|E(G)| - 1) + (|F(G)| - 1) \\ &= |V(G)| - |E(G)| + |F(G)| \end{aligned}$$

which contradicts the assumption that G is a counterexample.

Proposition 4.3.2
If f is a face of a plane graph G that is not a forest, then the boundary of f contains a circuit of G.

Proposition 4.3.3
If G is a simple planar graph with $|V(G)| \geq 3$, then

$$|E(G)| \leq 3|V(G)| - 6$$

If G is also triangle-free, then

$$|E(G)| \leq 2|V(G)| - 4$$

The number of edges in a general graph is $O(|V|^2)$ but it is $O(|V|)$ in planar graphs.

Proof
Note that if G is a forest then

$$|E(G)| \leq |V(G)| - 1 \leq 3|V(G)| - 6 \leq 2|V(G)| - 4$$

so the proposition holds.

Let us assume G contains a circuit so that

$$X := \{(f, e) : e \text{ is an edge in the boundary } f\}$$

Note that

$$|X| = \sum_{f \in F(G)} (\text{number of edges in the boundary of } f) \geq 3|F|$$

as each face boundary contains a circuit so hence has size at least 3.

Also

$$|X| = \sum_{e \in E} (\text{number of faces with } e \text{ in the boundary}) \leq 2|E(G)|$$

because each edge is in at most 2 face boundaries.

Thus

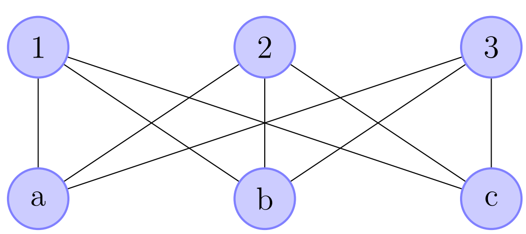
$$3|F| \leq |X| \leq 2|E|$$
$$|F| \leq \frac{2}{3}|E|$$

But

$$2 = |V| - |E| + |F| \leq |V| - |E| + \frac{2}{3}|E| \implies |E| \leq 3|V| - 6$$

Adjusting the lowerbound $3|F|$ above will show the proposition for triangle-free planar graphs.

Corollary 4.3.3.1
$K_{3,3}, K_5$ are non-planar.



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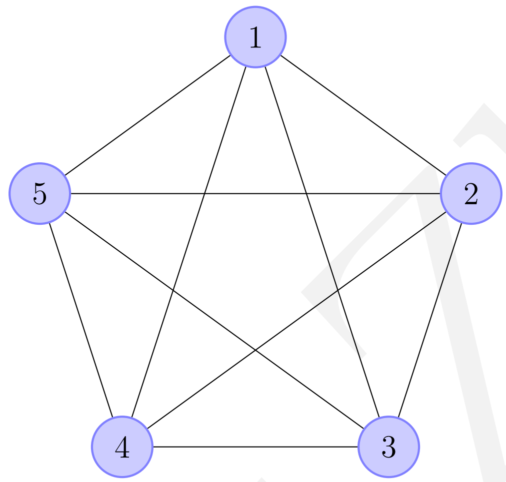
Proof
$K_{3,3}$ is triangle free with $|V| = 6, |E| = 9$ but

$$2|V| - 4 = 8 < 9$$

K_5 has $|V| = 5, |E| = 10$ so

$$3|V| - 6 = 9 < |E|$$



Note that we can draw both graphs after deleting an edge.

4.4 Subdivisions

Definition 4.4.1 (Edge Subdivision)
Let e be an edge of a graph G. The graph H obtained from G by subdividing e is the graph obtained from G by deleting the edge e, adding a new vertex v_e, and adding new edges $v_e u_1$ and $v_e u_2$ where u_1, u_2 (not necessarily distinct) were the original ends of e.

Definition 4.4.2 (Subdivision)
A subdivision of a graph G is a graph obtained from G by repeatedly subdividing edges.

Proposition 4.4.1
G is planar if and only if H is planar.
In fact, G, H have plane drawings coresponding to the same set of points in $\mathbb{R}^2$.

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Note that $\deg_H(v_e) = 2$ since we explicitly forced it to have two neighbours, the ends of e.

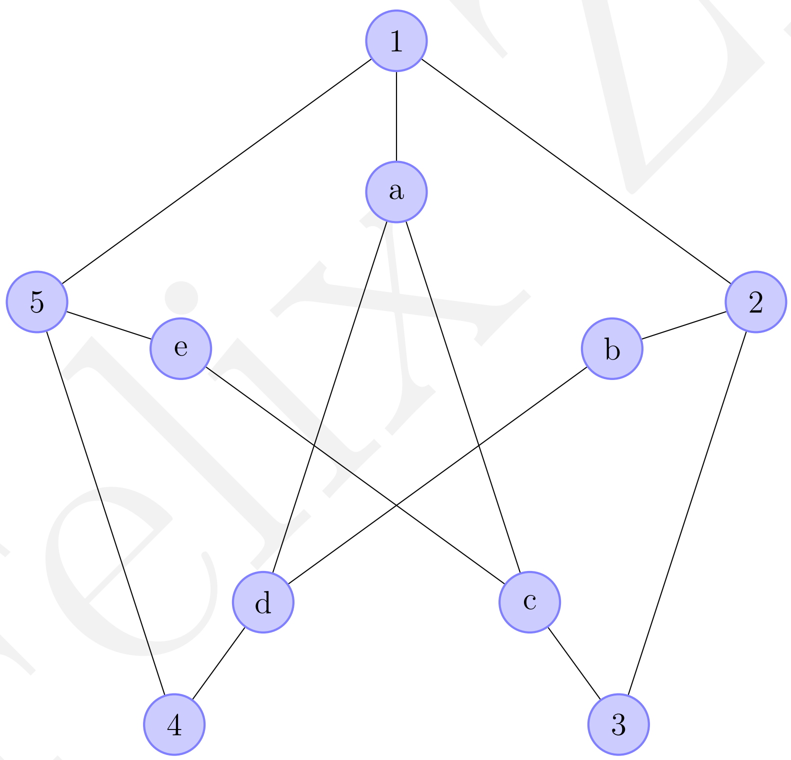
Also, G is isomorphic to $H/e_1, H/e_2$ as contracting either new edges "reverses" the subdivision.

Corollary 4.4.1.1
If H is non-planar and G is a subdivision of H, then G is non-planar.

Corollary 4.4.1.2
If G has a subdivision of a non-planar graph H as a subgraph, then G is non-planar.

Definition 4.4.3 (Topological Minor)
We say H as above to be a topological minor of G.

Notice that we can find a subdivision of $K_{3,3}$ in the Peterson Graph.



4.5 Facial Circuit: 3-Connected Graphs

Recall that G is a 2-connected loopless plane graph, every face boundary is a circuit.

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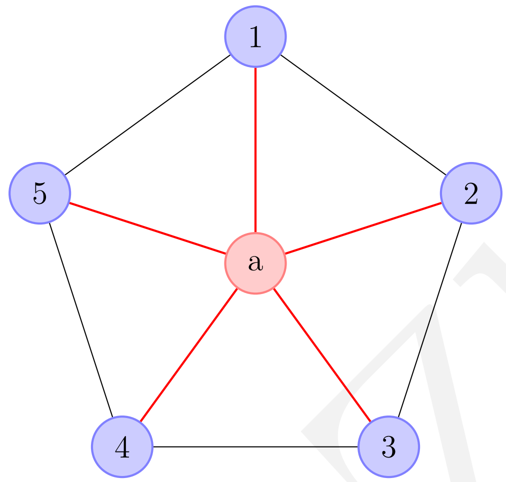
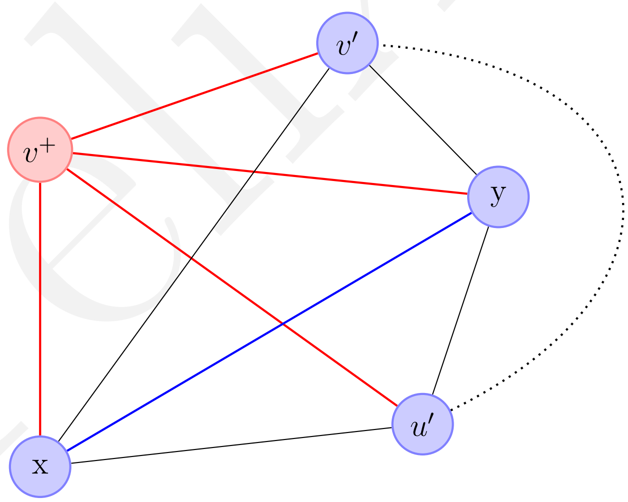
> **Lemma 4.5.1**
> If f is a face of a plane graph G, then there is a plane graph G^+ obtained by adding a vertex v inside the face f, and an edge from v to each vertex in the boudary of f.



> **Theorem 4.5.2**
> If G is a simple 3-connected plane graph, then a circuit C of G is the boundary of a face if and only if C is induced (chordless) and $G - C$ is connected (C is non-separating).

So the above is a purely combinatorial property and does not rely on the embedding in $\mathbb{R}^2$.



Proof ($\Longrightarrow$)
Let C be a circuit that is the boundary of a face f. We will construct the plane graph G^+ by adding a new vertex $v^+ \in f$ as in the lemma.

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Suppose that C has a chord xy. Therefore $|V(G)| \geq 4$ and there exists vertices u, v in different components of $C - \{x, y\}$.

By the 3-connectedness of G, there is a path P in $G - xy$ with one end u' and the other end v' in diffenent components of $C - \{x, y\}$.

Now, the path P, the edge xy, and the paths around C from x to u' to y to v', and the edges from v^+ to

$$x, u', y, v'$$

give a subgraph of G^+ that is a subdivision of K_5, where

$$x, u', y, v', v^+$$

are the "terminals".

Therefore, G^+ is nonplanar, which is a contradiction.

Suppose that $G - C$ is disconnected. Let xy be certices in different components of $G - C$.

By 3-connnectedness and Menger's Theorem, there are 3 internally disjoint xy-paths

$$P_1, P_2, P_3$$

in G.

None of these is a path in $G - C$ so there is a vertex

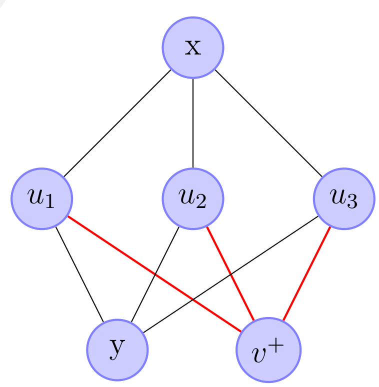
$$u_i \in V(C) \cap V(P_i)$$

for each $i \in \{1, 2, 3\}$.

Now the paths from x to u_i to y and the edges from v^+ to

$$u_1, u_2, u_3$$

form a $K_{3,3}$-subdivision. This is a subgraph of G^+, contradicting the planarity of G^+.



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Proof ($\Longleftarrow$)
Let C be a set so that C is an induced subgraph and $G - C$ is connected.

Let f_1, f_2 be the two faces of the plane graph C. If f_1, f_2 both contain points of the drawing of G, then since C has no chords, each contains a vertex of the drawing of G.

Let the vertices $v_1 \in f_1, v_2 \in f_2$. By the Jordan Curve Theorem, there is no v_1v_2-path in $G - C$, a contradiction since $G - C$ is connected.

It follows that either f_1, f_2 is a face of G as it contains no vertices of G with boundary C.

4.6 Graph Minors

Definition 4.6.1 (Minor)
A graph G has a graph H as a minor if H can be obtained from G by deleting vertices/edges and contracting edges.

Proposition 4.6.1
G has an H-minor if and only if there is a function φ that

(i) maps vertices of H to connected subgraphs of G

(ii) edges of H to edges of G

and

(a) the subgraphs $\varphi(v) : v \in V(H)$ are vertex-disjoint

(b) for each $e \in E(H)$ with ends u, v, the edge $\varphi(e)$ has ends in $\varphi(u), \varphi(v)$

(c) φ is injective

Definition 4.6.2 (Topological Minor)
A graph G has a graph H as a topological minor if some subdivision of H is contained in G as a subgraph.

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Proposition 4.6.2
H is a topological minor of G if and only if there is a function φ such that

(i) φ maps vertices of H to vertices of G

(ii) φ maps edges of H to paths of G

and

(a) the vertices $\varphi(v) : v \in V(H)$ are distinct vertices of G (terminals)

(b) for each edge $e \in E(H)$ with ends u, v, the path $\varphi(e)$ has ends $\varphi(u), \varphi(v)$ in $V(G)$ (or $\varphi(e)$ is a ciruit containing $\varphi(u)$ if e is a loop at u)

(c) paths $\varphi(e), \varphi(e')$ only intersect at a vertex x if $x = \varphi(u)$ and u is a common end of e, e' in H

Proposition 4.6.3
If G has an H-topological-minor, then G has an H-minor.

Proof
Assignment.

Theorem 4.6.4
G has K_5 **or** $K_{3,3}$ as a topological minor if and only if G has K_5 **or** $K_{3,3}$ as a minor.

Proof ($\implies$)
By previous proposition.

Proposition 4.6.5
If H has maximum degree at most 3 and G has an H-minor, then G has an H-topological-minor.

Proof
Assignment.

Proof (theorem, $\impliedby$)
If G has a $K_{3,3}$-minor, then it has a topological $K_{3,3}$-minor by another proposition since maximal degree of $K_{3,3}$ is 3.

It remains to show that if G has a K_5-minor, then it has a K_5 or $K_{3,3}$-topological minor.

Let G be a counterexample with as few edges as possible.

If G has at most 10 edges, G is just K_5 with some isolated vertices and has a topological K_5-minor.

Otherwise, G has at least 11 edges, so there is an edge e of G such that $G - e$ or G/e has a K_5-minor.

By the minimality of G, $G - e$ and G/e both have a K_5 or $K_{3,3}$-topological-minor.

But if H is a topological minor of $G - e$, it also has a topological minor G.

Elsewise, H is a topological minor of G/e. Let u, v be the ends of e, and $x = x_{uv}$ be the identified vertex in G/e.

Let T be the set of terminal vertices corresponding to the topological copy of H inside G. Furthermore, define $\mathcal{P}$ to be the set of paths betweeen the terminals that give H.

Case I: If $x \notin T$, nor is x in any of the paths in $\mathcal{P}$, then $T, \mathcal{P}$ give a topological copy of H inside G, which is contrary to the choice of G.

Case II: If x is an internal vertex of a path $P \in \mathcal{P}$ (ie $x \notin T$). Then, there is a path P' of G with the same ends as P such that

$$E(P) \subseteq E(P') \subseteq E(P') \cup \{e\}$$

Now, replacing $\mathcal{P}$ with $(\mathcal{P} \setminus \{P\}) \cup \{P'\}$ gives a topological copy of H within G, which is again a contradiction.

Case III: Otherwise, $x \in T$. So x corresponds to a vertex a of H and each edge of H incident with a corresponding to a path $P_f \in \mathcal{P}$ such that x is an end.

There is also a path P_f' of G with

$$E(P_f) = E(P_f')$$

and either u, v is an end.

If one of u, v (say u) is an end of at most 1 of the paths P_f', we can replace this P_f' with either itself or

$$P_f' \cup \{e\}$$

to give a topological copy of H in G.

Case IV: Finally, if this is not the case, then each of u, v is an end of at least 2 of the paths P_f'. Since the number of P_f' is equal to the degree of a in H, and $H \in \{K_{3,3}, K_5\}$, it follows that $H = K_5$ and each of u, v is an end of exactly two P_f'.

So G contains a topological copy of $K_{3,3}$.

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4.7 Kuratowski's Theorem

Theorem 4.7.1 (Kuratowski)
The following are equivalent

(1) G is planar

(2) G has no topological minor in $\{K_{3,3}, K_5\}$

(3) G has no minor in $\{K_{3,3}, K_5\}$

Lemma 4.7.2 (*A*-interval)
Let C be a circuit.
Given $A \subseteq V(C)$, a path contained in $V(C)$ with both endpoints in A but no internal vertices in A is an A-interval.

Lemma 4.7.3 (Circle Lemma)
Let C be a circuit.
If A, B are sets of vertices of C, only one of the following hold:

(1) $|A \cup B| \leq 2$

(2) $|A \cap B| \geq 3$

(3) There are distinct vertices a_1, b_2, a_2, b_2, in cyclic order around C such that $a_1, a_2 \in A, b_1, b_2 \in B$

(4) There is a A-interval containing B or vice versa.

Proof (circle lemma)
We may assume $|A| \leq |B|$.

Case I: If $|A| \leq 1$, then either $|B| = 1$ (outcome (1)) or there is a B-interval containing A (outcome (4)).

Case II: Now, suppose $|A| \geq 2$. If there is some $b_1 \in B \setminus A$, then $b_1 \in I$, an A-interval with ends a_1, a_2.

If $B \subseteq I$ (outcome (4)), we are done. Suppose not, there is some $b_2 \in B \setminus I$ (outcome (3)).

Case III: Elsewise, $B \subseteq A$ and by cardinality $A = B$. so $A \cap B = A \cup B$ (outcome (1) or (2)).

Lemma 4.7.4
If G is 3-connected and does not have a minor in $\{K_5, K_{3,3}\}$, then it is planar.

Proof (lemma)
Let G be a minimal counterexample (few edges as possible). Clearly, G is simple (parallel edges and loops do not affect planarity) and $|V(G)| \geq 5$ (K_4 is planar).

By the lemma a million years ago, there is an edge e such that G/e is 3-connected. So G/e does not have a minor in $K_{3,3}, K_5$ (or else G does), but is not a counterexample, Therefore, it is forced to be planar.

Let u, v be the ends of e in G. Let x be the identified vertex in G/e. Since G/e is 3-connected, $(G/e) - x$ is 2-connected and therefore, each face is bounded by a circuit.

Let C be the circuit bounding the face containing x in some planar embedding of G/e. So every neighbour of x in G/e is a vertex of C by planarity.

We show that we can contradict the either the nonplanarity of G, the 3-connectednes of G, or the assumption that we do not have a K_5 or $K_{3,3}$ minor.

Apply the circle lemma with $A = N_G(u), B = N_G(v)$ and consider the outcomes:

(1) $|A \cup B| \leq 2$ and $A \cup B$ is a separator in G/e, contradicting 3-connectedness.

(2) K_5-topological-minor

(3) $K_{3,3}$-topological-minior

(4) G is planar

In all cases, we have a contradiction, so G cannot be a counterexample.

Proposition 4.7.5
If G is a planar graph

(1) G has a planar embedding in $\{(x, y) \subseteq \mathbb{R}^2 : x > 0\}$

(2) For every vertex of $v \in V(G)$, G has a planar embedding so that v is at the origin and every other vertex has positive x-coordinate

(3) For every pair of adjacent vertices u, v of G, there is a planar embedding of G where u is at the origin, v is at $(0, 1)$, and every other point in the drawing has positive x-coordinates

and vice versa with "positive" replaced with "negative".

Proof (sketch)
We us stereographic projection to find a drawing of G where the edge from u to v is on the unbounded face.

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Shift this drawing so that all x-coordinates are positive.

Move u, v to the desired poisitions and reroute edges consistently.

Theorem 4.7.6
If G has no $K_{3,3}$-minor or K_5-minor, then G is planar.

Proof
Let G be a minimal counterexample.

Every proper minor of G is not a counterexample, and so is planar. If G is disconnected, then let C be a component of G. Now, $G - C$ and C are planar, and thus G is planar by the proposition.

We may proceed assuming G is connected. If G is not 2-connected, then G has a cut-vertex v. Therefore, let G_1, G_2 intersecting at only v, so that $G = G_1 \cup G_2$.

By the proposition, we can draw G_1 in the right of y-axis and G_2 on the other side, with v at the origin. This directly gives a drawing of G.

If G is not 3-connected, and $|V(G)| \geq 5$, then it is 2-connected and has vertices u, v such that $G - u - v$ is disconnected.

Let G_1, G_2 be graphs on at least 3 vertices, with intersection $\{u, v\}$ such that $G_1 \cup G_2 = G$

Let G_1', G_2' be obtained from G_1, G_2 by drawing an edge e from u to v. Since G is 2-connected, G_2 is connected, and so contains a path P_2 from u to v.

Therefore $G_1 \cup P_2$ has G_1' as a minor, so G_1' is a minor of G. Similary, G_2' is a minor of G.

Since G_1', G_2' are proper minors of G, they have no $K_{3,3}, K_5$ minors. They are not counterexamples, and thus are planar.

Now, we can glue embeddings G_1', G_2' with the proposition and this will give a drawing of a graph with G as a subgraph.

Otherwise, G is 3-connected and by our prior work, we are done.

4.8 Algorithmic Planarity Testing

Can we decide if a graph is planar in polynomial time?

We can test for 3-connectedness by deleting every pair of vertices and using BFS to check connectedness.

Given G that is 3-connected, we can always find an edge e such that G/e is 3-connected. Guess e, check if G/e is 3-connected, and so on.

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Now, recursively run the algorithm on G/e after simplifying to check if we have a $K_{3,3}$ or K_5 minor, then this is a minor of G.

Otherwise, the algorithm will give a planar drawing of G/e.

The recursion is by testing the identified vertex x in G/e and checking its bounding cycle and neighbours of original ends. Similar to our prior work in the proof of the 3-connected case of Kuratowski's Theorem

4.9 Planar Embeddings with Straight Line Edges

K_4 is planar and actually has an embedding with straight lines.

Theorem 4.9.1 (Fary, 1948)
Every simple planar graph has an embedding where edges are straight lines.

Theorem 4.9.2 (Tutte)
Every 3-connected simple planar graph has a drawing where edges are straight line segments and faces are convex arguments.

Definition 4.9.1 (Spring Embedding)
Given a circuit C of a 3-connected plane graph $G = (V, E)$, a spring embedding of G is a function

$$\varphi : V \to \mathbb{R}^2$$

such that

(1) the vertices of C are mapped to a prescribed convex polygon

(2) $\varphi(u) = \frac{1}{\deg u} \cdot \sum_{v \text{ adjacent to } u} \varphi(v)$ for $u \in V \setminus C$

To see this exists, we can use a physics argument, a linear algebraic argument using the Laplacian, or an analytic one with sequential compactness.

Theorem 4.9.3 (Tutte)
A spring embedding of a 3-connected simple planar graph G gives a straight line drawing with convex faces.

Lemma 4.9.4 (I)
In a spring embedding, for each $u \in V \setminus C$ and each line L through $\varphi(u)$, either

1. all neighbours of u live on L
2. u has a neighbour strictly on each side of L

Proof
If the first case does not hold, there must be neighbours on both sides of the line to be at "average" position.

Lemma 4.9.5 (II)
For each open half-plane P for which $P \cap \varphi(v) \neq \varnothing$, the vertices in $P \cap \varphi(v)$ induce a connected subgraph of G.

Proof
P will include some part of the bounding circuit C (connected).

Assume the there is another component from the connected subgraph including parts of C. We can find a linear function such that one vertex of that component is the unique optimal solution of $c^T x$.

So there is a line which on one side does not have neighbours.

Lemma 4.9.6 (III)
No vertices $u \notin C$ have all neighbours on a line through u (ie I(i) does not happen).

Proof
There must be three vertices on such a hypothetical line with neighbours on both sides of the hypothetical line. Or else there is a separator of size 2.

By Menger's Theorem, we can find internally disjoint paths from one neighbour on each side of the line to each of the 3 vertices on the line.

This is a $K_{3,3}$ minor.

Lemma 4.9.7 (IV)
For every pair C_1, C_2 of facial circuits of G intersecting in a edge $e = uv$, the line through $\varphi(u), \varphi(v)$ separates C_1 from C_2.

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Proof
Suppose not.

There is a uv-path Q that does not include e on the C_2 side. There is a path P disjoint from Q from a vertex in C_1 to a vertex in C_2.

We can add v^* to the ends of the P, Q and gives a K_5 minor. But that would give a planar embedding of some graph with a K_5 minor which is a contradiction.

Lemma 4.9.8 (V)
The facial circuits map to convex polygons, call them tiles. No tiles intersect.

Proof
There is a point "close" to the bounding circuit which is only in 1 tile. We can walk tile to tile by crossing an edge each time and not changing the amount of tiles which intersects the path (only 1).

This reduces to a smaller subgraph and we can proceed by induction.

Proof (theorem)
Argue by induction on the number of vertices or edges.

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5 Matchings

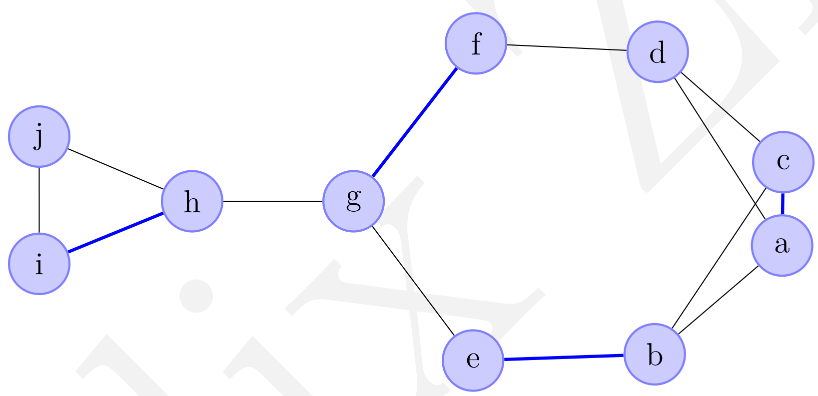
5.1 Definitions

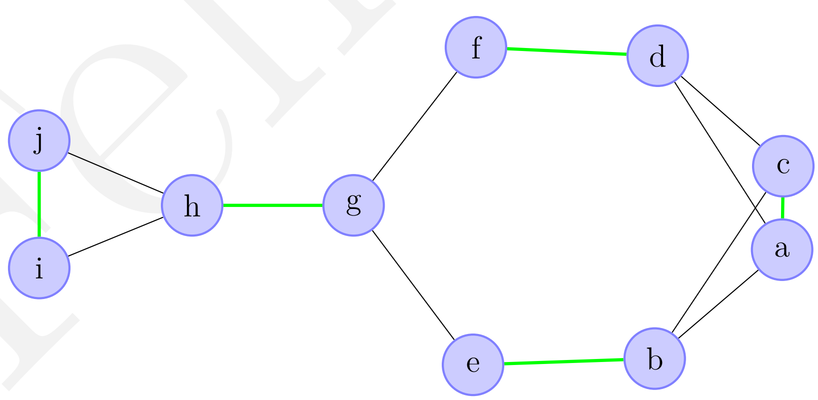
Definition 5.1.1 (Matching)
A matching in a graph G is a set of edges of G so that no two have an end in common.

Let $\nu(G)$ denotes the size of a maximum (largest) matching of G.

Definition 5.1.2 (Maximum Matching)
A maximum matching of G is a matching of size $\nu(G)$.

Remark that there is nothing to be gained from non-simple graphs since we really only care if an edge exists between two graphs and NOT how many edges.





Definition 5.1.3 (Saturated)
A vertex that is an end of an edge in a matching is saturated.

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Otherwise, the vertex is unsaturated.

Definition 5.1.4 (Vertex Cover)
A vertex Cover of G is a set $U \subseteq V(G)$ such that every edge has an end in U. (ie $G - U$ has no edgse).

For a graph G, $\tau(G)$ denote the size of a minimum (smallest) vertex cover.

5.2 Basic Results

Proposition 5.2.1
If U is a vertex cover and M is a matching, then every edge in M has an edge in U, and no two edges in M have such an end in common, so

$$|M| \leq |U|$$

thus

$$\nu(G) \leq \tau(G)$$

Proposition 5.2.2
If M is a matching and U is a cover with $|M| = |U|$, then M is a maximum matching and U is a minimum cover. In addition

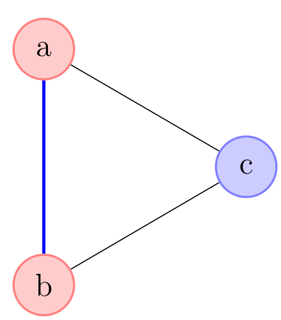
(i) Every vertexx in U is an end of an edge in M

(ii) every edge in M has exactly one end in U.

Proof
By previous proposition.

Theorem 5.2.3 (König, 1931)
If G is bipartite, then

$$\nu(G) = \tau(G)$$



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If a graph is not bipartite, then it has no odd cycles, which causes problems in general. For C_3 above $\nu(G) < \tau(G)$ strictly.

On the other hand, even circuits have perfect matchings equal to the minimal size of a vertex cover.

> **Proof (König)**
> We apply Menger's Theorem.
>
> Let A, B be the bipartition of $V(G)$.
>
> A vertex cover is the same as a set $X \subseteq V(G)$ so that $G - X$ has no edges.
>
> This is equivalent as saying $G - X$ has no AB-paths.
>
> So
> $$\tau(G) = \min_{X \subseteq V(G)} |X|$$
> such that $G - X$ has no AB-paths.
>
> By Menger's Theorem, this is precisely the number of vertex-disjoint AB-paths. This happens to be the maximum size of a matching of G. So
> $$\nu(G) = \tau(G)$$

There is an alternative proof that does not use Menger's Theorem NOR alternating paths. NICE!

> **Proof (König)**
> Let G be a counterexample with as few edges as possible. Note that if G has no edges, its minimal cover and maximum matching is 0. So $|E(G)| > 0$.
>
> In other words, $\nu(G) < \tau(G)$ but
> $$\nu(H) = \tau(H)$$
> for every proper subgraph H of G.
>
> G is connected, since if C is a component of G, then
> $$\nu(G) = \nu(C) + \nu(G - C) = \tau(C) + \tau(G - C)$$
> which is a contradiction.
>
> We claim that G is also not a path or a (even) circuit as they are both not counterexamples. (We considered these explicitly).
>
> Since G is connected but is not a path or circuit, it has a vertex u of degree at least 3.
>
> Let v be a neighbour of u.

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Case I: If $\nu(G-v) < \nu(G)$, then let U be a vertex cover of $G-v$. Now, $U \cup \{v\}$ is a cover of G. So

$$\begin{aligned}
\tau(G) &\leq |U \cup \{u\}| & \\
&= |U| + 1 & \\
&= \tau(G-v) + 1 & \\
&= \nu(G-v) + 1 & \text{induction} \\
&\leq \nu(G) & \text{assumption} \\
&< \tau(G) & \text{choice of } G
\end{aligned}$$

which is a contradiction.

Case II: So $\nu(G-v) = \nu(G)$. Let M be a maixmum matching of $G-v$, so M is also a max matching of G.

Since M is maximum in G but does not saturate v, it must saturate u.

Let f be an edge of G incident with u but not v such that $f \notin M$.

By the minimality of G we have

$$\nu(G-f) = \tau(G-f)$$

so M is a maximum matching of $G-f$, and $G-f$ has a vertex cover U such that

$$|U| = |M|$$

If U is a cover of G then $\nu(G) = |M| = |U| = \tau(G)$ which is a contradiction. Thus U is not a cover of G and $u \notin U$.

But the edge from u to v is an edge of $G-f$ so has on end in U, giving $v \in U$.

So v is a vertex of $G-f$ that is in the cover U, but is not saturated by M.

Since every edge in M contains a vertex in U, it follows that $|U| > |M|$, which is the desired contradiction.

5.3 Matchings in General Graphs

Proposition 5.3.1
If S is a set of vertices of a graph G such that $G-S$ has more than $|S|$ odd components, then G has no perfect matching.

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Definition 5.3.1 (Odd)
Let $\mathrm{odd}(G)$ denote the number of odd components of G.

Proposition 5.3.2
If S is a set of vertices in a graph G, and M is a matching of G, then G has at at least

$$\mathrm{odd}(G - S) - |S|$$

vertices that are not saturated.

Proof
Every odd component of $G - S$ that contains no unsaturated vertex has a vertex joined by an edge of M to a vertex in S.

There are at most $|S|$ edges of M with an end in S, so at least

$$\mathrm{odd}(G - S) - |S|$$

odd components of $G - S$ contain an unsaturated vertex.

When M is a maximum matching, this gives

$$\begin{aligned} n - 2\nu(G) &= n - 2|M| \\ &\geq \mathrm{odd}(G - S) - |S| \end{aligned}$$

for each $S \subseteq V(G)$.

So
$$\nu(G) \leq \min_{S \subseteq V(G)} \frac{1}{2}(|V(G)| - \mathrm{odd}(G - S) + |S|)$$

Let M, M' be matchings of G.

Definition 5.3.2
For $F \subseteq E(G)$ let

$$G[F] := \left(V(G), F, \phi\Big|_{V(G) \times F} \right)$$

be the graph obtained from G by deleting all edges not in F.

Consider $G[M \cup M']$. Every vertex has degree at most 2 by the definition of matchings and thus it is just paths and circuits.

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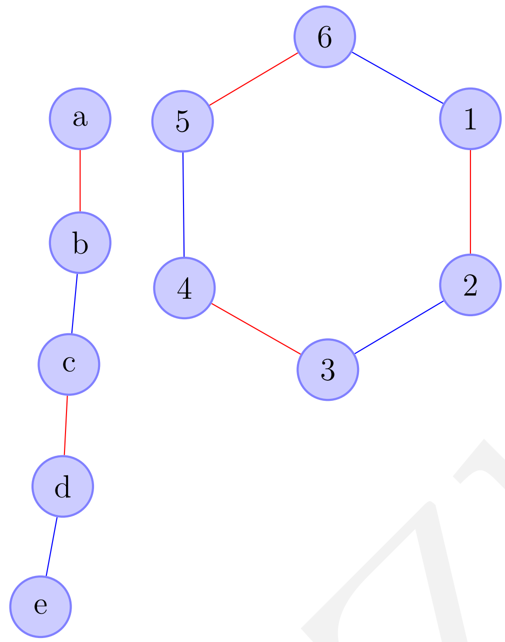

Each vertex in a circuit of $G[M \cup M']$ is saturated by both M, M'.

Each path either has length 1 (number of edges) and its edge is in M AND M', or all its interval vetices are saturated by both M and M', and all its end vertices are saturated by exactly 1, or it has length 0.

Each circuit and path of length at least 2 alternates between edges in M and M'. Therefore, each circuit is even.

If P is a path component of odd length that is NOT an edge of $M \cap M'$, either

$$|P \cap M| < |P \cap M'| \text{ or } |P \cap M| > |P \cap M'|$$

Definition 5.3.3 (Symmetric Difference)
Define

$$A \Delta B := (A \cup B) \setminus (A \cap B)$$

Say $|P \cap M'| < |P \cap M|$, then

$$M' \mapsto M' \underbrace{\Delta}_{\text{symmetric difference}} P$$

is a matching longer than M'.

So if M and M' are both maximum matchings, then every path component that is not an edge of $M \cap M'$ has even length!

Theorem 5.3.3 (Tutte, Berge)

$$\nu(G) = \frac{1}{2} \min_{S \subseteq V(G)} (|V(G)| + |S| - \mathrm{odd}(G - S))$$

In other words, for every maximum matching, there is a set S for which equality holds.
This S gives a concise certificate that there is no larger matching.

Definition 5.3.4 (Hypomatchable)
A graph G is hypomatchable if $G - u$ has a perfect matching for all $u \in V(G)$.

Lemma 5.3.4
If $\nu(G - v) = \nu(G)$ for every $v \in V(G)$, then every component H of G is hypomatchable.

Proof
Let $u \star v$ if

$$u = v \vee \nu(G - \{u, v\}) < \nu(G)$$

We show that $\star$ is an equivalence relation. It suffices to show transitivity.

Suppose that u_1, v, u_2 are distinct with

$$u_1 \star v, v \star u_2$$

Moreoever, suppose for a contradiction that $u_1 \not\star u_2$. (ie $\nu(G - (u_1 + u_2)) = \nu(G)$). In other words, there is a maximum matching M not saturating u_1, u_2.

Therefore, there is a maximum matching M' not saturating v.

Consider $G[M \cup M']$. Each of u_1, v, u_2 have degree at most 1 and so is an end of a path component of $G[M \cup M']$.

But at least one of u_1, u_2 is not connected to v, say u_1 so we can toggle the path with end u_1 to get a matching M'^*.

Notice that M'^* does not saturated v, u_1 and thus contradicts the fact that $v \star u_1$.

Now, suppose there is a path $u_1, \ldots, u_k$ then by transitivity

$$u_1 \star u_k$$

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and every pair of vertices in the same component are related by $\star$.

We claim that each component H of G has a matching saturating every vertex except u for every choice of u.

If not, then $H-u$ has a maximum matching avoiding another vertex v of H. This implies that

$$\nu(G-u-v)=\nu(G)$$

contradicting $u \star v$.

This gives the result.

Proof (theorem)
Let G be a minimal counterexample with respect to the number of vertices.

Clearly $|V(G)|>0$.

Claim I: G is connected.

If not, let H be a component of H. Since $H, G-H$ are not counterexamples, we have

$$\begin{aligned}
\nu(G) &= \nu(H)+\nu(G-H) \\
&= \frac{1}{2}\min_{S\subseteq V(H)}(|V(H)|+|S|-\mathrm{odd}(H-S))+\frac{1}{2}\min_{S'\subseteq V(G-H)}(|V(G-H)|+|S'|-\mathrm{odd}(G-H-S)) \\
&= \frac{1}{2}\min_{S\subseteq V(H), S'\subseteq V(G-H)}(|V(G)|+|S\cup S'|-\mathrm{odd}(H-S)-\mathrm{odd}(G-H-S')) \\
&= \frac{1}{2}\min_{S'\subseteq V(G)}(|V(G)|+|S|-\mathrm{odd}(G-S))
\end{aligned}$$

which is contrary to the choice of G.

Claim II: $\nu(G-v)=\nu(G)$ for each $v\in V(G)$.

Suppose that $v\in V(G)$ and $\nu(G-v)<\nu(G)$ ie

$$\nu(G-v)\le \nu(G)-1$$

Then

$$\nu(G-v)=\frac{1}{2}\min_{S\subseteq V(G-v)}(|V(G-v)|+|S|-\mathrm{odd}(G-v-S))$$

so there is a set $S_0\subseteq V(G-v)$ which achieves the minimum.

Let $S' := S_0\cup\{v\}$. We know that

$$\begin{aligned}
\nu(G-v) &\le \nu(G)-1 \\
|V(G-v)| &= |V(G)|-1 \\
|S'| &= |S_0|+1
\end{aligned}$$

so

$$\begin{aligned}\nu(G) - 1 \geq \nu(G-v) \\ &= \frac{1}{2}(|V(G-v)| + |S_0| - \text{odd}(G-v-S_0)) \\ &= \frac{1}{2}(|V(G)| - 1 + |S'| - 1 - \text{odd}(G-S'))\end{aligned}$$

This gives

$$\begin{aligned}\nu(G) &\geq \frac{1}{2}(|V(G)| + |S'| - \text{odd}(G-S')) \\ &\geq \frac{1}{2}\min_{S\subseteq V(G)}(|V(G)| + |S| - \text{odd}(G-S))\end{aligned}$$

and contradicts the choice of G as a counterexample.

So G is hypomatchable by the lemma and thus has an odd number of vertices (only 1 component). It has

$$\nu(G) = \frac{1}{2}(|V(G)| - 1)$$

because $\nu(G-v) = \frac{1}{2}|V(G-v)|$ for each v.

This is equal to

$$\frac{1}{2}(|V(G)| + |S| - \text{odd}(G-S))$$

when $S = \varnothing$.

Since this is an upper bound for $\nu(G)$, we have equality. It follows that G is not a counterexample and we conclude the result holds.

Corollary 5.3.4.1 (Tutte)
G has a perfect matching if and only if

$$|\text{odd}(G-S)| \leq |S|$$

for all $S \subseteq V(G)$.

Corollary 5.3.4.2 (Peterson)
If G is a 3-regular graph with no cut-edge, then G has a perfect matching.

Proof
Suppose not, there is a set S with

$$|\text{odd}(G-S)| > |S|$$

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Let $X \subseteq V(G)$ be a subset of the vertices with odd cardinality. We argue that X has an odd number of edges leaving X.

Let $F = \delta(X)$ be the set of outgoing edges from X. Since G has no cut edges, $|F| > 1$.

Next, remark that

$$\overbrace{\sum_{x\in X} \deg(x)}^{\text{odd}} = \underbrace{|F|}_{\text{outgoing}} + \overbrace{2|E(G[X])|}^{\text{internal edges, even}}$$

so $|F|$ is necessarily odd. This gives $|F| \geq 3$.

So every odd component of $\operatorname{odd}(G - S)$ has at least 3 outgoing edges.

Therefore, the number of edges with an end in S and an end in an odd component of $G - S$ is at least

$$3 \operatorname{odd}(G - S) > 3|S|$$

This contradicts the 3-regularity of G.

5.4 Matching Structure

Definition 5.4.1 (Berge Witness)
Let G be a graph. Then if $S \subseteq V(G)$ is such that

$$\nu(G) = \frac{1}{2}(|V(G)| + |S| - \operatorname{odd}(G - S))$$

then S is a berge witness.

What is the structure of G and its maximum matchings relative to a Berge Witness.

For any matching M, there are ar least $\operatorname{odd}(G - S) - |S|$ odd components of $G - S$ that contain an unsaturated vertex.

If M is a a maximum matching, then there are exactly $\operatorname{odd}(G-S)-|S|$ unsaturated vertices in M.

Therefore, the unsaturated vertices of M all lie in odd components of $G-S$, and no two are in the same component.

Every other vertex of G is saturated. Thus

(1) Every odd component of $G-S$ containing no unsaturated vertices must have a vertex that is matched by M to a vertex in S

(2) Every vertex in S is matched to a vertex in some odd component of $G-S$ in this way.

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(3) The even components of $G-S$ contain no unsaturated vertices and no vertices matched by M to a vertex in S, so they have a perfect matching

Definition 5.4.2 (Avoidable)
A vertex $v \in V(G)$ is avoidable if some maximum matching of G does NOT saturated v.
In other words
$$\nu(G-v) = \nu(G)$$

Theorem 5.4.1 (Gallai-Edmonds)
Let G be a graph, D the set of avoidable vertices of G, and A the neighbour of vertices that are not in D themself. Finally let $C = V(G) \setminus (A \cup D)$

(i) A is a Berge witness

(ii) Every odd component of $G-A$ is hypomatchable and contained in D

(iii) Every even component of $G-A$ has a perfec matching contained in C

Proof (Kotlov, 2000)
We will construct sets $\hat{A}, \hat{D}, \hat{C}$ and show taht they have the required properties.

Let $\hat{A}$ be a Berge witness, chosen so that

(a) The number of non-hypomatchable odd components of $G-\hat{A}$ is as small as possible

(b) The size of $\hat{A}$ is as small as possible (subject to the first condition)

Suppose inductively that the theorem holds for graphs with less than $|V(G)|$ vertices.

Lemma 5.4.2
Every odd component of $G-\hat{A}$ ia hypomatchable.

Proof
Let H be a non-hypomatchable odd component, and v be a vertex of H so that $H-v$ has no perfect matching.

Let X be a Berge Witness for $H-v$, chosen so that every odd component of $(H-v)-X$ is hypomatchable.

Since X is a Berge Witness in $H-v$ and $H-v$ is even, there are at least two vertices unsaturated in a maximum matching of $H-v$, so

$$\text{odd}(H-v-X) - |X| \geq 2$$

We now argue that $\{v\} \cup \hat{A} \cup X$ is a Berge Witness of G. This would be a contradiction since $G - (\{v\} \cup \hat{A} \cup X)$ has fewer non-hypomatchable odd components then $G - \hat{A}$ does.

We have

$$\begin{aligned}
\operatorname{odd}(G - (\{v\} \cup \hat{A} \cup X)) - |\{v\} \cup \hat{A} \cup X| &= \Big[(\operatorname{odd}(G - \hat{A}) - 1) + \operatorname{odd}(H - v - X)\Big] - 1 - |\hat{A}| - |X| \\
&= (\operatorname{odd}(G - \hat{A}) - |\hat{A}|) + [(\operatorname{odd}(H - v - X) - |X| - 2] \\
&\geq \operatorname{odd}(G - \hat{A}) - |\hat{A}|
\end{aligned}$$

So the fact that $\hat{A}$ is a Berge Witness show that

$$\hat{A} \cup \{v\} \cup X$$

is a Berge Witness whose deletion leaves fewer non-hypomatchable odd components than $\hat{A}$, a contradiction.

Lemma 5.4.3
For every non-empty set $A' \subseteq \hat{A}$, at least $|A'| + 1$ odd components of $G - \hat{A}$ have a neighbour in A'.

Proof
Let $A' \subseteq \hat{A}$ be a set violating this. Let

$$A' = \{v_1, \dots, v_k\}$$

In a maximum matching M, the vertices $v_1, \dots, v_k$ are matched to vertices in different odd components of $G - \hat{A}$.

Call them $H_1, \dots, H_k$ respectively. Since A' violates the claim, $\{H_i\}$ are the only odd components of $G - \hat{A}$ having a neighbour in A'.

We demonstrate that $\hat{A} \setminus A'$ is a Berge Witness in G.

$$\begin{aligned}
\operatorname{odd}(G - (\hat{A} \setminus A')) - |\hat{A} \setminus A'| &\geq (\operatorname{odd}(G - \hat{A}) - k) - |\hat{A}| + |A'| \\
&= \operatorname{odd}(G - \hat{A}) - |\hat{A}|
\end{aligned}$$

and $\hat{A} \setminus A'$ is a Berge Witness.

All odd components of $G - \hat{A}$ that are not in $\{H_i\}$ are odd components of $G - (\hat{A} \setminus A')$. The other vertices of $G - (\hat{A} \setminus A')$ are partitioned by the connected even sets

$$H_i \cup v_i$$

and the even components of $G - \hat{A}$.

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Therefore $G - (\hat{A} \setminus A')$ has no odd components that are not odd components of $G - \hat{A}$. But $|\hat{A} \setminus A'| < |\hat{A}|$ and $G - (\hat{A} \setminus A')$ have no more non-hypomatchable odd components than $G - \hat{A}$.

This contradicts the choice of $\hat{A}$.

Lemma 5.4.4
Every vertex in an odd component of $G - \hat{A}$ is avoidable.

Proof
Let v be a vertex of an odd component H of $G - \hat{A}$. We show that there is a maximum matching M_0 of G that matches every vertex in $\hat{A}$ to a vertex in an odd component of $G - \hat{A}$ other than H.

By the second lemma, each set $A' \subseteq \hat{A}$ has neighbours at least $|A'|$ odd components of $G - \hat{A}$ other than H. Then, there is a $\hat{A}$-saturating matching M_0 by Hall's Theorem which does not use vertices in H.

Now, since M_0 saturates at most 1 vertex from each odd component of $G - \hat{A}$, no vertices of any even component of $G - \hat{A}$, and saturates $\hat{A}$, we can use the second lemma and the fact that even components of $G - \hat{A}$ have perfect matchings to extend M_0 to a maximum matching of G, avoiding v.

We have shown the existence of a Berge Witness $\hat{A}$ such that

(1) Every odd component of $G - \hat{A}$ is hypomatchable

(2) Every vertex in an odd component of $G - \hat{A}$ is avoidable

(3) Each nonempty set $A' \subseteq A$ has edges to at least $|A'| + 1$ odd components of $G - \hat{A}$

Since every vertex in $\hat{A}$ and in an even component of $G - \hat{A}$ is unavoidable, claim 2 implies that the vertices in odd components of $G - \hat{A}$ are precisely the avoidable vertices of G.

By (2), every $x \in \hat{A}$ has a neighbour in D, the set of avoidable vertices of G and clearly no vertex outside of $\hat{A} \cup D$ has a neighbour in D, so $\hat{A} = A$, the set of neighbours of vertices in D that are not in D. This implies the result (using claim (1)).

Proposition 5.4.5
If M is a matching in a graph G that is NOT maximum, then there is a M-augmenting path in G.

Proof
Assignment 4.

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5.5 Slither

This is played on a graph G by two players Alice and Bob. They take turns choosing edges of G so that the chosen edges always form a path. The first player with no move loses.

5.5.1 Win Conditions

When can Alice force a win?

Proposition 5.5.1
If G has a perfect matching M, then Alice can force a win.

Proof
Consider the first turn where Alice cannot choose an edge from M.

Bob just extended the path P by a vertex w.

Every vertex of P has its matching in P, so Alice can extend using the match of w, contradicting the assumption that this turn is a counterexample.

When can Bob force a win?

Proposition 5.5.2
If G is hypomatchable, then Bob can force a win.

Proof
Symmetric to the previous proposition.

5.5.2 Gallai-Edmonds

There is a partition of $V(G)$ into three sets

$$C(G), A(G), D(G)$$

where C is the leftover, A is berge witness, and D has avoidable vertices.

Every component of $G[D(G)]$ is hypomatchable.

Since $A(G)$ is a Berge witness, every maximum matching of G induces a perfect matching of each component of $G[C(G)]$, and a maximum matching of each component of $G[D(G)]$.

Proposition 5.5.3
If $C(G) \neq \varnothing$, then Alice can win.

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Proof
Let M be a maximum matching of G, and Alice starts with $e_0 \in M$ in $C(G)$.

Say Alice uses vertex w and Alice is stuck. But then the portion of the path from e_0 to w is M-alternating. w is M-unsaturated, and both ends of e_0 are unavoidable, which is a contradiction.

Proposition 5.5.4
If $C(G) \neq \varnothing$, then the first player to choose an edge with an edn in $D(G)$ loses.

Proof
Similar to previous proposition.

We can continually reduce the game by consider the Gallai-Edmonds partition of $A(G)$, etc.

5.6 Blossom Algorithm

The idea is to take any matching M, look for an augmenting path. If there is not augmenting path, M is maximum. Otherwise, use the path to find a larger matching.

We can find an M-alternating tree.

Definition 5.6.1 (M-Alternating Tree)
An M-Alternating Tree rooted at u is a tree subgraph of G containing u so that

(i) for each $v \in V(T)$, the uv-path in T is M-alternating and v is saturated

(ii) every leaf vertex of T has even distance from u in T.

Note that by this definition, whenever $v \in V(T - u)$, the matching edge incident with v is an edge of T.

Definition 5.6.2 (Outer Vertices)
Given an M-alternating tree T rooted at u, the outer vertices of T are those at even distance from u (including u).

Definition 5.6.3 (Inner Vertices)
Similar.

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Definition 5.6.4 (M-Alternating Forest)
F, when it is a subgraph whose components are M-alternating trees.
Define the inner and outer vertices of F in the obvious way.

Proposition 5.6.1
Given a matching M is a graph G, we can find (efficiently) find either

(a) an M-augmenting path.

(b) an M-alternating forest so that

  (i) every M-unsaturated vertex of G is a vertex of F

  (ii) the neighbours of each outer vertex v in F are either inner vertices of F, or outer vertices in the same component of F as v

Proof
Start with $F = \{M\text{-unsaturated vertices}\}$.

While an outer vertex v of F has a neighbour v' outside F, since $v' \notin V(F)$, v' is saturated and therefore matched by M to some $w \notin V(F)$.

Let T_i be the component of F containing v. Replace T_i with

$$T_i \cup \{vv', v'w\}$$

to form a larger M-alternating forest F.

After the loop, all neighbours of outer vertices of F are in F. If v_i, v_j are outer vertices in distinct components T_i, T_j of F that are adjacent, then

$$P_i \cup \{v_i v_j\} \cup P_j$$

is an M-augmenting path, where P_i is the path in T_i from the root to v_i.

Thus, if we cannot find an M-augmenting path, F satisfies the alternative scenario of the proposition.

Corollary 5.6.1.1
If there are no edges between outer vertices in the same component, then G is bipartite and we are done.

Proof
Bipartiteness is clear.

Let F be given by the proposition. If there are no edges in F between outer vertices, then let

$$S := \{\text{inner vertices}\}$$

The outer vertices are isolated in $G - S$ so the outer vertices are precisely the odd components of $G - S$. Remark that

$$\begin{aligned}
\text{odd}(G - S) - |S| &= \#\text{ outer vertices} - \#\text{ inner vertices} \\
&= \#\text{ components of } F \\
&= \#\text{ unsaturated vertices of } F && \text{construction}
\end{aligned}$$

so S is a Berge Witness and M is a maximum matching.

Definition 5.6.5 (Blossom)
Tight odd circuit with only one unsaturated vertex.

Proposition 5.6.2
If C is an odd circuit in G and M is a matching of G such that M contains a maximum matching of C and the other vertex of C is unsaturated, then M is a maximum matching of G if and only if $M \setminus E(C)$ is a maximum matching of G/C.
Moreoever, if M_0 is any matching in G/C, then M_0 can be extended to a matching of G of size

$$|M_0| + \frac{1}{2}(|E(C)| - 1)$$

Proof
Assignment.

This allows us, given a matching M and a blossom C for M, to reduce the problem of finding a matching larger than M to find a maximum matching of G/C.

In other words, if $M \setminus E(C)$ is a maximum matching in G/C, stop; M is a maximum matching in G. Otherwise, let M_0 be a maximum matching in G/C; Extend M_0 to a matching of size

$$\begin{aligned}
|M_0| + \frac{1}{2}(|E(G)| - 1) &> |M \setminus E(C)| + \frac{1}{2}(|E(C)| - 1) \\
&= |M|
\end{aligned}$$

Proposition 5.6.3
If there is an edge e between outer vertices v, v' in the same component, we can recurse on a smaller graph.

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Proof
let C be the odd circuit of

$$T \cup \{e\}$$

containing e, let P be the shortest path in T from the circuit to C.

Now, C is a blossom in the matching

$$M \Delta P$$

which has size M. Use C recursively to either conclude that $M \Delta P$ is a maximum matching in G, or to find a larger matching in G.

Since matchings have size $< \frac{1}{2}|V(G)|$ we only find an augmenting path or recurse a maximum of $O(|V(G)|)$ times, gives a polytime algorithm.

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6 Bonus Material

6.1 Extremal Graph Theory

Does there exist a graph with chromatic colour 100?

The easy answer is K_{100}. But how but if we restrict to graphs which are only allowed circuits of size at least 1000?

6.2 Algebraic Graph Theory

From the assignment, we have seen the signed incidence matrix.

Let us consider the incidence matrix M of a graph G over the field $\mathbb{Z}_2$.

6.2.1 Cut Space

What is the row space of M?

Consider a sum of the rows of M. It is the sum of rows of $S \subseteq V(G)$. Any edge with both ends in S sum to 0 and any edge with no ends in S are no represented. It follows that $\sum S$ is the sum of edges with one edge in S ie the cut of S.

So the rows space of M is the space of all cuts.

Definition 6.2.1 (Cut Space)
The row space of the incidence matrix.

6.2.2 Cycle Space

What is the null space of M?

It is the set of vectors which are orthogonal to the row space. Let u be such that $Mu = 0$. u picks a combination of the edges such that every vertex is incident to an even number of edges in some subgraph of G.

So u corresponds to a cycle.

Definition 6.2.2 (Cycle Space)
The null space of the incidence matrix.

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We can define bipartite graphs and planar graphs using linear algebaric characterizations.

Theorem 6.2.1
Let $\mathcal{C}$ denote the cut space of a planar graph G. Then

$$\mathcal{C}^*(G^*) = C(G)$$

6.3 Bases of the Incidence Matrix

Proposition 6.3.1
Let B be the incidence matrix of a simple graph G. B is totally unimodular.

Proof
Induction.

Theorem 6.3.2 (Regular Matroid Decomposition Theorem)
Every regular matroid can be obtained from graphic matroids, their duals, and ...

What is $BB^T =: L$?

$$L_{u,w} = B_u^T B_w = \begin{cases} -1, & uw \in E \\ 0, & uw \notin E \\ \deg(u), & u = w \end{cases}$$

Definition 6.3.1 (Laplacian)
$L(G) = BB^T$ of the incidence matrix B of a simple graph G.

$uB = 0 \implies uBB^T = 0$ so it has determinant 0.

Let

$$L_v = B_v B_v^T$$

where B_v is the incidence matrix less the row corresponding to v.

Proposition 6.3.3 (Cauchy-Binet Formula)

$$\det AA^T = \sum_{r \times r \text{ submatrices } A' \text{ of } A} \det(A')^2$$

so

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Theorem 6.3.4 (Matrix Tree Theorem, Kirchoff)

$$\begin{aligned}\det L_v &= \det B_v B_v^T \\ &= \sum_{(|V|-1)^2 \text{ submatrices } B' \text{ of } B} \det(B')^2 \\ &= \#\text{ of spanning trees of } G\end{aligned}$$

Corollary 6.3.4.1 (Cayley)
The number of spanning trees on K_n is n^{n-2}.

Proof
L_v of K_n is $n-1$ on the diagonal and -1 elsewise.

$$L_v = nI_{n-1} - 1_{(n-1)^2}$$

Say v is an eigenvector of L_v

$$\begin{aligned}(nI_{n-1} - J)x &= \lambda x \\ -Jx &= (\lambda - n)x \\ Jx &= (n-\lambda)x\end{aligned}$$

so x is an eigenvector of J with eigenvalue $n-\lambda$.

Now the nullspace of J_{n-1} has dimension $n-2$ so 0 is an eigenvalue with multiplicity $n-2$. The only other eigenvalue is $n-1$ as the trace is the sum of the eigenvalues.

So L_v has eigenvalues

$$n, \ldots, n, 1$$

and so

$$\det L_v = n^{n-2}$$

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